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New lump solutions to a (3+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation

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A B S T R A C T

The aim of this paper is to show the existence of three-wave lump solutions to a (3+1)-dimensional generalized CBS (gCBS) equation. Based on the Hirota method, the quadratic functions of the form $f = f_1^2 + f_2^2 + f_3^2 + d$ with nondegenerate condition are applied to solve the corresponding bilinear equation and to generate lump solutions to the gCBS equation. We present two examples of such nonlinear equations and their lump solutions. Moreover, 3-dimensional plots and contour plots are exhibited for three reduced lump solutions.

1. Introduction

Lump solutions are a kind of analytical solutions, which are rationally localized in all directions in space. The study of lump solutions has a long history and recently new solutions have been presented for (2+1)-dimensional integrable equations including the KP and the BKP equations [1–5]. The main idea to get lumps in soliton theory is to take long wave limit of multi-soliton solutions obtained by the inverse scattering transformation (IST) [6] or Hirota bilinear method [7–9] and a new algorithm to look for lumps is given via symbolic computations [10]. Based on the Hirota method, there are many discussions for (2+1)-dimensional systems recently for lump and interaction solutions [11–26] and other kinds of exact solutions (see, e.g, [27–40]).

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In this paper, we try to find lump solutions to (3+1)-dimensional evolution equations by the Hirota bilinear method. Two approaches can be applied: direct search and taking long wave limit of 2N-solitons. Many studies [37–42] found two-wave lump solutions to (3+1)-dimensional nonlinear evolution equations which can also be obtained by taking long wave limit of two-solitons. These solutions are not rationally localized in all directions in space and we refer to them as lump-type solutions [11]. The aim of our current paper is to find three-wave lump solutions which are rationally localized in all directions in space and them cannot be long wave limit of any 2N-solitons. In the rest of paper, we simply call three-wave lump solutions to $(3+1)$-dimensional evolution equations as lump solutions.

The research [11,12] shows that the existence of lump solutions for higher dimension is very restricted. For example, in [11], we showed that any $(N+1)$-dimensional KP equation with $N>2$ cannot have lump solutions generated by quadratic functions. However, only recently lump solutions have been found widely exist in (3+1)-dimensional linear partial differential equations [43,44]. It is extremely important to explore lump solutions to nonlinear partial differential equations in (3+1)-dimensions. In the following, we will show the existence of lump solutions to a generalized Calogero–Bogoyavlenskii–Schiff (gCBS) equation in (3+1) dimensions.

The (2+1)-dimensional integrable Calogero–Bogoyavlenskii–Schiff (CBS) equation [45–47] reads

$$v_t + 4vv_y + 2v_x\partial_x^{-1}v_y + v_{xxy} = 0, \tag{1.1}$$

where

$$\partial_x^{-1}f := \int f dx.$$

It can be reduced to the famous KdV equation by setting $\partial_y = \partial_x$.

If we introduce the potential $v = u_x$, we get the potential form of the (2+1)-dimensional CBS equation

$$u_{xt} + 4u_xu_{xy} + 2u_{xx}u_y + u_{xxxy} = 0. \tag{1.2}$$

It is the special case of the (2+1)-dimensional gCBS equation [21,48]

$$u_{xt} + au_xu_{xy} + bu_{xx}u_y + u_{xxxy} = 0. \tag{1.3}$$

We are interested in the following gCBS equation in (3+1) dimensions

$$u_{xxxy} + 3u_xu_{xy} + 3u_{xx}u_y + 2\alpha_1u_{xt} + 2\alpha_2u_{yt} + 2\alpha_3u_{zt} + \alpha_4u_{tt} = 0, \tag{1.4}$$

where $\alpha_1,\dots,\alpha_4$ are constants. It is not hard to show that (1.4) has the Hirota bilinear form

