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Finite-dimensional integrable systems associated with the Davey–Stewartson I equation

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Abstract
For the Davey–Stewartson I equation, which is an integrable equation in 1 + 2 dimensions, we have already found its Lax pair in (1 + 1)-dimensional form by nonlinear constraints. This paper deals with the second nonlinearization of this (1 + 1)-dimensional system to obtain three (1 + 0)-dimensional Hamiltonian systems with a constraint of Neumann type. The full set of involutive conserved integrals is obtained and their functional independence is proved. Therefore, the Hamiltonian systems are completely integrable in the Liouville sense. A periodic solution of the Davey–Stewartson I equation is obtained by solving these classical Hamiltonian systems as an example.

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1. Introduction
The Davey–Stewartson I (DSI) equation is a famous (1 + 2)-dimensional integrable equation describing the motion of a water wave [6]. It has been discussed using various methods. Soliton solutions were obtained by the inverse scattering method [7, 13], Bäcklund transformation [1], binary Darboux transformation [12], nonlinearization method [17, 18], etc. Almost-periodic solutions were also obtained [11].

In [17, 18], this (1 + 2)-dimensional problem was nonlinearized to an essentially (1 + 1)-dimensional linear system (1) where all the differentials are separated. This system is very useful for obtaining localized soliton solutions using the Darboux transformation in 1 + 1 dimensions.

On the other hand, many (1 + 1)-dimensional integrable systems can be nonlinearized to (1 + 0)-dimensional (or so-called finite-dimensional) integrable systems [2, 9, 15, 16], and the
idea of nonlinearization was proposed by Cao [2]. Some \((1 + 1)\)-dimensional problems were completely solved and the periodic or quasi-periodic solutions were obtained.

In [3], the KP equation, which is \((1 + 2)\) dimensional, was nonlinearized not only to \((1 + 1)\)-dimensional [4, 8], but also to \((1 + 0)\)-dimensional integrable Hamiltonian systems.

In the present paper, we show that the \((1 + 1)\)-dimensional system obtained by nonlinearizing the Lax pair of the DSI equation can also be nonlinearized to three \((1 + 0)\)-dimensional Hamiltonian systems. We find a full set of involutive conserved integrals and prove their functional independence. Therefore, these systems are completely integrable in the Liouville sense. As an example, when the number of eigenvalues is two, we solve the systems directly to obtain a periodic solution of the DSI equation.

It is well known that the DSI equation has a Lax pair in \(1 + 2\) dimensions. In [17, 18], a new integrable system was presented, which is essentially \((1 + 1)\) dimensional, since all the differentials are separated. This system can be written explicitly as

\[
\Phi_y = V \Phi = \begin{pmatrix} i \lambda & u & if \\ -\bar{u} & -i \lambda & -ig \\ if & -ig & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi \\ \Phi \end{pmatrix} \\
\Phi_y = U \Phi = \begin{pmatrix} i \lambda & 0 & if \\ 0 & i \lambda & ig \\ if & ig & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \Phi \\ \Phi \end{pmatrix} \\
\Phi_y = W \Phi = \begin{pmatrix} -2i \lambda^2 + i|u|^2 + iv_1 \\ 2i \lambda^2 - i|u|^2 - iv_2 \\ -2i \lambda \bar{\lambda} + 2f_y \\ 2i \lambda \bar{\lambda} - 2g_y \end{pmatrix} \Phi.
\]

Here \(u, f\) and \(g\) are complex functions, \(v_1\) and \(v_2\) are real functions.

Its integrability conditions \(\Phi_{xy} = \Phi_{yx}, \Phi_{zt} = \Phi_{ty}\) and \(\Phi_{xt} = \Phi_{tx}\) consist of the following three parts.

1. DSI equation

\[
-i u_t = u_{xx} + u_{yy} + 2|u|^2 u + 2(v_1 + v_2) u \\
v_{1,x} - v_{1,y} = v_{2,x} + v_{2,y} = -(|u|^2)_x.
\]

2. Standard Lax pair of the DSI equation

\[
F_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F_x + \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} F
\]

\[
F_t = 2i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F_{xx} + 2i \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} F_x + i \begin{pmatrix} |u|^2 + 2v_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & -|u|^2 - 2v_2 \end{pmatrix} F
\]

where

\[
F = \begin{pmatrix} f \\ g \end{pmatrix}.
\]

3. Nonlinear constraint

\[
FF^* = \frac{1}{2} \begin{pmatrix} v_1 \\ \bar{u}_x \\ v_2 \end{pmatrix}.
\]

Therefore, the nonlinear equations we will consider consist of the DSI equation, its standard Lax pair in \(1 + 2\) dimensions and the nonlinear constraint. As soon as this problem is solved, we obtain the solution of the DSI equation, although only part of the solutions can be
obtained by this method. In [18], localized \((M, N)\) soliton solutions were obtained from the Darboux transformation for (1).

As for many other \((1+1)\)-dimensional problems, here we want to find the nonlinear constraint for (1) and perform nonlinearization again to obtain \((1+0)\)-dimensional Hamiltonian systems.

