# Combined Wronskian solutions to the 2D Toda molecule equation

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**Abstract**

By combining two pieces of bi-directional Wronskian solutions, molecule solutions in Wronskian form are presented for the finite, semi-infinite and infinite bilinear 2D Toda molecule equations. In the cases of finite and semi-infinite lattices, separated-variable boundary conditions are imposed. The Jacobi identities for determinants are the key tool employed in the solution formulations.

**Key words:** Toda lattice, Wronskian solution, Soliton equation

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## 1 Introduction

The study of soliton equations presents interesting mathematical theories to deal with nonlinear equations. Wronskian determinants, double Wronskian determinants and bi-directional Wronskian determinants are used to construct exact solutions to soliton equations, among which are the KdV equation, the Boussinesq equation, the KP equation, the Toda lattice equation and the 2D Toda lattice equation (see, e.g., [1]-[10]). The Plücker relations for determinants and the Jacobi identities for determinants are the key tools employed in formulating exact solutions to soliton equations [1, 11].

Generic multi-exponential wave solutions can be constructed by the multiple exp-function method [12]. The approach generalizes the transformed rational function method [13] and the Hirota perturbation technique [1], and it is very powerful while applying computer algebra systems [12]. The resulting multiple wave solutions contain linear combination solutions of exponential waves [14, 15] and resonant solitons [16].

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This also shows that soliton equations can possess linear superpositions among particular solutions [14, 15], and thus possess linear subspaces of solutions. Therefore, though soliton equations are nonlinear, they are good neighbors to linear equations.

However, given the complexity that nonlinear equations bring, there is a need to develop more explicit and systematic formulations for generating exact solutions. This paper is one of such explorations.

In this paper, we would like to formulate molecule bi-directional Wronskian solutions for the 2D Toda molecule (2DTM) equation in bilinear form:

$$\frac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1}$$

in three cases of the finite lattice: $1 \le n \le N$, the semi-infinite lattice: $1 \le n < \infty$, and the infinite lattice: $-\infty < n < \infty$. For the first two cases, we impose the separated-variable boundary conditions:

$$\tau_0 = \phi_1(x)\chi_1(y),\ \tau_{N+1} = \phi_2(x)\chi_2(y);$$

and

$$\tau_0 = \phi(x)\chi(y);$$

respectively, where all $\phi$- and $\chi$-functions are arbitrarily given. The difference among these three cases is that we simply don't require any boundary conditions at $n = \pm\infty$. We will show that combining two pieces of bi-directional Wronskian solutions [17] yields a required molecule solution. By molecule solutions, we mean a kind of determinant solutions whose determinants have orders depending on the discrete independent variable $n$. The Jacobi identities for determinants are the key tool employed in the solution formulations

# 2 Bi-directional Wronskians and the Jacobi identity

We provide the definition of the bi-directional Wronskian determinant and discuss the Jacobi identity for determinants for the reader's convenience and ease of reference.

A bi-directional Wronskian determinant is defined as follows.

**Definition 2.1** *A bi-directional Wronskian determinant of order $n$ associated with*

$\Upsilon = \Upsilon(x,y)$ *is defined by*

$$\left|\left(\frac{\partial}{\partial x}\right)^{i-1}\left(\frac{\partial}{\partial y}\right)^{j-1}\Upsilon\right|_{1\le i,j\le n} = \begin{vmatrix} \Upsilon & \frac{\partial}{\partial y}\Upsilon & \cdots & \frac{\partial^{n-1}}{\partial y^{n-1}}\Upsilon \\ \frac{\partial}{\partial x}\Upsilon & \frac{\partial^2}{\partial x\partial y}\Upsilon & \cdots & \frac{\partial^n}{\partial x\partial y^{n-1}}\Upsilon \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{n-1}}{\partial x^{n-1}}\Upsilon & \frac{\partial^n}{\partial x^{n-1}\partial y}\Upsilon & \cdots & \frac{\partial^{2n-2}}{\partial x^{n-1}\partial y^{n-1}}\Upsilon \end{vmatrix}. \quad (2.1)$$

The determinant in (2.1) is a Wronskian determinant in both horizontal and vertical directions. That is why it is called bi-directional.

Let us next state the Jacobi identity and give a direct proof by using the Laplace Expansion Theorem. Let $n > 2$ be an integer, $A = (a_{i,j})_{1\le i,j\le n}$ be a square matrix of order $n$ and $D$ denote the determinant of $A$, that is,

$$D = \det(A) = |a_{i,j}|_{1\le i,j\le n}. \quad (2.2)$$

So $D$ is an $n$th-order determinant.