$$(D_x^3D_y + 2\alpha_1D_xD_t + 2\alpha_2D_yD_t + 2\alpha_3D_zD_t + \alpha_4D_t^2)f\cdot f = 0 \tag{1.5}$$

under the Hopf–Cole transformation

$$u = 2(\ln f)_{xx}. \tag{1.6}$$

Based on the Hirota method, we use the approach proposed in [11] to find lump solutions to the (3+1)-dimensional generalized CBS Eq. (1.4). We first study properties of lump solutions to (3+1)-dimensional evolution equations generated by quadratic polynomials. Then we present the calculation process to get lump solutions for the (3+1)-dimensional gCBS equation. Next we have two examples with 3d and contour plots. Finally, we have some concluding remarks.

2. Lump solutions in (3+1)-dimensions and applications to the (3+1)-dimensional gCBS equation

In order to solve the bilinear Eq. (1.5), we can choose any polynomial for f. Then the function u defined by (1.6) is a rational function solution to the gCBS Eq. (1.4). In particular, u is analytic (with non–vanishing denominator) when $f>0$. In this work, we only consider positive quadratic functions solving (1.5). By the research in [11], the positive quadratic function solutions to Eq. (1.5) can be written in the form of

$$f=f_1^2+f_2^2+f_3^2+d, \tag{2.1}$$

where $f_j=a_jx+b_jy+c_jz+d_jt+e_j$ with $a_j,b_j,c_j,d_j,e_j, j=1,2,3$ being constants. Then

$$f_x=2a_1f_1+2a_2f_2+2a_3f_3,\quad f_{xx}=2a_1^2+2a_2^2+2a_3^2.$$

In this case

$$v=(\ln f)_x=\frac{f_x}{f}=\frac{2(a_1f_1+a_2f_2+a_3f_3)}{f_1^2+f_2^2+f_3^2+d} \tag{2.2}$$

and

$$\begin{aligned} u=(\ln f)_{xx}&=\frac{f_{xx}f-f_x^2}{f^2}\\ &=\frac{2[(a_1^2+a_2^2+a_3^2)(f_1^2+f_2^2+f_3^2+d)-2(a_1f_1+a_2f_2+a_3f_3)^2]}{(f_1^2+f_2^2+f_3^2+d)^2}. \end{aligned} \tag{2.3}$$

We assume that functions f_1,f_2,f_3 are linearly independent. Otherwise, function f can be reduced to the case $f_3=0$.

Define

$$D:=\det\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}. \tag{2.4}$$

We prove an important property on a function of f_1,f_2 and f_3.

Theorem 1. *Suppose $D\neq 0$. Then any continuous function $w(f_1,f_2,f_3)$ has the property that it maintains its shape while it moves at a fixed velocity.*

Proof. By the condition that $D\neq 0$, we know the system of equations

$$\begin{cases} a_1x+b_1y+c_1z=-d_1t,\\ a_2x+b_2y+c_2z=-d_2t,\\ a_3x+b_3y+c_3z=-d_3t.\end{cases} \tag{2.5}$$

has a unique solution $(x_0t,y_0t,z_0t)=(x_0,y_0,z_0)t$ for any fixed t.

Therefore we have $f_i=a_i(x-x_0t)+b_i(y-y_0t)+c_i(z-z_0t)+e_i$ for $i=1,2,3$. This imply any function w keeps its shape and shift steadily at a velocity (x_0,y_0,z_0). □

In [2], two-dimensional lump solutions were described as “the solutions that decay to a uniform state in all directions and the amplitude of these solutions is rational in its independent variables”. Therefore, three-dimensional lump solutions are rational function solutions that decay to a uniform state in all directions. This state usually is the origin $(0,0,0)$.

In the following, we will give the conditions that u (or v) is a lump solution of an evolution equation.