2. Nonlinearization and Hamiltonians

In order to obtain a nonlinear constraint which is compatible with all the \(x, y\) and \(t\) equations in (1), we must consider the \(y\) equation first.

Note that if \(\Phi\) is a solution of (1) for real \(\lambda\), then

\[
\bar{\Phi}_y = V^T \Phi, \quad \bar{\Phi}_x = U^T \Phi, \quad \bar{\Phi}_t = W^T \Phi.
\]

Suppose \(\Phi = (\phi_1, \phi_2, \phi_3)^T\), then we may choose \((i\bar{\phi}_1, i\bar{\phi}_2, i\bar{\phi}_3)^T\) to be the corresponding conjugate coordinates.

The first element of the Lenard sequence corresponding to \(w = (u, -\bar{u}, i\bar{f}, -ig, -i\bar{g})\) is [14]

\[
G_0 = (0, 0, i\bar{f}, if, -ig, -i\bar{g}).
\]

By computing the variation of \(\lambda\) [9], we have

\[
\delta \lambda / \delta \omega = (i\bar{\phi}_1 \phi_2, i\bar{\phi}_2 \phi_1, i\bar{\phi}_3 \phi_1, i\bar{\phi}_2 \phi_3, i\bar{\phi}_3 \phi_2).
\]

Now take \(N\) distinct eigenvalues \(\lambda_j \in \mathbb{R} (N \geq 2)\). Suppose the corresponding solution of the Lax pair for \(\lambda = \lambda_j\) is \((\phi_{1j}, \phi_{2j}, \phi_{3j})^T\). Then define

\[
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \quad \Phi_j = (\phi_{1j}, \ldots, \phi_{3j})^T \quad (j = 1, 2, 3).
\]

Each \(\Phi_j\) is a column vector, and \(\Lambda\) is a constant real diagonal matrix.

We choose \((\phi_{1j}, \phi_{2j}, \phi_{3j}, i\bar{\phi}_{1j}, i\bar{\phi}_{2j}, i\bar{\phi}_{3j})\) as the coordinates instead of the real ones \(\text{Re}(\phi_{jk})\) and \(\text{Im}(\phi_{jk})\) \((1 \leq j \leq 3, 1 \leq k \leq N)\). The standard symplectic form of \(\mathbb{R}^{6N}\) is given by

\[
\omega = 2 \sum_{1 \leq j \leq N} \sum_{1 \leq \alpha \leq 3} d \text{Im}(\phi_{j\alpha}) \wedge d \text{Re}(\phi_{j\alpha}) + \sum_{1 \leq j \leq N} \sum_{1 \leq \alpha \leq 3} i d\bar{\phi}_{j\alpha} \wedge d\phi_{j\alpha}.
\]

Denote \(\langle V_1, V_2 \rangle = V_1^* V_2\) for two column vectors \(V_1\) and \(V_2\).

Define the nonlinear constraint

\[
G_0 = \sum_j \delta \lambda_j / \delta \omega
\]

which is

\[
\langle \Phi_1, \Phi_2 \rangle = 0 \quad \langle \Phi_3, \Phi_1 \rangle = f \quad \langle \Phi_3, \Phi_2 \rangle = g.
\]

By differentiating these constraints and using (1), we have

\[
\begin{align*}
\langle \Phi_1, \Phi_1 \rangle_x &= \langle \Phi_2, \Phi_2 \rangle_x = \langle \Phi_3, \Phi_3 \rangle_x = 0 \\
\langle \Phi_1, \Phi_1 \rangle_y &= \langle \Phi_2, \Phi_2 \rangle_y = \langle \Phi_3, \Phi_3 \rangle_y = 0 \\
\langle \Phi_1, \Phi_1 \rangle_t &= \langle \Phi_2, \Phi_2 \rangle_t = \langle \Phi_3, \Phi_3 \rangle_t = 0
\end{align*}
\]
where \( a_{\omega \beta} \)

Theorem 1. The Hamiltonians for (19)–(21) are given by

\[
H = -\langle \Phi_1, \Lambda \Phi_1 \rangle - \langle \Phi_2, \Lambda \Phi_2 \rangle - \langle \Phi_1, \Phi_2 \rangle^2 - i \langle \Phi_1, \Phi_3 \rangle^2 \quad \text{(23)}
\]

\[
H' = 2\langle \Phi_1, \Lambda^2 \Phi_1 \rangle - 2\langle \Phi_2, \Lambda^2 \Phi_2 \rangle + 2 \Re(\langle \Phi_1, \Phi_2 \rangle \langle \Phi_3, \Lambda \Phi_1 \rangle) \quad \text{(24)}
\]

\[
H'' = \quad -4 \Re(\langle \Phi_2, \Phi_3 \rangle \langle \Phi_3, \Lambda \Phi_2 \rangle) + 2 \Re(\langle \Phi_2, \Phi_3 \rangle \langle \Phi_3, \Phi_1 \rangle) \quad \text{(25)}
\]
respectively, which satisfy
\[ \{H^x, \langle \Phi_1, \Phi_2 \rangle \} = \{H^y, \langle \Phi_1, \Phi_2 \rangle \} = \{H^t, \langle \Phi_1, \Phi_2 \rangle \} = 0. \tag{26} \]

Here \( u \) is given by (15) and \( q \) is given by (17).