The $(i,j)$ minor of $A$ is defined as the $(n-1)$th-order determinant obtained by striking out the $i$th row and the $j$th column of $D$, denoted by $D\begin{bmatrix} i \\ j \end{bmatrix}$. All such minors are called first minors. The $(i,j;k,l)$ minor of $A$ is defined as the $(n-2)$th-order determinant obtained by striking out the $i$th and $j$th rows and the $k$th and $l$th columns of $D$, denoted by $D\begin{bmatrix} i,j \\ k,l \end{bmatrix}$. All such minors are called second minors.

Now we can state the Jacobi identity [1, 18] as follows.

**Theorem 2.1** *Let* $n > 3$, $A = (a_{i,j})_{1\le i,j\le n}$ *and* $D = \det(A)$. *For* $1 \le i \ne j \le n$, *we have*

$$D\begin{bmatrix} i \\ i \end{bmatrix} D\begin{bmatrix} j \\ j \end{bmatrix} - D\begin{bmatrix} i \\ j \end{bmatrix} D\begin{bmatrix} j \\ i \end{bmatrix} = D\begin{bmatrix} i,j \\ k,l \end{bmatrix} D, \quad (2.3)$$

*where* $D\begin{bmatrix} i \\ j \end{bmatrix}$ *and* $D\begin{bmatrix} i,j \\ k,l \end{bmatrix}$ *are the* $(i,j)$ *minor and the* $(i,j;k,l)$ *minor of* $A$, *respectively.*

*Proof*: By the properties of determinants, without loss of generality, we only need to verify the Jacobi identity for $i = 1$ and $j = 2$. Let us denote the $(i,j)$ cofactor of $A$ by

$C_{i,j}$:

$$C_{i,j} = (-1)^{i+j} D \begin{bmatrix} i \\ j \end{bmatrix}. \tag{2.4}$$

We partition the matrix $A$ into four blocks as follows:

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right], \ A_1 = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}. \tag{2.5}$$

By the Laplace Expansion Theorem, we have

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{cc|ccc} C_{1,1} & C_{1,2} & & 0 & \\ C_{2,1} & C_{2,2} & & & \\ \hline C_{3,1} & C_{3,2} & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ C_{n,1} & C_{n,2} & 0 & & 1 \end{array}\right] = \left[\begin{array}{cc|c} D & 0 & \\ 0 & D & A_2 \\ \hline & 0 & A_4 \end{array}\right].$$

Taking determinants on both sides leads to

$$D(C_{1,1}C_{2,2} - C_{1,2}C_{2,1}) = D \begin{bmatrix} 1,2 \\ 1,2 \end{bmatrix} D^2. \tag{2.6}$$

If $D \neq 0$, this gives the desired Jacobi identity, upon using (2.4). If $D = 0$, we take another matrix $A' = A + \varepsilon I_n$, where $I_n$ is the $n$th-order identity matrix, and then taking the limit $\varepsilon \to 0$ of the resulting identity (2.6) associated with $A'$ yields the desired Jacobi identity. Note that we have used a fact that if $\varepsilon$ is small enough, the matrix $A'$ is invertible. □

# 3 Combined bi-directional Wronskian solutions

Let us now start to construct combined bi-directional Wronskian solutions to the 2D Toda molecule (2DTM) equation

$$\frac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1}, \tag{3.1}$$

which is equivalent to

$$D_x D_y \tau_n \cdot \tau_n = 2\tau_{n+1}\tau_{n-1}, \tag{3.2}$$

where $D_x$ and $D_y$ are Hirota's differential operators [1, 19]. We will present the solution formulations in the finite, semi-infinite and infinite cases separately.

### 3.1 Finite lattice

We consider the finite 2DTM equation

$$\frac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1},\ 1 \le n \le N, \tag{3.3}$$

with the following separated-variable boundary conditions:

$$\tau_0 = \phi_1(x)\chi_1(y),\ \tau_{N+1} = \phi_2(x)\chi_2(y), \tag{3.4}$$

where $\phi_i$ and $\chi_i$, $i = 1, 2$, are four arbitrarily given functions of the indicated variables. We apply the Jacobi identities for determinants to guarantee a class of combined molecule bi-directional Wronskian solutions to this boundary problem.

