A COMBINED SHORT PULSE-MKDV EQUATION AND ITS EXACT SOLUTIONS BY TWO-DIMENSIONAL INVARIANT SUBSPACES

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In this paper, a multi-component combined short pulse-mKdV equation is constructed as a nonstreching invariant curve flow in the n-dimensional unit sphere $\mathbb{S}^n(1) = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. The invariant subspace method is used to solve the resulting combined short pulse-mKdV equation with $n = 1$. Through symbolic computation with the aid of mathematical software Maple, new exact solutions are obtained from its two-dimensional invariant subspaces formed in forms of the exponential functions and trigonometric functions.

Keywords: short pulse-mKdV equation, invariant subspace, curve flow, exact solution.

1. Introduction

Integrable equations arise in shallow water wave, condensed matter physics, quantum field theory, optical communication and other applied sciences. Typical ones include the KdV equation, the mKdV equation, the nonlinear Schrödinger (NLS) equation, and the short pulse equation. Recently, it has been of great interest to study geometric characteristics of integrable equations. In [1], Hasimoto has published a pioneering result that the NLS equation

$$i\phi_t + \phi_{ss} + |\phi|^2\phi = 0 \tag{1.1}$$

is equivalent to the system for the curvature κ and τ of curve γ in $\mathbb{R}^3$,

$$\begin{aligned} \kappa_t &= -2\tau\kappa_s - \kappa\tau_s, \\ \tau_t &= \frac{\kappa_{sss}}{\kappa} - \frac{\kappa_s\kappa_{ss}}{\kappa^2} - 2\tau\tau_s + \kappa\kappa_s \end{aligned} \tag{1.2}$$

via the so-called Hasimoto transformation $\phi = \kappa\exp(i\int^s \tau(t,z)dz)$. Indeed, the system (1.2) is equivalent to the vortex filament equation $\gamma_t = \gamma_s \times \gamma_{ss} = \kappa \mathbf{b}$, where $\mathbf{b}$ is the binormal vector of γ. In [2], the authors found that Hasimoto transformation is a gauge transformation relating the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ to the parallel frame $\{\mathbf{t}^1, \mathbf{n}^1, \mathbf{b}^1\}$. The Hasimoto transformation has been generalized to the Riemannian manifold with constant curvature, which is used to obtain the corresponding integrable equations associated with the invariant nonstretching curve flows [3]. It is also a Poisson map that transforms the Hamiltonian structure of the NLS equation to that of the vertex filament flow [4]. The KdV equation, the mKdV equation, the Camassa-Holm equation, the Sawada–Kotera equation and the Kaup–Kuperschmidt equation have also been shown to arise from the invariant curve flows in centro-equiaffine geometry, Euclidean geometry, special affine geometry and projective geometry, respectively [5–12].

In Section 2, we will show that the following multi-component equation

$$\vec{u}_{xt} + \frac{\alpha}{2}(|\vec{u}|^2 \vec{u}_x)_x + \alpha \vec{u} + \beta \vec{u}_{xxxx} + \frac{3}{2}\beta |\vec{u}_x|^2 \vec{u}_{xx} = 0 \tag{1.3}$$

can be generated from a nonstreching invariant curve flow in the n-dimensional unit sphere $\mathbb{S}^n(1) = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. When $\alpha \neq 0$ and $\beta = 0$, Eq. (1.3) becomes the multi-component short pulse equation

$$\vec{u}_{xt} + \frac{\alpha}{2}(|\vec{u}|^2 \vec{u}_x)_x + \alpha \vec{u} = 0,$$

which has been presented in [12]. When $\alpha = 0$ and $\beta \neq 0$, Eq. (1.3) reduces to the well-known multi-component mKdV equation

$$\vec{v}_t + \beta \vec{v}_{xxx} + \frac{3}{2}\beta |\vec{v}|^2 \vec{v}_x = 0, \qquad \vec{v} = \vec{u}_x.$$

Thus, we call (1.3) a multi-component combined short pulse-mKdV equation when $\alpha \neq 0$ and $\beta \neq 0$.

Many symmetry-related methods provide efficient tools to reduce and solve nonlinear partial differential equations (PDEs), which contain the Lie point symmetry, conditional symmetry, Lie–Bäcklund symmetry, and C^∞-symmetry methods. Correspondingly, group-invariant solutions stemming from symmetries play important roles in the study of their asymptotical behaviours, blow up phenomena and geometric properties. These exact solutions can also be used to justify the numerical scheme of solving nonlinear PDEs [13–18]. The invariant subspace method, related to the conditional Lie–Bäcklund symmetry method, can be used to construct different types of exact solutions of evolution equations [19–29]. For example, N-soliton solutions of integrable equations, derived by Hirota's bilinear method, belong to a linear subspace of exponential functions, upon some change of dependent variables. In particular, Galaktionov and Svirshchevskii [19] proposed a systematic approach to invariant subspaces of evolution equations, and they obtained many interesting exact solutions of NLEEs in mechanics and physics.