The proof is obtained by direct computation.

Therefore, we obtain Hamiltonian systems (23)–(25) with Neumann-type constraint, \((\Phi_1, \Phi_2) = 0\). Any solution of the Hamiltonian equations
\[
\begin{align*}
\dot{\phi}_{jk,x} &= \frac{\partial H^x}{\partial \phi_{jk}} \\
\dot{\phi}_{jk,y} &= \frac{\partial H^y}{\partial \phi_{jk}} \\
\dot{\phi}_{jk,t} &= \frac{\partial H^t}{\partial \phi_{jk}}
\end{align*}
\]
gives a solution of the DSI equation (2), where \( u, v_1, v_2 \) are given by (15) and (18), respectively.

3. Integrability

Now we consider the integrability of the Hamiltonian systems given by theorem 1 on the submanifold
\[ S = \{ (\Phi_1, \Phi_2) \in \mathbb{R}^{6N} | \langle \Phi_1, \Phi_2 \rangle = 0, \langle \Phi_1, \Phi_1 \rangle \neq \langle \Phi_2, \Phi_2 \rangle \}. \tag{28} \]

Here we still use \( 3N \) complex numbers and their complex conjugates \((\Phi_1, \Phi_2, \Phi_3, i\bar{\Phi}_1, i\bar{\Phi}_2, i\bar{\Phi}_3)\) to represent a point in \( \mathbb{R}^{6N} \). Clearly, \( S \) has two connected components characterized by \( \langle \Phi_1, \Phi_1 \rangle > \langle \Phi_2, \Phi_2 \rangle \) and \( \langle \Phi_2, \Phi_2 \rangle > \langle \Phi_1, \Phi_1 \rangle \), respectively. 

Since \( \langle \Phi_1, \Phi_1 \rangle \neq \langle \Phi_2, \Phi_2 \rangle, 0 \notin S \). \( S \) is a \((6N - 2)\)-dimensional real analytic manifold, on which the coordinates can be given by \( \phi_{11}, \ldots, \phi_{3N} \) (with their complex conjugates) and \( 2N - 1 \) of \( \phi_{11}, \ldots, \phi_{3N}, \phi_{21}, \ldots, \phi_{2N} \) (with their complex conjugates) whenever the remaining one is non-zero.

Define
\[
\begin{align*}
\gamma_1 &= \text{Re} (\Phi_1, \Phi_2) = \frac{1}{2}((\Phi_1, \Phi_2) + (\Phi_2, \Phi_1)) \\
\gamma_2 &= \text{Im} (\Phi_1, \Phi_2) = \frac{1}{2i}((\Phi_1, \Phi_2) - (\Phi_2, \Phi_1))
\end{align*}
\]
then \( S \) is defined by two real-valued functions \( \gamma_1 = \gamma_2 = 0 \). Since
\[
\{ \gamma_1, \gamma_2 \} = \frac{1}{2}(\langle \Phi_1, \Phi_1 \rangle - \langle \Phi_2, \Phi_2 \rangle)
\]
is never zero on \( S \), the symplectic form (10) on \( \mathbb{R}^{6N} \) naturally induces a \((\text{non-degenerate})\) symplectic form on \( S \). The corresponding Poisson bracket of two functions \( \xi, \eta \) on \( S \) is still given by (22) if they satisfy \([\xi, \eta] = 0, [\eta, \gamma_j] = 0 (j = 1, 2)\).

Hereafter, the Poisson bracket \{ \} always denotes the standard Poisson bracket (22) on \( \mathbb{R}^{6N} \).

Let
\[
L(\lambda) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j} \phi_{1j} & \phi_{2j} \phi_{1j} & \phi_{3j} \phi_{1j} \\ \phi_{1j} \phi_{2j} & \phi_{2j} \phi_{2j} & \phi_{3j} \phi_{2j} \\ \phi_{1j} \phi_{3j} & \phi_{2j} \phi_{3j} & \phi_{3j} \phi_{3j} \end{pmatrix}. \tag{31} \]
Then we have

**Lemma 1.** \( L(λ) \) satisfies the Lax equations

\[
L_y = [V, L] \quad L_z = [U, L] \quad L_t = [W, L]
\]  

if and only if the constraints (12) hold.