Theorem 2. *If $D\neq 0$ then $u(x,y,z,t)$ defined by (2.3) and $v(x,y,z,t)$ defined by (2.2) satisfy*

$$\lim_{\|(x,y,z)\|\to+\infty}u(x,y,z,t)=0,\qquad \lim_{\|(x,y,z)\|\to+\infty}v(x,y,z,t)=0, \tag{2.6}$$

for any fixed $t>0$.

Proof. Under the above condition, along any direction in space, the numerate of (2.3) is a polynomial of degree 2 whereas the denominator of (2.3) is of degree 4. So u will decay to 0 along any direction in space. The same is true for v. $\square$

Corollary 1. *When $a_3 = b_3 = c_3 = d_3 = e_3 = 0$, u and v do not decay along all directions in space.*

In this case, $f = f_1^2 + f_2^2 + d$. More generally, if u (or v) is a function of f_1 and f_2, then it can not be a lump solution in (3+1)-dimensions.

Corollary 2. *Suppose a nonzero function $w(\cdot,\cdot)$ is a differentiable function in two variables. Then $w(f_1, f_2)$ does not decay along all directions in space.*

Proof. Suppose that $w = h(f_1, f_2)$. We can easily find a direction (a, b, c) which is perpendicular to (a_1, b_1, c_1) and (a_2, b_2, c_2). When t is fixed, f_1 and f_2 keep unchanged when (x, y, z) goes along the direction (a, b, c). Since w is not zero, it cannot always decay to a zero along the direction (a, b, c). Therefore, w is not a lump solution. $\square$

Remark 1. Lump solutions of any (3+1)-dimensional nonlinear evolution equation cannot be obtained by taking long wave limit of a two-soliton solutions.

Unlike many researches for (2+1)-dimensional equations, it is not easy to search for f as sum of three squares via (2.1) to present a lump solution in (3+1)-dimensions. Here we use the algorithm proposed in [11].

Let

$$f(x, y, z, t) = w^T A w + d, \tag{2.7}$$

where $A = [a_{ij}]$ is a 4×4 symmetric matrix, $w = (x, y, z, t)^T$, d is a constant.

Introduce two matrices

$$Q = \begin{bmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 0 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_5 & a_6 & a_7 \\ a_3 & a_6 & a_8 & a_9 \\ a_4 & a_7 & a_9 & a_{10} \end{bmatrix}. \tag{2.8}$$

Let $Q(i, j), A(i, j)$ be the entry of ith row and jth column in Q and A respectively. Then we define

$$k := \frac{1}{2} \sum_{i,j=1}^{4} Q(i, j)A(i, j) = \frac{1}{2}(2\alpha_1 a_4 + 2\alpha_2 a_7 + 2\alpha_3 a_9 + \alpha_4 a_{10}). \tag{2.9}$$

According to the Theorem 3.4 in [11], the function f defined by (2.7) is a solution to the bilinear Eq. (3.5) if and only if

$$3a_1 a_2 + kd = 0, \tag{2.10}$$

$$kA - AQA = 0. \tag{2.11}$$

By using Maple with condition Rank$(A) = 3$, we get $k = 0$. Plug into (2.10), we have $d > 0$ is arbitrary and $a_1 a_2 = 0$. However, all principal minors of A are non-negative since A is positive semi-definite. If $a_1 = 0$, we have

$$\det \begin{bmatrix} a_1 a_2 & \\ a_2 & a_5 \end{bmatrix} = a_1 a_5 - a_2^2 = -a_2^2 \geq 0,$$

$a_2 = 0$ follows. By the same reason, $a_3 = a_4 = 0$. Then we get a trivial solution for f which does not depend on x and no lump solutions can be expected since $D = 0$. Therefore, we must have $a_2 = 0$. The nontrivial solution of (2.11) can be expressed as

$$\begin{aligned} a_1 &= -\frac{2\alpha_3 a_3 + \alpha_4 a_4}{2\alpha_1}, \\ a_5 &= -\frac{2\alpha_3 a_6 + \alpha_4 a_7}{2\alpha_2}, \\ a_8 &= -\frac{2\alpha_1 a_3 + 2\alpha_2 a_6 + \alpha_4 a_9}{2\alpha_3}, \\ a_{10} &= -\frac{2(\alpha_1 a_4 + \alpha_2 a_7 + \alpha_3 a_9)}{\alpha_4}, \end{aligned} \tag{2.12}$$

where a_3, a_4, a_6, a_7, a_9 are free parameters such that $A \geq 0$ and $\text{Rank}(A) = 3$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are all nonzero constants.