Set $N = N_1 + N_2 + 4$, where $N_1$ and $N_2$ are non-negative integers. Let us combine two pieces of bi-directional Wronskian determinant functions to introduce $\tau_n$ as follows:

$$\begin{cases} \tau_n = \left| \left(\frac{\partial}{\partial x}\right)^{i-1} \left(\frac{\partial}{\partial y}\right)^{j-1} \Phi(x,y) \right|_{1\le i,j\le N_1-n+1}, \quad 0 \le n \le N_1, \\ \tau_{N_1+1} = 1,\ \tau_{N_1+2} = 0,\ \tau_{N_1+3} = 0,\ \tau_{N_1+4} = 1, \\ \tau_n = \left| \left(\frac{\partial}{\partial x}\right)^{i-1} \left(\frac{\partial}{\partial y}\right)^{j-1} \Psi(x,y) \right|_{1\le i,j\le n-N+N_2}, \quad N_1 + 5 \le n \le N + 1. \end{cases} \tag{3.5}$$

One piece is defined over the $N_1$ lattice points: $1 \le n \le N_1$, and the other piece, over the $N_2$ lattice points: $N - N_2 + 1 = N_1 + 5 \le n \le N$. In between, set $\tau_n$ as either zero or one.

We will prove that this combined Wronskian determinant function solves the 2DTM equation (3.3) with the boundary conditions (3.4). Note that the two involved determinants in the presented solution formulation are bi-directional Wronskian determinants and their orders depend on the discrete independent variable $n$. Therefore, (3.5) presents combined molecule bi-directional Wronskian solutions.

**Solving the 2DTM equation:** Let us first prove that $\tau_n$ defined by (3.5) solves the 2DTM equation (3.3) when $1 \le n \le N_1$. For brevity, we assume that

$$\Phi_{i,j} = \left(\frac{\partial}{\partial x}\right)^{i} \left(\frac{\partial}{\partial y}\right)^{j} \Phi(x,y),\ i, j \ge 0. \tag{3.6}$$

If $n = N_1$, the 2DTM equation (3.3) becomes

$$\Phi_{1,1}\Phi_{0,0} - \Phi_{1,0}\Phi_{0,1} = \begin{vmatrix} \Phi_{0,0} & \Phi_{0,1} \\ \Phi_{1,0} & \Phi_{1,1} \end{vmatrix}, \tag{3.7}$$

which is obviously true. Let $1 \le n \le N_1 - 1$. We introduce three kinds of determinants:

$$D_1 = \left| \left(\frac{\partial}{\partial x}\right)^{i-1} \left(\frac{\partial}{\partial y}\right)^{j-1} \Phi(x,y) \right|_{1 \le i,j \le N_1-n+2} = \tau_{n-1}, \tag{3.8}$$

$$D_1 \begin{bmatrix} i \\ j \end{bmatrix} = \text{the determinant obtained by striking out the } i\text{th row and } j\text{th column of } D_1, \tag{3.9}$$

$$D_1 \begin{bmatrix} i,j \\ k,l \end{bmatrix} = \text{the determinant obtained by striking out the } i\text{th and } j\text{th rows and the } k\text{th and } l\text{th columns of } D_1, \tag{3.10}$$

which are the determinant, and a first minor and a second minor of a corresponding matrix, respectively. Using this determinant notation, we can easily compute

$$\tau_n = D_1 \begin{bmatrix} N_1-n+2 \\ N_1-n+2 \end{bmatrix}, \ \tau_{n+1} = D_1 \begin{bmatrix} N_1-n+1, N_1-n+2 \\ N_1-n+1, N_1-n+2 \end{bmatrix},$$
$$\frac{\partial \tau_n}{\partial x} = D_1 \begin{bmatrix} N_1-n+1 \\ N_1-n+2 \end{bmatrix}, \ \frac{\partial \tau_n}{\partial y} = D_1 \begin{bmatrix} N_1-n+2 \\ N_1-n+1 \end{bmatrix},$$
$$\frac{\partial^2 \tau_n}{\partial x \partial y} = D_1 \begin{bmatrix} N_1-n+1 \\ N_1-n+1 \end{bmatrix}.$$

Now it follows that for each $1 \le n \le N_1 - 1$, the 2DTM equation (3.3) is equivalent to

$$D_1 \begin{bmatrix} N_1-n+1 \\ N_1-n+1 \end{bmatrix} D_1 \begin{bmatrix} N_1-n+2 \\ N_1-n+2 \end{bmatrix} - D_1 \begin{bmatrix} N_1-n+1 \\ N_1-n+2 \end{bmatrix} D_1 \begin{bmatrix} N_1-n+2 \\ N_1-n+1 \end{bmatrix}$$
$$= D_1 \begin{bmatrix} N_1-n+1, N_1-n+2 \\ N_1-n+1, N_1-n+2 \end{bmatrix} D_1.$$

These are simply the Jacobi identities for determinants. Therefore, $\tau_n$ defined by (3.5) solves (3.3) when $1 \le n \le N_1$.