**Proof.** Let \( J = \text{diag}(1, -1, 0) \), \( L_0 = \text{diag}(1, 1, 0) \), \( \phi_j = (\phi_{1j}, \phi_{2j}, \phi_{3j})^T \), then

\[
L(λ) = L_0 + \sum_{j=1}^{N} \frac{1}{λ - λ_j} φ_j φ_j^*.
\]  

Since \( V(λ_j)^* = -V(λ_j) \),

\[
L_y(λ) = \sum_{j=1}^{N} \frac{1}{λ - λ_j} [V(λ_j), φ_j φ_j^*]
\]

\[= \sum_{j=1}^{N} \frac{1}{λ - λ_j} [V(λ), φ_j φ_j^*] - \sum_{j=1}^{N} \frac{1}{λ - λ_j} [V(λ) - V(λ_j), φ_j φ_j^*]
\]

\[= [V(λ), L(λ) - L_0] - i \left[J, \sum_{j=1}^{N} φ_j φ_j^* \right].
\]  

Hence \( L_y(λ) = [V(λ), L(λ)] \) if and only if

\[
[L_0, V(λ)] = i \left[J, \sum_{j=1}^{N} φ_j φ_j^* \right].
\]  

Written in the components, this is exactly the constraints (12). This proves that the first equation of (32) is equivalent to (12). When (12) holds, the other two equations of (32) are obtained similarly to the first one. The lemma is proved.

By lemma 1, \( \text{tr} \ L_k \) (\( k \geq 1 \)) are all conserved. Expand \( \text{tr} \ L_k \) as a Laurent series

\[
\text{tr} \ L_k = 2 + \sum_{j=0}^{∞} \frac{\tilde{E}^{(k)}_j}{λ^{j+1}}
\]  

which is convergent absolutely and uniformly as \(|λ| > \max_{1 \leq j \leq N} |λ_j|\), then all \( \{\tilde{E}^{(k)}_j\} \) are conserved.

Moreover, we can show that any two of \( \{\tilde{E}^{(k)}_j\} \) commute with each other. This follows from the following more general lemma.

**Lemma 2.** Suppose \( \mathbb{R}^{2nr} = \{\{q_{11}, \ldots, q_{1n}, q_{21}, \ldots, q_{2n}, \ldots, q_{r1}, \ldots, q_{rn}, p_{11}, \ldots, p_{1n}, \ldots, p_{r1}, \ldots, p_{rn}\}\} \) is equipped with the standard symplectic form

\[
ω = \sum_{1 \leq j \leq r \atop 1 \leq α \leq n} dp_{jα} \wedge dq_{jα}.
\]  

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\[
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\( \text{tr} \ L_k \) (\( k \geq 1 \)) are all conserved. Expand \( \text{tr} \ L_k \) as a Laurent series

\[
\text{tr} \ L_k = 2 + \sum_{j=0}^{∞} \frac{\tilde{E}^{(k)}_j}{λ^{j+1}}
\]  

which is convergent absolutely and uniformly as \(|λ| > \max_{1 \leq j \leq N} |λ_j|\), then all \( \{\tilde{E}^{(k)}_j\} \) are conserved.

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\[
ω = \sum_{1 \leq j \leq r \atop 1 \leq α \leq n} dp_{jα} \wedge dq_{jα}.
\]  

\( \text{tr} \ L_k \) (\( k \geq 1 \)) are all conserved. Expand \( \text{tr} \ L_k \) as a Laurent series

\[
\text{tr} \ L_k = 2 + \sum_{j=0}^{∞} \frac{\tilde{E}^{(k)}_j}{λ^{j+1}}
\]  

which is convergent absolutely and uniformly as \(|λ| > \max_{1 \leq j \leq N} |λ_j|\), then all \( \{\tilde{E}^{(k)}_j\} \) are conserved.

Moreover, we can show that any two of \( \{\tilde{E}^{(k)}_j\} \) commute with each other. This follows from the following more general lemma.
Denote $q_j = (q_{j1}, \ldots, q_{jn})^T$, $p_j = (p_{j1}, \ldots, p_{jn})^T$. Let $\lambda_1, \ldots, \lambda_n$ be $n$ distinct real numbers and $A$ be an $r \times r$ constant matrix,

$$M(\lambda) = A + \sum_{a=1}^{n} \frac{1}{\lambda - \lambda_a} \begin{pmatrix} p_{1a}q_{1a} & p_{2a}q_{1a} & \cdots & p_{ra}q_{1a} \\ p_{1a}q_{2a} & p_{2a}q_{2a} & \cdots & p_{ra}q_{2a} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1a}q_{ra} & p_{2a}q_{ra} & \cdots & p_{ra}q_{ra} \end{pmatrix}.$$  \hfill (38)

Then for any two complex numbers $\lambda, \mu$ and any positive integers $k, l, a, b$ with $1 \leq a, b \leq r$,

$$\{ \text{tr } M^k(\lambda), \text{tr } M^l(\mu) \} = 0$$

$$\{ \text{tr } M^k(\lambda), \langle p_b, q_a \rangle \} = -k [A, M^{k-1}(\lambda)]_{ab}.$$  \hfill (39)

Here $\langle p_b, q_a \rangle = \sum_{a=1}^{n} p_{ba}q_{aa}$.