We get the matrix A by substituting (2.12) into A in (2.8):

$$\begin{bmatrix} -\frac{\alpha_3 a_3 + \alpha_4 a_4}{2\alpha_1} 0 & a_3 & a_4 & \\ 0 & -\frac{2\alpha_3 a_6 + \alpha_4 a_7}{2\alpha_2} & a_6 & a_7 \\ a_3 & a_6 & -\frac{2\alpha_1 a_3 + 2\alpha_2 a_6 + \alpha_4 a_9}{2\alpha_3} & a_9 \\ a_4 & a_7 & a_9 & -\frac{2(\alpha_1 a_4 + \alpha_2 a_7 + \alpha_3 a_9)}{\alpha_4} \end{bmatrix}$$

and the function defined by (2.7) reads

$$\begin{aligned} f(x,y,z,t) = &-\frac{\alpha_3 a_3}{\alpha_1}\left(x - \frac{\alpha_1}{\alpha_3} z\right)^2 - \frac{\alpha_4 a_4}{2\alpha_1}\left(x - \frac{2\alpha_1}{\alpha_4} t\right)^2 - \frac{\alpha_3 a_6}{\alpha_2}\left(y - \frac{\alpha_2}{\alpha_3} z\right)^2 \\ &-\frac{\alpha_4 a_7}{2\alpha_2}\left(y - \frac{2\alpha_2}{\alpha_4} t\right)^2 - \frac{\alpha_4 a_9}{2\alpha_3}\left(z - \frac{2\alpha_3}{\alpha_4} t\right)^2 + d. \end{aligned} \tag{2.13}$$

If we choose a_3, a_4, a_6, a_7, a_9 and d such that

$$\frac{\alpha_3 a_3}{\alpha_1} < 0,\ \frac{\alpha_4 a_4}{\alpha_1} < 0,\ \frac{\alpha_3 a_6}{\alpha_2} < 0,\ \frac{\alpha_4 a_7}{\alpha_2} < 0,\ \frac{\alpha_4 a_9}{\alpha_3} < 0,\ d > 0 \tag{2.14}$$

then function f is positive.

By symbolic computation, we derive from (2.13)

$$\begin{aligned} f(x,y,z,t) \quad &= A_1\left(x + \frac{a_3}{A_1} z + \frac{a_4}{A_1} t\right)^2 + A_2\left(y + \frac{a_6}{A_2} z + \frac{a_7}{A_2} t\right)^2 \\ &+ A_3\left(z - \frac{2\alpha_3}{\alpha_4} t\right)^2 + d, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} A_1 &:= -\frac{2\alpha_3 a_3 + \alpha_4 a_4}{2\alpha_1}, \\ A_2 &:= -\frac{2\alpha_3 a_6 + \alpha_4 a_7}{2\alpha_2}, \\ A_3 &:= -\frac{\alpha_4 a_9}{2\alpha_3} - \frac{\alpha_1 \alpha_4 a_3 a_4}{\alpha_3(2\alpha_3 a_3 + \alpha_4 a_4)} - \frac{\alpha_2 \alpha_4 a_6 a_7}{\alpha_3(2\alpha_3 a_6 + \alpha_4 a_7)}. \end{aligned} \tag{2.16}$$

