Algebraic Structures Derived from Essential Surfaces and Foams

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Outline

1 Motivations
2 Essential surfaces
3 Frobenius pairs
4 Constructions
5 Foams
6 Lie algebras
7 Bialgebras
8 Skein modules
Background

\{ (1 + 1)-TQFTs \} \overset{1-1}{\leftrightarrow} \{ \text{Commutative Frobenius algebras} \}

Diagrammatics of Frobenius algebras, generating operators:

- Multiplication \(\mu\)
- Comultiplication \(\Delta\)
- Unit \(\eta\)
- Counit \(\varepsilon\)
- Transposition \(\tau\)

Relations:

- Associativity
- Coassociativity
- Compatibility
- Unit
- Counit
- Commutative
Problem: Characterize TQFTs of essential surface cobordisms and foams.

We focus on thickened surfaces $F \times [0, 1]$ and $sl(3)$-foams, and investigate their algebraic structures.
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Motivations

- A natural question as generalizations of $(1+1)$-TQFT.
- Essential cobordisms are used in global Khovanov homology by Asaeda–Przytycki-Sikora [APS] (for thickened surfaces $F \times [0,1]$), Turaev-Turner [TT] (unoriented TQFT) and Ishii-Tanaka [IT] (for virtual knots, cf. [Manturov]). They should have algebraic structures.
- Foams appear in $sl(3)$ Khovanov homology [Mackaay-Vaz].
- By considering such algebraic structures of generalized TQFTs, new generalizations of KhoHo may be found.
- Possible refinements of Bar-Natan modules by Asaeda-Frohman, Kaiser.
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Generating maps for some saddle points:

These suggest module and comodule structures over Frobenius algebras.
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Relations are checked case by case. For example, if a handle is present as depicted, we obtain a relation:
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1 Motivations

2 Essential surfaces

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4 Constructions

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7 Bialgebras

8 Skein modules
Frobenius pairs

By checking all conditions in [APS],

**Definition**

We propose a *commutative Frobenius pair* \((A, E)\):

(i) \(A = (m_A, \Delta_A, \iota_A, \epsilon_A)\) is a commutative Frobenius algebra over \(k\).

(ii) \(E\) is an \(A\)-bimodule and \(A\)-bicomodule, with the same right and left actions and coactions.
Frobenius pairs (cont.)

These should satisfy variety of conditions:

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics{fn2.png}}
\end{array}
\end{align*}
\]

Roughly, the dotted line does not end at a trivalent vertex, while a solid line can. All relations that satisfy this condition hold.
Möbius maps

Three maps are called **Möbius maps** if they satisfy:

\[ \begin{align*}
(1) &: E \rightarrow A \\
(2) &: A \rightarrow E \\
(3) &: E \rightarrow E \\
(4) &: Y \rightarrow Y \\
(5) &: Y \rightarrow Y \\
(6) &: Y \rightarrow Y \\
(7) &: Y \rightarrow Y \\
(8) &: Y \rightarrow Y \\
(9) &: Y \rightarrow Y \\
(10) &: Y \rightarrow Y
\end{align*} \]

corresponding to non-orientable cobordisms.
Möbius maps

For example, Case (4) of [APS] gives rise to one of the conditions:
Examples

Example

For [APS]: $A = \mathbb{Z}[X]/(X^2)$, $E = \langle Y, Z \rangle_\mathbb{Z}$ with correspondence: $1 \leftrightarrow -$, $X \leftrightarrow +$, $Y \leftrightarrow -0$, $Z \leftrightarrow +0$. Multiplications: $XY = XZ = Y^2 = Z^2 = 0$ and $YZ = X$. Comultiplications:

$$\Delta(1) = 1 \otimes X + X \otimes 1 + Y \otimes Z + Z \otimes Y,$$

$$\Delta(X) = X \otimes X, \quad \Delta(Y) = X \otimes Y, \quad \Delta(Z) = X \otimes Z.$$

Möbius maps: $\nu(1) = Y + Z$, $\nu(X) = 0$, $\nu(Y) = \nu(Z) = X$.

Example

For [TT]: $A = E = \mathbb{Z}_2[X, \lambda^{\pm 1}]/(X^2 - hX)$ with $h = \lambda^2$. All multiplications and comultiplications are those of $A$. All Möbius maps are defined by multiplication by $\lambda$.