**Proof.** The $j$th row of

$$\frac{\partial M(\lambda)}{\partial q_{ja}}$$

is

$$\frac{1}{\lambda - \lambda_a} (p_{1a}, \ldots, p_{ra})$$

and the other rows are zero. Similarly, the $j$th column of

$$\frac{\partial M(\lambda)}{\partial p_{ja}}$$

is

$$\frac{1}{\lambda - \lambda_a} (q_{1a}, \ldots, q_{ra})^T$$

and the other columns are zero. Hence

$$\frac{1}{kl} \{ \text{tr } M^k(\lambda), \text{tr } M^l(\mu) \} = \frac{1}{kl} \sum_{j,a} \left( \frac{\partial}{\partial q_{ja}} \left( \text{tr } M^k(\lambda) \right) \frac{\partial}{\partial p_{ja}} \left( \text{tr } M^l(\mu) \right) \right)$$

$$- \frac{\partial}{\partial q_{ja}} \left( \text{tr } M^l(\mu) \right) \frac{\partial}{\partial p_{ja}} \left( \text{tr } M^k(\lambda) \right)$$

$$= \sum_{j,a} \left( \text{tr } \left( M^{k-1}(\lambda) \frac{\partial M(\lambda)}{\partial q_{ja}} \right) \text{tr } \left( M^{l-1}(\mu) \frac{\partial M(\mu)}{\partial p_{ja}} \right) \right)$$

$$- \text{tr } \left( M^{l-1}(\mu) \frac{\partial M(\mu)}{\partial q_{ja}} \right) \text{tr } \left( M^{k-1}(\lambda) \frac{\partial M(\lambda)}{\partial p_{ja}} \right)$$
This proves the first part. The second part is proved as follows:

\[
\frac{1}{k} \{ \text{tr } M^k(\lambda), \langle p_b, q_a \rangle \} = \frac{1}{k} \sum_a \left( \frac{\partial}{\partial q_{ba}} (\text{tr } M^k(\lambda)) p_{ba} - \frac{\partial}{\partial p_{aa}} (\text{tr } M^k(\lambda)) q_{ba} \right)
\]

\[
= \sum_{i,a} \left( \frac{1}{\lambda - \lambda_a} (M^{k-1}(\lambda))_{ib} p_{ia} - \frac{1}{\lambda - \lambda_a} (M^{k-1}(\lambda))_{ai} q_{ia} \right)
\]

\[
= \left( (M(\lambda) - A)M^{k-1}(\lambda) - M^{k-1}(\lambda)(M(\lambda) - A) \right)_{ab}
\]

\[
= -[A, M^{k-1}(\lambda)]_{ab}.
\]

(41)

The lemma is proved.

From this lemma, we know that the set \( \{ \tilde{E}_j(\lambda) \} \) is in involution.

Remark 1. For the three-wave equation, [14] wrote down such \( \{ \tilde{E}_j(\lambda) \} \). It was proved that they were in involution and 3N of them were independent. However, for the three-wave equation, the first term of \( L \) is \( \text{diag}(\beta_1, \beta_2, \beta_3) \) (\( \beta_1, \beta_2, \beta_3 \) are distinct) rather than \( \text{diag}(1, 1, 0) \) here. In the case \( \beta_1 = 1, \beta_2 = 1, \beta_3 = 0 \), those \( \{ \tilde{E}_j(\lambda) \} \) are obviously still in involution. Hence we can also use the result of [14] to find the involution. However, we prove it more directly and easily here. On the other hand, with the constraint \( \langle \Phi_1, \Phi_2 \rangle = 0 \), the independence of \( \{ \tilde{E}_j(\lambda) \} \) changes completely. Hence we should prove the independence in this constrained case.

For real \( \lambda \), suppose the eigenvalues of the Hermitian matrix \( L(\lambda) \) are \( \nu_1, \nu_2 \) and \( \nu_3 \), then

\[
\text{tr } L^k = \sum_{j=1}^3 \nu_j^k,
\]

while

\[
\det(\mu - L(\lambda)) = \prod_{j=1}^3 (\mu - \nu_j)
\]

\[
= \mu^3 - (\nu_1 + \nu_2 + \nu_3)\mu^2 + (\nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1)\mu - \nu_1\nu_2\nu_3
\]

(42)

for any \( \mu \). Hence \( \text{tr } L^k \) can be expressed by the coefficients of \( \mu \) in \( \det(\mu - L(\lambda)) \).

Suppose \( L(\lambda) = (L_{jk})_{1 \leq j,k \leq 3} \), then

\[
v_1 + v_2 + v_3 = \text{tr } L \quad v_1v_2v_3 = \det L
\]

\[
v_1v_2 + v_2v_3 + v_3v_1 = \sum_{1 \leq j,k \leq 3} \left| \begin{array}{cc} L_{jj} & L_{jk} \\ L_{kj} & L_{kk} \end{array} \right|
\]

(43)
Define
\[
\begin{align*}
\text{tr} L &= \sum_{k=1}^{\infty} \frac{e_k^{(0)}}{\lambda_{k+1}} \quad \text{det} L = \sum_{k=1}^{\infty} \frac{e_k^{(2)}}{\lambda_{k+1}} \\
\sum_{1 \leq i < j \leq 3} \begin{vmatrix} L_{ij} & L_{jk} \\ L_{kj} & L_{kk} \end{vmatrix} &= \sum_{k=1}^{\infty} \frac{e_k^{(1)}}{\lambda_{k+1}}.
\end{align*}
\]
(44)

These \(e_k^{(j)}\) differ from the \(F_k^{(j)}\) in [14] with a multiple \(i\).