In order to make a new application example of the invariant subspace method, we will analyze Eq. (1.3) with $n = 1$, namely, the scale equation

$$u_{xt} + \frac{\alpha}{2}(u^2 u_x)_x + \alpha u + \beta u_{xxxx} + \frac{3}{2}\beta u_x^2 u_{xx} = 0. \tag{1.4}$$

We will show that this scalar equation admits two-dimensional invariant subspaces in Section 3. Through symbolic computation with the aid of mathematical software Maple, this equation will be reduced to some two-dimensional dynamical systems. Then new exact solutions will be obtained, in terms of the exponential functions and trigonometric functions. Section 4 will be devoted to conclusions and discussions.

2. A combined short pulse-mKdV equation

Firstly, we give a brief account of the curve flow theory on $\mathbb{S}^n(1)$ (see [12] for details).

Assume that $\gamma(x,t)$ is a curve flow on the unit sphere $\mathbb{S}^n(1) = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, which satisfies $\|\gamma\| = 1$, where x is the invariant arc-length parameter and t is the time. The natural frame of the curve $\gamma \in \mathbb{S}^n(1)$ is $\{e_1 = \gamma_x, e_2, \ldots, e_n\}$. Let $\rho = (e_0 = \gamma, e_1, \ldots, e_n) \in \mathrm{SO}(n+1)$ be the lift from $\mathbb{S}^n(1)$ to the bundle space $\mathrm{SO}(n+1)$, and D_x and D_t denote the tangent and evolutionary vector fields, respectively. It follows that

$$\rho_x = \rho\hat{\omega}(D_x), \qquad \hat{\omega}(D_x) = \begin{pmatrix} 0 & -1 & \vec{0}^T \\ 1 & 0 & -\vec{k}^T \\ \vec{0} & \vec{k} & O \end{pmatrix}, \qquad O \in \mathfrak{so}(n-1),$$

where $\hat{\omega}$ is the Cartan connection and $\vec{k} = (k_1, k_2, \ldots, k_{n-1})$ is the natural curvature vector of γ.

Assume that the curve flow is governed by

$$\gamma_t = f e_1 + h_1 e_2 + h_2 e_3 + \cdots + h_{n-1} e_n,$$

where the tangent velocity f and the normal velocities h_i $(i = 1, 2, \ldots, n-1)$ depend on the curvatures and their derivatives with respect to the arc-length x. So we let the time evolution for the frame read

$$\rho_t = \rho\hat{\omega}(D_t), \qquad \hat{\omega}(D_t) = \begin{pmatrix} 0 & -f & -\vec{h}^T \\ f & 0 & -\vec{\xi}^T \\ \vec{h} & \vec{\xi} & \Theta \end{pmatrix}, \qquad \Theta \in \mathfrak{so}(n-1),$$

where $\vec{h} = (h_1, \cdots, h_{n-1})^T$ and $\vec{\xi} \in \mathbb{R}^{n-1}$ is an unknown vector to be determined later by the structure equation.

Furthermore, we assume that the flow is intrinsic, namely, the distribution $\{D_x, D_t\}$ satisfies $[D_x, D_t] = 0$ so that the integral submanifold is a smooth two-dimensional surface on the Lie group $\mathrm{SO}(n+1)$. By means of the Cartan structure equation

$$\frac{d}{dt}\hat{\omega}(D_x) - \frac{d}{dx}\hat{\omega}(D_t) - \big[\hat{\omega}(D_x), \hat{\omega}(D_t)\big] = 0,$$

one gets the following determining equations:

$$f_x = \langle \vec{k}, \vec{h} \rangle, \tag{2.1}$$

$$\vec{\xi} = \vec{h}_x + f\vec{k}, \tag{2.2}$$

$$\Theta_x = \vec{k} \otimes \vec{\xi} - \vec{\xi} \otimes \vec{k}, \tag{2.3}$$

$$\vec{k}_t = \vec{\xi}_x - \Theta\vec{k} + \vec{h}, \tag{2.4}$$

where (2.1) is the arc-length preserving condition, $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T\vec{b}$ denotes the usual Euclidean inner product, and $\vec{a} \otimes \vec{b}$ denotes the tensor product, namely,

$$\vec{a} \otimes \vec{b} = \begin{pmatrix} a_1b_1 & a_1b_2 & \dots & a_1b_{n-1} \\ a_2b_1 & a_2b_2 & \dots & a_2b_{n-1} \\ \dots & \dots & \dots & \dots \\ a_{n-1}b_1 & a_{n-1}b_2 & \dots & a_{n-1}b_{n-1} \end{pmatrix}.$$