We can choose parameters a_3, a_4, a_6, a_7 and a_9 such that

$$A_1 > 0, \qquad A_2 > 0, \qquad A_3 > 0 \tag{2.17}$$

then when $d > 0$ function f is positive. It is not hard to check that (2.17) is guaranteed by (2.14). Since functions $f_1 = x+\frac{a_3}{A_1}z+\frac{a_4}{A_1}t$, $f_2 = y+\frac{a_6}{A_2}z+\frac{a_7}{A_2}t$ and $f_3 = z-\frac{2\alpha_3}{\alpha_4}t$ are linearly independent. If condition (2.17) is satisfied then $D \neq 0$ is also true. Therefore f defined by (2.15) generates a lump solution to the gCBS equation.

Remark 2. The decomposition (2.15) is not unique simply because

$$a^2+b^2 = \left(\frac{a+b}{\sqrt{2}}\right)^2 + \left(\frac{a-b}{\sqrt{2}}\right)^2.$$

In general, if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are fixed nonzero real numbers then according to [11] we have the following lump solution for Eq. (1.4)

$$u(x,y,z,t) = 2\ln(f(x,y,z,t))_{xx}$$

with

$$\begin{aligned} f(x,y,z,t) &= A_1\left(x+\frac{a_3}{A_1}z+\frac{a_4}{A_1}t+e_1\right)^2 + A_2\left(y+\frac{a_6}{A_2}z+\frac{a_7}{A_2}t+e_2\right)^2 \\ &\quad +A_3\left(z-\frac{2\alpha_3}{\alpha_4}t+e_3\right)^2+d, \end{aligned} \tag{2.18}$$

where a_3, a_4, a_6, a_7, a_9 are real numbers such that (2.17) is true for A_1, A_2, A_3 given by (2.16), e_1, e_2, e_3 are arbitrary real numbers and d is a positive real number.

There are an interesting class of solutions which are static (solutions that do not depend on t)

$$\begin{aligned} f(x,y,z,t) &= a_1x^2+a_5y^2+a_8z^2+a_3xz+a_6yz \\ &= a_1\left(x+\frac{a_3}{a_1}z\right)^2 + a_5\left(y+\frac{a_6}{a_5}z\right)^2 + \left(a_8-\frac{a_3^2}{a_1}-\frac{a_6^2}{a_5}\right)z^2+d. \end{aligned} \tag{2.19}$$

The aforementioned function f generates a lump solution to the gCBS equation if and only if

$$a_1 > 0, \quad a_5 > 0, \quad a_8-\frac{a_3^2}{a_1}-\frac{a_6^2}{a_5} > 0, \quad d > 0. \tag{2.20}$$

3. Two illustrative examples

In this section, we will show examples of lump solutions to the (3+1)-dimensional gCBS equation according to previous discussions.

Example 1. We take $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ and $a_3 = 0, a_4 = -2, a_6 = -1, a_7 = a_9 = -2, d = 1$. Then

$$A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 2 & -1 & -2 \\ 0 & -1 & 2 & -2 \\ -2 & -2 & -2 & 12 \end{bmatrix}$$

possesses eigenvalues $0, 1, 3, 13$.

By the eigenvalue decomposition of quadratic forms, we know

$$\begin{aligned} f(x,y,z,t) &= w^TAw+d = x^2+2y^2+2z^2+12t^2-4xt-2yz-4yt-4zt+1 \\ &= \frac{(2x-y-z)^2}{6}+\frac{3(y-z)^2}{2}+\frac{(x+y+z-6t)^2}{3}+1. \end{aligned} \tag{3.1}$$