To simplify the expressions of \(E_k^{(j)}\), we define the following simpler but equivalent conserved integrals \(E_k^{(j)}\), which are non-degenerate linear combinations of \(e_k^{(j)}\). This also makes the expressions (50)–(52) of the Hamiltonians and the proof of theorem 4 simpler. For \(m \geq 0\), define
\[
E_m^{(1)} = 2e_m^{(0)} - e_m^{(1)} = \langle \Phi_1, \Lambda^m \Phi_1 \rangle + \langle \Phi_2, \Lambda^m \Phi_2 \rangle \\
- \sum_{1 \leq i < j \leq 3} \sum_{l=1}^{m} \begin{vmatrix} \langle \Phi_1, \Lambda^{l-1} \Phi_i \rangle & \langle \Phi_j, \Lambda^{m-l} \Phi_j \rangle \\ \langle \Phi_1, \Lambda^{l-1} \Phi_j \rangle & \langle \Phi_j, \Lambda^{m-l} \Phi_j \rangle \end{vmatrix}
\]
(45)

\[
E_m^{(2)} = -e_m^{(2)} + e_m^{(1)} - e_m^{(0)} = \sum_{i=0}^{m} \begin{vmatrix} \langle \Phi_1, \Lambda^i \Phi_1 \rangle & \langle \Phi_2, \Lambda^m-\Phi_1 \rangle \\ \langle \Phi_1, \Lambda^i \Phi_2 \rangle & \langle \Phi_2, \Lambda^m-\Phi_2 \rangle \end{vmatrix}
- \sum_{1 \leq i < j \leq 3} \sum_{l=1}^{m} \begin{vmatrix} \langle \Phi_1, \Lambda^l \Phi_1 \rangle & \langle \Phi_2, \Lambda^m-\Phi_1 \rangle \\ \langle \Phi_1, \Lambda^l \Phi_2 \rangle & \langle \Phi_2, \Lambda^m-\Phi_2 \rangle \end{vmatrix}
\]
(46)

\[
E_m^{(3)} = e_m^{(1)} - e_m^{(0)} = \langle \Phi_3, \Lambda^m \Phi_3 \rangle + \sum_{1 \leq i < j \leq 3} \sum_{l=1}^{m} \begin{vmatrix} \langle \Phi_1, \Lambda^{l-1} \Phi_i \rangle & \langle \Phi_j, \Lambda^{m-l} \Phi_j \rangle \\ \langle \Phi_1, \Lambda^{l-1} \Phi_j \rangle & \langle \Phi_j, \Lambda^{m-l} \Phi_j \rangle \end{vmatrix}
\]
(47)

The above sums are zero if the upper bound is smaller than the lower bound. According to lemma 2, we have

**Theorem 2.** \(\{E_k^{(j)}, E_k^{(k)}\} = 0\) and \(\{E_k^{(j)}, \langle \Phi_1, \Phi_2 \rangle \} = 0\) for any \(j, k = 1, 2, 3\) and \(m, n \geq 0\). Therefore, \(\{E_m^{(j)}\}\) are in involution on \(S\).

Define
\[
\Omega_1 = \langle \Phi_1, \Phi_1 \rangle \quad \Omega_2 = \langle \Phi_2, \Phi_2 \rangle \quad \Omega_3 = \langle \Phi_3, \Phi_3 \rangle.
\]
(48)

By (45)–(47),
\[
E_0^{(1)} = \Omega_1 + \Omega_2 \quad E_0^{(2)} = \Omega_1 \Omega_2 - |\langle \Phi_1, \Phi_2 \rangle|^2 \quad E_0^{(3)} = \Omega_3.
\]
(49)

Hence \(\Omega_1, \Omega_2, \Omega_3\) are expressed by \(E_k^{(j)}\) and \(|\langle \Phi_1, \Phi_2 \rangle|^2\).

The Hamiltonians in theorem 1 can be expressed as
\[
H^x = -E_1^{(1)} - E_0^{(1)} E_0^{(3)} - E_0^{(2)}
\]
(50)

\[
H^y = \frac{1}{\Omega_1 - \Omega_2} (-E_1^{(1)} E_1^{(1)} + 2E_1^{(2)} - E_0^{(1)} E_0^{(2)} - E_0^{(1)} E_0^{(3)} + 2E_0^{(2)} E_0^{(3)}).\]
(51)
Theorem 3. Three Hamiltonians $H^i$, $H^j$, $H^k$ defined by theorem 1 commute with each other:

$$\{H^i, H^j\} = \{H^j, H^k\} = \{H^k, H^i\} = 0$$

(57)

and they satisfy

$$\{H^i, \{\Phi_1, \Phi_2\}\} = \{H^j, \{\Phi_1, \Phi_2\}\} = \{H^k, \{\Phi_1, \Phi_2\}\} = 0.$$  

(58)

Moreover, each $E^{(j)}_m$ is conserved under the Hamiltonian flows given by $H^i$, $H^j$, $H^k$, respectively.