From (2.1) and (2.2), it follows that

$$f = \partial_x^{-1}\langle \vec{k}, \vec{h} \rangle, \qquad \vec{\xi} = \vec{h}_x + \left(\partial_x^{-1}\langle \vec{k}, \vec{h} \rangle\right)\vec{k}. \tag{2.5}$$

Integrating (2.3) with respect to x once, we have

$$\Theta = \partial_x^{-1}(\vec{k} \wedge \vec{\xi}), \tag{2.6}$$

upon setting $\vec{a} \wedge \vec{b} = \vec{a} \otimes \vec{b} - \vec{b} \otimes \vec{a}$ for convenience. Substituting (2.5) and (2.6) into (2.4) and using the identity for vectors

$$(\vec{a} \wedge \vec{b}) \cdot \vec{c} = \langle \vec{b}, \vec{c} \rangle \vec{a} - \langle \vec{a}, \vec{c} \rangle \vec{b},$$

we can obtain an equation for the curvature vector

$$\vec{k}_t = \vec{h}_{xx} + \langle \vec{k}, \vec{k} \rangle \vec{h} + \left(\partial_x^{-1}\langle \vec{k}, \vec{h} \rangle\right)\vec{k}_x + \left(\partial_x^{-1}(\vec{k}_x \wedge \vec{h})\right)\vec{k} + \vec{h}. \tag{2.7}$$

In [12], the authors have considered the cases $\{\vec{h} = \vec{u}, \vec{k} = -\vec{u}_x\}$ and $\{\vec{h} = \vec{u}_x, \vec{k} = \vec{u} \mp \vec{u}_{xx}\}$ which are used to construct the multi-component short pulse equation and the Camassa–Holm type equation, respectively. In what follows, let us study a new case $\{\vec{h} = \alpha\vec{u} + \beta\vec{u}_{xx}, \vec{k} = -\vec{u}_x\}$, which has not been considered in the literature including [12]. In this case, the tangent velocity f is determined by

$$\begin{aligned} f &= \partial_x^{-1}\langle \vec{k}, \vec{h}\rangle \\ &= \partial_x^{-1}\langle -\vec{u}_x, \alpha\vec{u} + \beta\vec{u}_{xx}\rangle \\ &= -\frac{\alpha}{2}|\vec{u}|^2 - \frac{\beta}{2}|\vec{u}_x|^2 + c_0, \end{aligned} \tag{2.8}$$

where c_0 is an integration constant. Substituting (2.8) with $c_0 = \alpha + \beta$ into (2.7), and noting that $\Theta = -\beta\partial_x^{-1}(\vec{u}_x \wedge \vec{u}_{xxx}) = \beta\vec{u}_{xx} \wedge \vec{u}_x$, we obtain the multi-component combined short pulse-mKdV equation (1.3). When $n = 1$, the multi-component equation (1.3) becomes (1.4). To summarize, we have the following result.

THEOREM 1. *Assume that curves $\gamma(x,t)$ on the sphere $\mathbb{S}^n(1)$ $(n \geq 1)$ are governed by the flow*

$$\gamma_t = \left(\alpha + \beta - \frac{\alpha}{2}|\vec{u}|^2 - \frac{\beta}{2}|\vec{u}_x|^2\right)e_1 + \sum_{j=1}^{n-1} u_{j,x}e_{j+1}, \tag{2.9}$$

where $\{e_1, e_2, \ldots, e_n\}$ is the natural frame of the curve $\gamma(x,t)$, and $(u_1, u_2, \ldots, u_{n-1})$ is defined by the curvatures $\vec{k} = -\vec{u}_{xx}$. Then, the flow (2.9) *is intrinsic and the curvature vector $\vec{u}$ satisfies the combined short pulse-mKdV equation* (1.3).