Note: $(\mu \Delta)(1) = h = \lambda^2$.
Examples

Example

For [APS]: \( A = \mathbb{Z}[X]/(X^2), \ E = \langle Y, Z \rangle_\mathbb{Z} \) with correspondence: \( 1 \leftrightarrow -, \ X \leftrightarrow +, \ Y \leftrightarrow -0, \ Z \leftrightarrow +0. \)

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Comultiplications:

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\Delta(1) = 1 \otimes X + X \otimes 1 + Y \otimes Z + Z \otimes Y,
\]

\[
\Delta(X) = X \otimes X, \ \Delta(Y) = X \otimes Y, \ \Delta(Z) = X \otimes Z.
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Note: \( (\mu \Delta)(1) = h = \lambda^2, \ \begin{array}{c}
\includegraphics[width=0.1\textwidth]{example_diagram_1}
\end{array} = \sqrt{\begin{array}{c}
\includegraphics[width=0.1\textwidth]{example_diagram_2}
\end{array}.}
Virtual circles with poles

A circle with pairs of opposite poles are considered essential. The two smoothings correspond to a cobordism between essential and inessential circles.
Virtual circles with poles

Smoothings for the Miyazawa polynomial

Smoothings of a virtual Hopf link

A circle with pairs of opposite poles are considered essential. The two smoothings correspond to a cobordism between essential and inessential circles.
Virtual circles with poles

Relations among poles in the Miyazawa polynomial

Hemmed cobordisms formed by poles

Cobordisms of virtual circles with poles have a structure of commutative Frobenius pairs with Möbius maps.
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A generalization of [APS]:

**Theorem**

Let $A = \mathbb{Z}[X,h,t]/(X^2 - hX - t)$ and $E = \langle Y,Z \rangle$. If $(A,E)$ is a commutative Frobenius pair with Möbius maps such that $\mu^E_{E,E} = \Delta^E_{E,E} = 0$, then $A$ must be $A = \mathbb{Z}[X,a]/(X - a)^2$. For this $A$, operations are defined by

$$XY = aY, \quad XZ = aZ, \quad Y^2 = c_{YY}(X - a),$$

$$YZ = c_{YZ}(X - a), \quad Z^2 = c_{ZZ}(X - a),$$

$$\Delta^A_{E,E}(Y) = (X - a) \otimes Y, \quad \Delta^A_{E,E}(Z) = (X - a) \otimes Z,$$

$$\Delta^E_{A,E}(1) = d_{YY} Y \otimes Y + d_{YZ} Y \otimes Z + d_{YZ} Z \otimes Y$$

$$+ d_{ZZ} Z \otimes Z, \quad \Delta^E_{A,E}(X) = a \Delta^E_{A,E}(1),$$

for some constants that satisfy certain conditions.
### Constructions

<table>
<thead>
<tr>
<th></th>
<th>[APS]</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\mathbb{Z}[X]/(X^2)$</td>
<td>$\mathbb{Z}[X, a]/(X - a)^2$</td>
</tr>
<tr>
<td>$XY$</td>
<td>0</td>
<td>$aY$</td>
</tr>
<tr>
<td>$XZ$</td>
<td>0</td>
<td>$aZ$</td>
</tr>
<tr>
<td>$Y^2$</td>
<td>0</td>
<td>$c_{YY}(X - a)$</td>
</tr>
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</tr>
<tr>
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</tr>
<tr>
<td>$\Delta^{A,E}_E(Y)$</td>
<td>$X \otimes Y$</td>
<td>$(X - a) \otimes Y$</td>
</tr>
<tr>
<td>$\Delta^{A,E}_E(Z)$</td>
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</tr>
<tr>
<td>$\Delta^{A,E}_E(1)$</td>
<td>$Y \otimes Z + Z \otimes Y$</td>
<td>$d_{YY} Y \otimes Y + d_{YZ} Y \otimes Z$ $+ d_{YZ} Z \otimes Y + d_{ZZ} Z \otimes Z$</td>
</tr>
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<td>$a \Delta^{E,E}_A(1)$ $a$</td>
</tr>
<tr>
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Constructions

A generalization of [TT]:

**Theorem**

Let $A$ be a commutative Frobenius algebra with handle element $\phi$, such that there exists an element $\xi \in A$ with $\xi^2 = \phi$. Then there exists a commutative Frobenius pair with Möbius maps.

**Corollary**

Let $A = E = k[X]/(X^2 - hX - t)$, where $k = \mathbb{Z}[a^\pm 1, b^\pm 1]$, and $h = -2b^{-1}(a - b^{-1})$, $t = -b^{-2}(a^2 + h)$. Then $(A, E)$ gives rise to a commutative Frobenius pair with Möbius maps.

*Proof.* Let $\xi = a + bX$, and one computes that $\xi^2 = \phi = 2X - h$. 
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A generalization of [IT]:

**Theorem**

Let $A$ be a commutative Frobenius algebra with handle element $\phi$ such that its handle element $\phi \in A$ is invertible, then there exists a commutative Frobenius pair $(A, E)$ with Möbius maps.

**Idea of Proof:** Take $E = A \otimes A$ and define maps as follows:

There is a solution for constants: $e_0 = -1$, $e_1 = e_2 = -2$, $\nu_0 = 1$, $\nu_1 = -1$ and $\nu_2 = 0$. 
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Example

In [Mackaay-Vaz], the $sl(3)$-invariant was described by foams, with Frobenius algebra structure defined by
\[ A = \mathbb{Z}[a, b, c][X]/(X^3 - aX^2 - bX - c) \]
with the multiplication and the unit are defined by those for polynomials,
the Frobenius form (counit) $\epsilon$ is defined by
\[ \epsilon(1) = \epsilon(X) = 0, \quad \epsilon(X^2) = -1, \]
the comultiplication is computed as
\[ \Delta(1) = -(1 \otimes X^2 + X \otimes X + X^2 \otimes 1) + a(1 \otimes X + X \otimes 1) + b(1 \otimes 1), \]
\[ \Delta(X) = -(X \otimes X^2 + X^2 \otimes X) + a(X \otimes X) - c(1 \otimes 1), \]
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What does the branch line represent?
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What does the branch line represent?
Example (cont.)