Next we shall prove the integrability of these Hamiltonian systems. That is

Theorem 4. $3N - 1$ real-valued functions $\{E^{(j)}_m\}$ ($j = 1, 3; 0 \leq m \leq N - 1$) and $\{E^{(3)}_m\}$ ($0 \leq m \leq N - 2$) are functionally independent in a dense open subset of $S$. 

\[ H^i = \frac{1}{\Omega_1 - \Omega_2} \left( 2E_0^{(1)}E_2^{(1)} - 4E_2^{(2)} + 2(E_0^{(1)} - 2E_0^{(2)})(E_1^{(1)} + E_1^{(3)}) \right) \]
\[ + (E_0^{(1)} - 4E_0^{(2)})H^i + (E_0^{(1)} - 2E_0^{(3)})(\Omega_1 - \Omega_2)H^j + (H^i)^2 - (H^j)^2 \]
\[ - (u(\Phi_1, \Phi_2) - \bar{u}(\Phi_2, \Phi_1))^2 + 4|\langle \Phi_1, \Phi_2 \rangle|^2 + |\langle \Phi_2, \Phi_2 \rangle|^2. \]
Proof. Let
\[ \widetilde{S}_1 = \{ (\Phi_1, \Phi_2, \Phi_3, i\Phi_1, i\Phi_2, i\Phi_3) \in S \mid \phi_{1,N} \neq 0, \tilde{\phi}_{1,N} \neq 0, (\Phi_1, \Phi_1) > (\Phi_2, \Phi_2) \} \]
\[ \widetilde{S}_2 = \{ (\Phi_1, \Phi_2, \Phi_3, i\Phi_1, i\Phi_2, i\Phi_3) \in S \mid \phi_{1,N} \neq 0, \tilde{\phi}_{1,N} \neq 0, (\Phi_1, \Phi_1) < (\Phi_2, \Phi_2) \} \]
(59)
\[ \widetilde{S} = \widetilde{S}_1 \cup \widetilde{S}_2 \]
then \( \widetilde{S} \) is a dense open subset of \( S \). Similar to \( S \), \( \widetilde{S} \) has also two connected components, which are \( \widetilde{S}_1 \) and \( \widetilde{S}_2 \). In \( \widetilde{S} \), we can solve \( \phi_{2,N}, \tilde{\phi}_{2,N} \) from the constraint \( (\Phi_1, \Phi_2) = 0 \) as
\[ \phi_{2,N} = -\sum_{j=1}^{N-1} \frac{\partial E}{\partial \phi_{1j}} \phi_{2j}; \quad \tilde{\phi}_{2,N} = -\sum_{j=1}^{N-1} \frac{\partial E}{\partial \phi_{1j}} \tilde{\phi}_{2j}. \] (60)
Hence \( \widetilde{S}_1 \) has global coordinates
\[ \Theta = \{ \phi_{1j}, i\tilde{\phi}_{1j} (1 \leq j \leq N); \phi_{3j}, i\tilde{\phi}_{3j} (1 \leq j \leq N - 1); \phi_{3j}, i\tilde{\phi}_{3j} (1 \leq j \leq N) \}. \] (61)
Let \( P_0 \in \widetilde{S}_1 \) be given by \( \Phi_1 = (1, 1, \ldots, 1)^T, \Phi_2 = \epsilon(1, 1, \ldots, 1, -N + 1)^T, \Phi_3 = (1, 1, \ldots, 1)^T \), where \( \epsilon \) is a small real constant. Here \( \Phi_3 \) is chosen to be parallel with \( \Phi_1 \) so that the following computation will be simplified. Since \( \phi_{2N} \) and \( \tilde{\phi}_{2N} \) are functions of the variables in \( \Theta \), we have, at \( P_0 \),
\[ \frac{\partial E^{(1)}}{\partial \phi_{1j}} = \lambda_j^{m} + O(\epsilon) \quad \frac{\partial E^{(1)}}{\partial \phi_{2j}} = O(\epsilon) \quad \frac{\partial E^{(1)}}{\partial \phi_{3j}} = O(\epsilon) \]
\[ \frac{\partial E^{(2)}}{\partial \phi_{1j}} = O(\epsilon^2) \quad \frac{\partial E^{(2)}}{\partial \phi_{2j}} = O(\epsilon^3) \]
\[ \frac{\partial E^{(2)}}{\partial \phi_{2j}} = \sum_{i=0}^{m} \left| \langle \Phi_1, \Lambda^i \Phi_1 \rangle \lambda_i^{m-i} \phi_{1j} \right| - \phi_{1j} \sum_{i=0}^{m} \left| \langle \Phi_1, \Lambda^i \Phi_1 \rangle \lambda_i^{m-i} \phi_{1N} \right| + O(\epsilon^3) \] (62)
\[ = N \epsilon \sum_{i=0}^{m} \lambda_i^{m-i} (\lambda_i^{1} + \cdots + \lambda_i^{N-1}) + O(\epsilon^3) \]
\[ \frac{\partial E^{(3)}}{\partial \phi_{1j}} = O(\epsilon^2) \quad \frac{\partial E^{(3)}}{\partial \phi_{2j}} = O(\epsilon) \quad \frac{\partial E^{(3)}}{\partial \phi_{3j}} = \epsilon \lambda_j^{m} + O(\epsilon^3). \]
Here the subscript \( j \) is taken from 1 to \( N \) for \( \phi_{1j}, \phi_{3j}, \) and from 1 to \( N - 1 \) for \( \phi_{2j} \). It can be checked that
\[ \det \left( \sum_{i=0}^{m} \lambda_i^{m-i} (\lambda_i^{1} + \cdots + \lambda_i^{N-1}) + \lambda_i^{j} \right) \]
\[ \prod_{0 \leq m \leq N-2, 1 \leq j \leq N-1} (\lambda_j - \lambda_k) = N \prod_{1 \leq j < k \leq N-1} (\lambda_j - \lambda_k). \] (63)
Hence the Jacobian determinant
\[ J = \frac{\partial (E^{(1)}_0, \ldots, E^{(1)}_{N-1}, E^{(2)}_0, \ldots, E^{(3)}_{N-1})}{\partial (\phi_{11}, \ldots, \phi_{1N}, \phi_{21}, \ldots, \phi_{2N-1}, \phi_{31}, \ldots, \phi_{3N})} \]
\[ = N^N \epsilon^{2N-1} \left( \prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k) \right)^2 \prod_{1 \leq j < k \leq N-1} (\lambda_j - \lambda_k) + O(\epsilon^{2N}). \] (64)
When $\epsilon$ is small enough and $\epsilon \neq 0$, $J$ is non-zero near the point $P_0$. Since $J$ is a rational function of