3. Exact solutions in the scalar case

Let us give a brief account of the invariant subspace method [19, 21–23, 25, 26, 29], which was carefully refined in [24]. Consider a $(1+1)$-dimensional evolution equation

$$u_t = F(x, u, u_x, \cdots, u_{kx}), \tag{3.1}$$

where $F(x, u, u_x, \cdots, u_{kx})$ is a given sufficiently smooth function of its arguments and

$$u_{jx} = \frac{\partial^j u}{\partial x^j} \qquad (j = 1, \cdots, k).$$

Let $\{g_1(x), g_2(x), \cdots, g_m(x)\}$ be a finite set of linearly independent functions and W_m denote their linear span $W_m = \mathcal{L}\{g_1(x), g_2(x), \cdots, g_m(x)\}$. The subspace W_m is said to be invariant under the given operator F, namely, F is said to preserve W_m if $F(W_m) \subseteq W_m$, that means

$$F\Big[\sum_{j=1}^{m} C_j g_j(x)\Big] = \sum_{j=1}^{m} \Psi_j(C_1, C_2, \cdots, C_m)g_j(x)$$

for any $(C_1, C_2, \cdots, C_m) \in \mathbb{R}^m$. It follows that if the linear subspace W_m is invariant with respect to F, then Eq. (3.1) has an exact solution of the form

$$u(x,t) = \sum_{j=1}^{m} C_j(t)g_j(x),$$

where the coefficients $\{C_j(t), (j = 1, 2, \cdots, m)\}$ satisfy an m-dimensional dynamical system

$$\frac{\mathrm{d}C_j(t)}{\mathrm{d}t} = \Psi\left(C_1(t), C_2(t), \cdots, C_m(t)\right), \qquad j = 1, 2, \cdots, m.$$

Assume that the invariant subspace W_m is defined as the space of solutions to a linear mth-order ordinary differential equation (ODE),

$$L[y] \equiv y^{(m)} + a_{m-1}(x)y^{(m-1)} + \cdots + a_1(x)y' + a_0(x)y = 0, \tag{3.2}$$

then the invariant condition with respect to F takes the form

$$L\left[F[u]\right]\Big|_{[H]} = 0, \tag{3.3}$$

where $[H]$ denotes the equation $L[u] = 0$ and its differential consequences with respect to x. This invariant condition (3.3) can be interpreted in terms of Lie–Bäcklund symmetry of (3.1) of linear ODEs. Thus the maximal dimension of the invariant subspaces preserved by F can be determined, which is included in the following proposition proved by Galaktionov and Svirshchevskii [19].

PROPOSITION 1. *Assume that a linear subspace W_m is invariant under a nonlinear ordinary differential operator F of the order k, then $m \leq 2k + 1$.*

From the above proposition, we only need to consider the cases $W_2, W_3, \cdots, W_9$ defined by an ODE (3.2) with constant coefficients $a_0, a_1, \cdots, a_{m-1}$.

We first analyze the case of $m = 2$. By setting

$$L[y] \equiv y'' + a_1 y' + a_0 y = 0$$

in (3.2) and

$$F = \frac{\alpha}{2}(u^2 u_x)_x + \alpha u + \beta u_{xxxx} + \frac{3}{2}\beta u_x^2 u_{xx}$$

in (1.4), we have

$$\begin{aligned}&\left(9a_0a_1\beta - 9a_1^3\beta - 5a_1\alpha\right)u_x^3 + \left(9a_0^2\beta + 4a_1^2\alpha - 21a_0a_1^2\beta - 9a_0\alpha\right)uu_x^2\\ &\quad + \left(7a_0a_1\alpha - 15a_0^2a_1\beta\right)u^2u_x + \left(3a_0^2\alpha - 3a_0^3\beta\right)u^3 = 0\end{aligned} \tag{3.4}$$

from the invariant condition (3.3). Here we use the software Maple to deal with complicated calculations. To vanish all the coefficients of (3.4), we have three kinds of invariant subspaces determined by the linear ODEs:

$$y'' + \frac{\alpha}{\beta}y = 0 \qquad \text{and} \qquad y'' = 0.$$

Case 1: When $\alpha/\beta < 0$, from $y'' + \alpha y/\beta = 0$, we have an invariant subspace

$$\mathcal{L}\left\{\exp\left(-\sqrt{-\frac{\alpha}{\beta}}x\right), \exp\left(\sqrt{-\frac{\alpha}{\beta}}x\right)\right\}. \tag{3.5}$$

Thus the corresponding exact solution is given by

$$u(x,t) = C_1(t)\exp\left(-\sqrt{-\frac{\alpha}{\beta}}x\right) + C_2(t)\exp\left(\sqrt{-\frac{\alpha}{\beta}}x\right), \tag{3.6}$$

where $C_1(t)$ and $C_2(t)$ satisfy the two-dimensional dynamical system

$$\begin{aligned} C_1' &= \sqrt{-\frac{\alpha}{\beta}}C_1\left(2C_1C_2\alpha - \alpha - \beta\right), \\ C_2' &= -\sqrt{-\frac{\alpha}{\beta}}C_2\left(2C_1C_2\alpha - \alpha - \beta\right). \end{aligned}$$