When $n = N_1 + i$, $1 \le i \le 4$, it is direct to check that the 2DTM equation (3.3) holds.

Let us now similarly prove that $\tau_n$ defined by (3.5) solves the 2DTM equation (3.3) when $N_1 + 5 \le n \le N$. Assume for brevity that

$$\Psi_{i,j} = \left(\frac{\partial}{\partial x}\right)^{i} \left(\frac{\partial}{\partial y}\right)^{j} \Psi(x,y), \ i,j \ge 0. \tag{3.11}$$

If $n = N_1 + 5$, the 2DTM equation (3.3) reduces to

$$\Psi_{1,1}\Psi_{0,0} - \Psi_{1,0}\Psi_{0,1} = \begin{vmatrix} \Psi_{0,0} & \Psi_{0,1} \\ \Psi_{1,0} & \Psi_{1,1} \end{vmatrix}, \tag{3.12}$$

which is clearly right. Let $N_1+6 \leq n \leq N$. To apply the Jacobi identities for determinants, we introduce three kinds of determinants:

$$D_2 = \left| \left(\frac{\partial}{\partial x}\right)^{i-1} \left(\frac{\partial}{\partial y}\right)^{j-1} \Psi(x,y) \right|_{1\leq i,j\leq n-N+N_2+1} = \tau_{n+1}, \tag{3.13}$$

$$D_2 \begin{bmatrix} i \\ j \end{bmatrix} = \text{the determinant obtained by striking out the } i\text{th row and } j\text{th column of } D_2, \tag{3.14}$$

$$D_2 \begin{bmatrix} i,j \\ k,l \end{bmatrix} = \text{the determinant obtained by striking out the } i\text{th and } j\text{th rows and the } k\text{th and } l\text{th columns of } D_2, \tag{3.15}$$

which are the determinant, and a first minor and a second minor of a corresponding matrix, respectively. In terms of this determinant notation, we can easily obtain

$$\tau_n = D_2 \begin{bmatrix} n-N+N_2+1 \\ n-N+N_2+1 \end{bmatrix}, \ \tau_{n-1} = D_2 \begin{bmatrix} n-N+N_2, n-N+N_2+1 \\ n-N+N_2, n-N+N_2+1 \end{bmatrix},$$

$$\frac{\partial \tau_n}{\partial x} = D_2 \begin{bmatrix} n-N+N_2 \\ n-N+N_2+1 \end{bmatrix}, \ \frac{\partial \tau_n}{\partial y} = D_2 \begin{bmatrix} n-N+N_2+1 \\ n-N+N_2 \end{bmatrix},$$

$$\frac{\partial^2 \tau_n}{\partial x \partial y} = D_2 \begin{bmatrix} n-N+N_2 \\ n-N+N_2 \end{bmatrix}.$$

Then it follows from these formulas that for each $N_1+6 \leq n \leq N$, the 2DTM equation (3.3) is equivalent to

$$\begin{aligned} & D_2 \begin{bmatrix} n-N+N_2 \\ n-N+N_2 \end{bmatrix} D_2 \begin{bmatrix} n-N+N_2+1 \\ n-N+N_2+1 \end{bmatrix} \\ & -D_2 \begin{bmatrix} n-N+N_2 \\ n-N+N_2+1 \end{bmatrix} D_2 \begin{bmatrix} n-N+N_2+1 \\ n-N+N_2 \end{bmatrix} \\ = \ & D_2 \begin{bmatrix} n-N+N_2, n-N+N_2+1 \\ n-N+N_2, n-N+N_2+1 \end{bmatrix} D_2. \end{aligned}$$

These are exactly the Jacobi identities for determinants. Therefore, $\tau_n$ defined by (3.5) solves the 2DTM equation (3.3) when $N_1+5 \leq n \leq N$.