Following our approach, evaluate the branch line operation by skein relation. For $A = \mathbb{Z}[X]/(X^N)$, for example, it looks like:

\[
[X^j, X^k] = \sum_{i=0}^{N-1} \theta(X^i, X^j, X^k)X^{N-1-i}
\]

**Theorem**

The branch curve operation $[\ , \ ]$ is skew-symmetric and satisfies the Jacobi identity:

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Using the Frobenius form, define $\Delta : A \to A \otimes A$ by
$$\Delta(V) = \sum [V, 1_{(1)}] \otimes 1_{(2)}, \text{where } \Delta(u) = \sum u_{(1)} \otimes u_{(2)}.$$ Similarly, for $\Delta$, denote $\Delta(u) = \sum u_{((1))} \otimes u_{((2))}$. 

![Diagram](image-url)
Example (cont.)

**Theorem**

For $A = \mathbb{Z}[a, b, c]/(X^3 - aX^2 - bX - c)$ with $\theta$ values as above, the map $\Delta : A \to A \otimes A$ satisfies the following identities:

\[
(m \otimes id)(id \otimes \Delta) = \Delta(1)(\epsilon \mu) - \tau,
\]

\[
( (m \otimes id)(id \otimes \Delta) )^2 = id + \Delta(1)(\epsilon \mu),
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\[
m\Delta = 2 \text{ id}.
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Same as $sl(3)$ relations!
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Example (cont.)

Relations to foam skeins:

\[ \begin{array}{ccc}
\text{Lie alg.} & \rightarrow & \text{Rep. of } q\text{-gp} \\
\text{Categorify by foams} & & \text{quantum knot inv.} \\
& & \text{TQFT and foams} \\
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Categorify by foams

KhoHo
Example (cont.)

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Are there other TQFTs and theta foam values that produce Lie algebras?

**Theorem**

For any positive odd integer $N > 1$, there exists a TQFT that induces a Lie algebra structure along branch circles.

**Sketch proof.** Let $A = R[X]/(X^N)$ for an odd integer $N > 1$. For $N > 3$, define

$$\theta(X^a, X^b, X^c) = \begin{cases} 
1 & \text{if } a = 0, b + c = N, 1 < b < c, \\
-1 & \text{if } a = 0, b + c = N, 1 < c < b, \\
0 & \text{otherwise}
\end{cases}$$

and the same values for cyclic permutations.
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and the same values for cyclic permutations.
Let $A = R[G]$ be the group ring with a commutative unital ring $R$.

\[ \Delta(x) = x \otimes x : \text{bialgebra comultiplication} \]

\[ \Delta(x) = \sum_{x=yz} y \otimes z : \text{Frobenius algebra comultiplication} \]

In this case, foams admit both structures (saddle $\leftrightarrow$ Frobenius, branch lines $\leftrightarrow$ bialgebra) only when every element of $G$ is of order 2.

Compatibility $\Delta(xy) = \Delta(x)\Delta(y)$
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In this case, foams admit both structures (saddle \( \leftrightarrow \) Frobenius, branch lines \( \leftrightarrow \) bialgebra) only when every element of \( G \) is of order 2.

Compatibility

\[ \Delta(xy) = \Delta(x)\Delta(y) \]
Bialgebras

Let $A = R[G]$ be the group ring with a commutative unital ring $R$.

- $\Delta(x) = x \otimes x$: bialgebra comultiplication
- $\Delta(x) = \sum_{x=yz} y \otimes z$: Frobenius algebra comultiplication

In this case, foams admit both structures (saddle $\leftrightarrow$ Frobenius, branch lines $\leftrightarrow$ bialgebra) only when every element of $G$ is of order 2.

Compatibility $\Delta(xy) = \Delta(x)\Delta(y)$
Definition

\( M \): a compact 3-manifold

\( \text{FF}(M) \): free module on foams in \( M \)

\( \text{FS}(M) \): submodule generated by skein relations in the figure, where a dot represents \( x \in A = R[x]/(x^2 - 1) \)

\( \text{F}(M) = \text{FF}(M)/\text{FS}(M) \): the surface skein module with \( A \)

An analogue of Bar-Natan skein module by Asaeda-Frohman, Kaiser.
Skein modules

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Conclusion:

For essential cobordisms:

We proposed an algebraic structure, commutative Frobenius pairs, that describes essential cobordisms in thickened surfaces and cobordisms formed by virtual circles with poles.

Constructions and new examples are provided, that may be useful in further generalizing Khovanov homology for thickened surfaces and virtual knots.

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We studied Lie algebra and bialgebra structures along branch circles.

Skein modules and more general foams need more study.
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