Exact solution of this dynamical system can be obtained as

$$\begin{aligned} C_1 &= c_2 e^{c_1 t}, \\ C_2 &= \frac{1}{2c_2\alpha}\left(c_1\sqrt{-\frac{\beta}{\alpha}} - \alpha - \beta\right)e^{-c_1 t} \end{aligned} \tag{3.7}$$

with two arbitrary constants c_1 and c_2.

Case 2: When $\alpha/\beta > 0$, from $y'' + \alpha y/\beta = 0$, we have an invariant subspace

$$\mathcal{L}\left\{\sin\left(\sqrt{\frac{\alpha}{\beta}}x\right), \cos\left(\sqrt{\frac{\alpha}{\beta}}x\right)\right\}. \tag{3.8}$$

Thus the corresponding exact solution is given by

$$u(x,t) = C_1(t)\sin\left(\sqrt{\frac{\alpha}{\beta}}x\right) + C_2(t)\cos\left(\sqrt{\frac{\alpha}{\beta}}x\right), \tag{3.9}$$

where $C_1(t)$ and $C_2(t)$ satisfy the two-dimensional dynamical system

$$\begin{aligned} C_1' &= \sqrt{\frac{\alpha}{\beta}}C_2\left(\frac{C_1^2 + C_2^2}{2}\alpha - \alpha - \beta\right), \\ C_2' &= -\sqrt{\frac{\alpha}{\beta}}C_1\left(\frac{C_1^2 + C_2^2}{2}\alpha - \alpha - \beta\right). \end{aligned}$$

Exact solution of this dynamical system can be obtained as

$$\begin{aligned} C_1 &= c_1\sin\left(\sqrt{\frac{\alpha}{\beta}}|\xi\alpha - \alpha - \beta|t\right) \pm \sqrt{2\xi - c_1^2}\cos\left(\sqrt{\frac{\alpha}{\beta}}|\xi\alpha - \alpha - \beta|t\right), \\ C_2 &= \mp\sqrt{2\xi - c_1^2}\sin\left(\sqrt{\frac{\alpha}{\beta}}|\xi\alpha - \alpha - \beta|t\right) + c_1\cos\left(\sqrt{\frac{\alpha}{\beta}}|\xi\alpha - \alpha - \beta|t\right) \end{aligned} \tag{3.10}$$

with two arbitrary constants c_1 and ξ.

Case 3: From $y'' = 0$, although we have an invariant subspace

$$\mathcal{L}\{1, x\},$$

no nontrivial exact solution of (1.4) can be obtained by $u(x,t) = C_1(t) + C_2(t)x$ because $\alpha C_2(C_2^2 + 1) = 0$ and $C_2' = -\alpha C_1(C_2^2 + 1)$.

Moreover, we can show that there is no invariant subspace with $m \geq 3$, through complicated calculations. So we have established the following result.

THEOREM 2. *The combined short pulse-mKdV equation* (1.4) *have two classes of exact solutions,* (3.6)–(3.7) *and* (3.9)–(3.10)*, which are generated from two-dimensional invariant subspaces* (3.5) *and* (3.8)*, respectively.*

4. Concluding remarks

In this paper, a geometrical formulation for the multi-component curve flow equation (1.3) has been established. More specifically, we have shown that the combined short pulse-mKdV equation (1.3) is equivalent to a nonstreching invariant curve flow in the n-dimensional unit sphere $\mathbb{S}^n(1)$. Eq. (1.3) provides a generalization of the multi-component short pulse equation and the multi-component mKdV equation. It is not clear to us, however, if there is any more general equation generated from a curve flow in the n-dimensional unit sphere $\mathbb{S}^n(1)$, which admits invariant subspaces of solutions with separated variables.

Furthermore, the invariant subspace method has been used to solve the scale equation (1.4) through symbolic computation with the aid of mathematical software Maple. Novel exact solutions (3.6)–(3.7) and (3.9)–(3.10), generated from two-dimensional invariant subspaces, have been presented, in terms of the exponential functions and trigonometric functions. On the other hand, we would like to point out that a general question is more challenging: how to determine invariant subspaces of the form $W = W_{m_1}^1 \times W_{m_2}^2 \times \cdots \times W_{m_n}^n$ for the combined short pulse-mKdV equation (1.3).

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