**Satisfying the boundary conditions:** To satisfy two boundary conditions in (3.4), we require that

$$\Phi(x,y) = \sum_{j=1}^{N_1+1} u_j(x) v_j(y), \ \Psi(x,y) = \sum_{j=1}^{N_2+1} r_j(x) s_j(y), \tag{3.16}$$

where all functions $u_j, v_j, r_j$ and $s_j$ are to be determined.

Let us first compute $\tau_0$ as follows:

$$\begin{aligned}
\tau_0 &= \left|\left(\frac{\partial}{\partial x}\right)^{i-1}\left(\frac{\partial}{\partial y}\right)^{k-1}\Phi(x,y)\right|_{1\le i,k\le N_1+1} \\
&= \left|\left(\frac{\partial}{\partial x}\right)^{i-1}\left(\frac{\partial}{\partial y}\right)^{k-1}\sum_{j=1}^{N_1+1} u_j(x)v_j(y)\right|_{1\le i,k\le N_1+1} \\
&= \left|\sum_{j=1}^{N_1+1}\left(\frac{\partial}{\partial x}\right)^{i-1}u_j(x)\left(\frac{\partial}{\partial y}\right)^{k-1}v_j(y)\right|_{1\le i,k\le N_1+1} \\
&= \det(U_{N_1+1}V_{N_1+1}) = \det(U_{N_1+1})\det(V_{N_1+1}),
\end{aligned} \tag{3.17}$$

where

$$U_{N_1+1} = \left(\frac{\partial^{i-1}}{\partial x^{i-1}}u_j(x)\right)_{1\le i,j\le N_1+1},\ V_{N_1+1} = \left(\frac{\partial^{k-1}}{\partial x^{k-1}}v_j(y)\right)_{1\le j,k\le N_1+1}. \tag{3.18}$$

We can now take

$$\phi_1(x) = \det(U_{N_1+1}),\ \ \chi_1(y) = \det(V_{N_1+1}). \tag{3.19}$$

For two given functions $\phi_1(x)$ and $\chi_1(y)$, we fix $N_1$ functions among $u_j$ and $v_j$, $1 \le j \le N_1 + 1$, and then the conditions in (3.19) present two linear ordinary differential equations on the unfixed functions, let us say $u_k$ and $v_k$, respectively. The existence theory of linear differential equations guarantees that we have solutions for $u_k$ and $v_k$. Therefore, the first boundary condition in (3.4) can be satisfied.

Similarly, it can be shown that

$$\tau_{N+1} = \det(R_{N_2+1})\det(S_{N_2+1}), \tag{3.20}$$

where

$$R_{N_2+1} = \left(\frac{\partial^{i-1}}{\partial x^{i-1}}r_j(x)\right)_{1\le i,j\le N_2+1},\ S_{N_2+1} = \left(\frac{\partial^{k-1}}{\partial x^{k-1}}s_j(y)\right)_{1\le j,k\le N_2+1}. \tag{3.21}$$

By the same reason, we can achieve

$$\phi_2(x) = \det(R_{N_2+1}),\ \ \chi_2(y) = \det(S_{N_2+1}). \tag{3.22}$$

Therefore, the second boundary condition in (3.4) can be satisfied, too.

To conclude, $\tau_n$ defined by (3.5) and (3.16) solves the 2DTM equation (3.3) and satisfies the boundary conditions in (3.4).

### 3.2 Semi-infinite lattice

There are two semi-infinite lattice equations: one is with $-\infty < n \leq K$ and the other is with $L \leq n < \infty$, where $K, L \in \mathbb{Z}$ are arbitrarily fixed. Note that the 2DTM equation is invariant under the reflection $n \to -n$ and the translation $n \to n+m$ with any given $m \in \mathbb{Z}$. Thus we only need to consider the following semi-infinite 2DTM equation with one separated-variable boundary condition at $n = 0$:

$$\begin{cases} \dfrac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \dfrac{\partial \tau_n}{\partial x}\dfrac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1}, \ 1 \leq n < \infty, \\ \tau_0 = \phi(x)\chi(y), \end{cases} \tag{3.23}$$

where $\phi$ and $\chi$ are two arbitrarily given functions of the indicated variables.