Also, by (2.15)

$$f(x,y,z,t) = (x-2t)^2+2(y-\frac{z}{2}-t)^2+\frac{3}{2}(z-2t)^2+1. \tag{3.2}$$

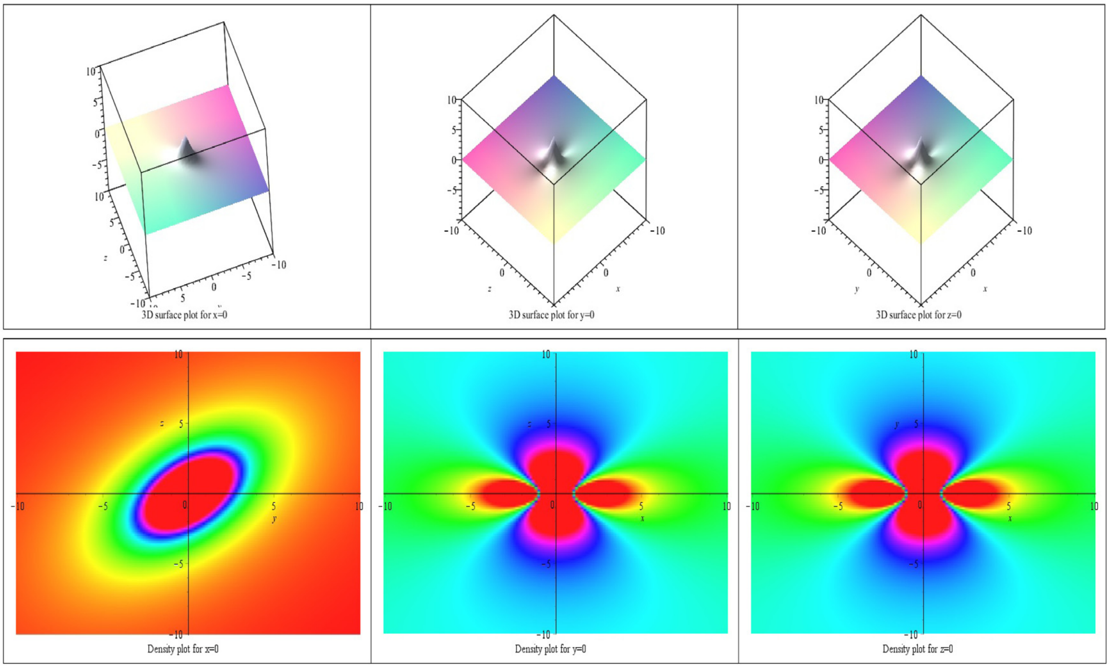

Fig. 1. 3d plots (top) and density plots (bottom) of u for $(a)x = 0, (b)y = 0, (c)z = 0$ when $t = 0$.

Then the function

$$\begin{aligned} u(x,y,z,t) &= (2\ln f(x,y,z,t))_{xx} \\ &= \frac{4(-x^2+2y^2+2z^2+4t^2+4tx-4ty-4tz-2yz+1)}{(x^2+2y^2+2z^2+12t^2-4xt-2yz-4yt-4zt+1)^2} \end{aligned}$$

is a lump solution to the following special gCBS equation

$$(D_x^3D_y + 2D_xD_t + 2D_yD_t + 2D_zD_t + D_t^2)f \cdot f = 0. \tag{3.3}$$

Fig. 1 illustrates 3d plots and contour plots of the lump solution projecting to (a) $x = 0$, (b) $y = 0$, (c) $z = 0$, when $t = 0$.

Example 2. We take $\alpha_1 = \alpha_2 = 1, \alpha_3 = \alpha_4 = -1$ and $a_3 = 1, a_4 = 0, a_6 = 1, a_7 = 0, a_9 = -1, d = 2$. Then

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 5/2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

possesses eigenvalues $0, 1, (11 \pm \sqrt{17})/4$.