In (3.5), setting $M = N_1 \geq 0$ and letting $N \to \infty$, we obtain the required combined molecule bi-directional Wronskian solution:

$$\begin{cases} \tau_n = \left| \left(\dfrac{\partial}{\partial x}\right)^{i-1} \left(\dfrac{\partial}{\partial y}\right)^{j-1} \Phi(x,y) \right|_{1 \leq i,j \leq M-n+1}, \quad 0 \leq n \leq M, \\ \tau_{M+1} = 1, \ \tau_{M+2} = 0, \ \tau_{M+3} = 0, \ \tau_{M+4} = 1, \\ \tau_n = \left| \left(\dfrac{\partial}{\partial x}\right)^{i-1} \left(\dfrac{\partial}{\partial y}\right)^{j-1} \Psi(x,y) \right|_{1 \leq i,j \leq n-M-4}, \quad M+5 \leq n < \infty, \end{cases} \tag{3.24}$$

where $\Psi(x,y)$ is arbitrary but $\Phi(x,y)$ is defined by

$$\Phi(x,y) = \sum_{j=1}^{M+1} u_j(x) v_j(y), \tag{3.25}$$

which satisfies

$$\left| \frac{\partial^{i-1}}{\partial x^{i-1}} u_j(x) \right|_{1 \leq i,j \leq M+1} = \phi(x), \ \left| \frac{\partial^{k-1}}{\partial x^{k-1}} v_j(y) \right|_{1 \leq j,k \leq M+1} = \chi(y). \tag{3.26}$$

As shown before, there is no problem for existence of those functions $u_j$'s and $v_j$'s.

### 3.3 Infinite lattice

The infinite 2DTM equation is

$$\frac{\partial^2 \tau_n}{\partial x \partial y}\tau_n - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1}, \ -\infty < n < \infty. \tag{3.27}$$

Similarly, by extending two boundaries 0 and $N$ to $-\infty$ and $\infty$, respectively, we can obtain a class of combined molecule bi-directional Wronskian solutions:

$$\begin{cases} \tau_n = \left|\left(\frac{\partial}{\partial x}\right)^{i-1}\left(\frac{\partial}{\partial y}\right)^{j-1}\Phi(x,y)\right|_{1\le i,j\le M-n+1}, & -\infty < n \le M, \\ \tau_{M+1}=1,\ \tau_{M+2}=0,\ \tau_{M+3}=0,\ \tau_{M+4}=1, & \\ \tau_n = \left|\left(\frac{\partial}{\partial x}\right)^{i-1}\left(\frac{\partial}{\partial y}\right)^{j-1}\Psi(x,y)\right|_{1\le i,j\le n-M-4}, & M+5 \le n < \infty, \end{cases} \tag{3.28}$$

where $M \in \mathbb{Z}$, $\Phi(x,y)$ and $\Psi(x,y)$ are all arbitrary.

# 4 Concluding remarks

The combined molecule bi-directional Wronskian solutions have been presented for the finite, semi-infinite and infinite bilinear 2D Toda molecule (2DTM) equations. In the first two cases, separated-variable boundary conditions were imposed. The Jacobi identities for determinants are the key tool employed. The success is to combine two pieces of molecule bi-directional Wronskian solutions in formulating the solutions. Between the two pieces of molecule bi-directional Wronskian solutions, we defined $\tau_n$ as either zero or one to move from one piece to the other piece following the 2DTM equations.

It is known that the finite 2DTM equation (3.3) has double Wronskian solutions which satisfy the boundary conditions [20]:

$$\tau_0 = \phi(x),\ \tau_{N+1} = \chi(y), \tag{4.1}$$

where $\phi$ and $\chi$ are arbitrary functions of the indicated variables. Our construction tells that there exist combined molecule bi-directional Wronskian solutions to the finite 2DTM equation (3.3) which satisfy the above boundary conditions. These solutions correspond to the case of $\chi_1(y) = 1$ and $\phi_2(x) = 1$ in our formulation of solutions for (3.3). Similarly, we can get combined molecule bi-directional Wronskian solutions to the finite 2DTM equation (3.3) which satisfy the following boundary conditions:

$$\tau_0 = \phi(x),\ \tau_{N+1} = \psi(x)\chi(y), \tag{4.2}$$

where $\phi, \psi$ and $\chi$ are arbitrary functions of the indicated variables. Moreover, forcing one of the boundary conditions in (4.1) to be constant (there is no problem for existence of such double Wronskian solutions, based on the previous discussion on the separated-variable boundary conditions using the existence theory of solutions of linear differential equations), the same idea in our construction can be used to connect the corresponding double Wronskian solution with a molecule bi-directional Wronskian solution to form a new solution to the finite, semi-infinite or infinite 2DTM equations. But this kind of solutions is not molecule.

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