By (2.15)

$$f(x,y,z,t) = (x+z)^2 + 2(t-z/2)^2 + (y+z)^2 + 2. \tag{3.4}$$

Then the rational function

$$\begin{aligned} u(x,y,z,t) &= (2\ln f(x,y,z,t))_{xx} \\ &= \frac{-4(x+z)^2+2(2t-z)^2+4(y+z)^2+8}{[(x+z)^2+2(t-z/2)^2+(y+z)^2+2]^2} \end{aligned}$$

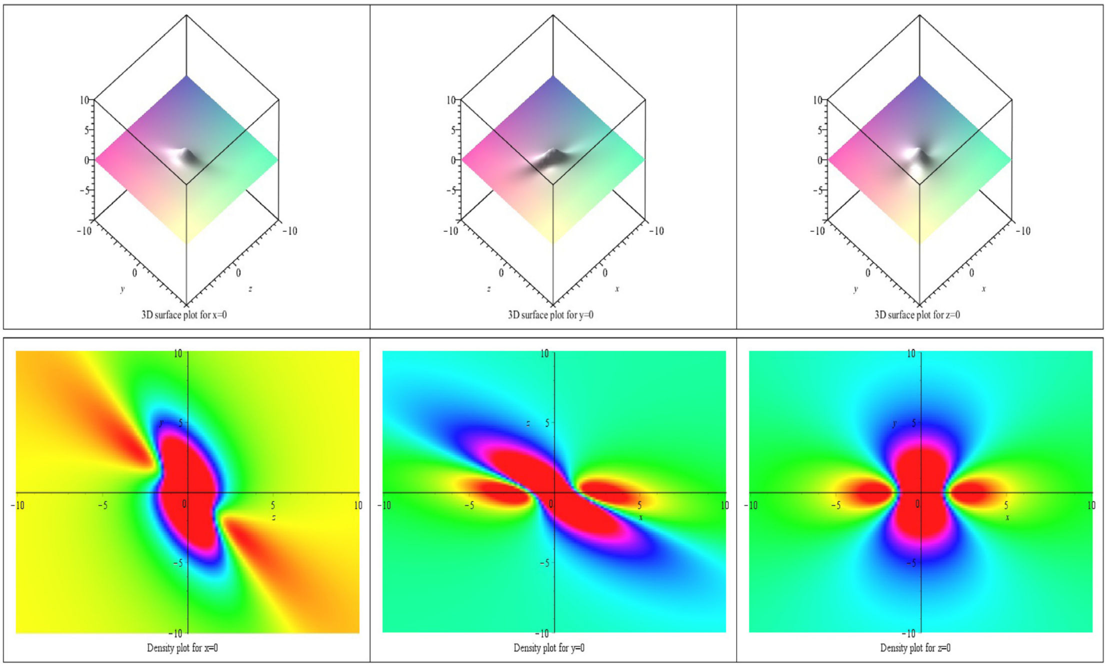

Fig. 2. 3d plots (top) and density plots (bottom) of u for $(a)x = 0, (b)y = 0, (c)z = 0$ when $t = 0$.

is a lump solution to the following special gCBS equation

$$(D_x^3 D_y + 2D_x D_t + 2D_y D_t - 2D_z D_t - D_t^2) f \cdot f = 0. \tag{3.5}$$

Fig. 2 illustrates 3d plots and contour plots of the lump solution projecting to (a) $x = 0$, (b) $y = 0$, (c) $z = 0$, when $t = 0$.

4. Conclusion

In this paper, we first introduce a (3+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation and then explored the three-wave lump solutions generated by positive quadratic functions solutions to the corresponding bilinear equation. We find that these three-wave lump solutions are localized in all directions in space which is different from two-wave lump solutions. We also analyzed the conditions for parameters which lead to lump solutions. A set of three-wave lump solutions including a class of static ones to the (3+1)-dimensional gCBS was discovered and two illustrative examples are given with 3d-plots and density plots. Our computations are based on Hirota bilinear equations and the symbolic computation software Maple.

The formulation of lump solutions is closed to that of rogue waves. In recent study, we find lump solutions and line rogue waves exist in some (2+1) evolution equations [49,50]. We point out that it is possible for line rogue waves appear in some (3+1) systems and we will study the problem in future.

Data availability

No data was used for the research described in the article.

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