$$\{ \phi_{1j}, i\phi_{1j} (1 \leq j \leq N); \phi_{2j}, i\phi_{2j} (1 \leq j \leq N - 1); \phi_{3j}, i\phi_{3j} (1 \leq j \leq N) \}$$

$J = 0$ identically if $J$ is zero in an open subset of $\mathbb{S}_1$. Therefore, $J$ is non-zero in a dense open subset of $\mathbb{S}_1$. Similarly, $J$ is non-zero in a dense open subset of $\mathbb{S}_2$. Since all $E_{m}^{(j)}$ are real-valued functions, the Jacobian matrix of $(E_{m}^{(1)}, \ldots, E_{m}^{(3)}, E_{m-1}^{(2)}, \ldots, E_{m-2}^{(2)}, E_{m-1}^{(3)}, \ldots, E_{m-1}^{(3)})$ with respect to the real coordinates

$$\text{Re}(\phi_{11}), \ldots, \text{Re}(\phi_{1N}), \text{Re}(\phi_{21}), \ldots, \text{Re}(\phi_{2N-1}), \text{Re}(\phi_{31}), \ldots, \text{Re}(\phi_{3N})$$

$$\text{Im}(\phi_{11}), \ldots, \text{Im}(\phi_{1N}), \text{Im}(\phi_{21}), \ldots, \text{Im}(\phi_{2N-1}), \text{Im}(\phi_{31}), \ldots, \text{Im}(\phi_{3N})$$

is of full rank $3N = 1$. The theorem is proved.

**Theorem 5.** The Hamiltonian systems given by theorem 1 are completely integrable on $S$ in the Liouville sense.

**Proof.** We have proved: (1) $\{E_{m}^{(j)} (j = 1, 2, 3; m = 0, 1, 2, \ldots)\}$ are in involution on $S$ (theorem 2). (2) $\{E_{m}^{(j)} (j = 1, 3; 0 \leq m \leq N - 1)\}$ and $\{E_{m}^{(2)} (0 \leq m \leq N - 2)\}$ are functionally independent in a dense open subset of $S$ (theorem 4). It remains to prove that the Hamiltonian vector fields of all $E_{m}^{(j)}$ are complete. This follows from the compactness of each level set, which is a closed subset of the compact set

$$\{(\Phi_1, \Phi_2, \Phi_3, i\Phi_1, i\Phi_2, i\Phi_3) \in S \mid (\Phi_1, \Phi_j) = \Omega_{j0}, j = 1, 2, 3\}$$

where $\Omega_{j0} (j = 1, 2, 3)$ are constants. Therefore, the Hamiltonian systems given by theorem 1 are completely integrable [5].

**4. Example: an explicit solution for $N = 2$**

Now suppose $N = 2$,

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \Phi_j = \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix}.$$ 

Let $R_1, R_2, R_3, G$ and $K$ be defined by

$$R_j = |\phi_j|^2 + |\psi_j|^2 \quad (j = 1, 2, 3)$$

$$G = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - R_2^2$$

$$K = (\langle \Phi_1, \Lambda \Phi_1 \rangle + \langle \Phi_1, \Phi_3 \rangle \langle \Phi_3, \Phi_1 \rangle)/R_1^2$$

then from the above list of conserved integrals, we know that $R_1, R_2, R_3, G$ and $K$ are all constants. Moreover,

$$\langle \Phi_2, \Lambda \Phi_2 \rangle + \langle \Phi_2, \Phi_3 \rangle \langle \Phi_3, \Phi_2 \rangle = R_2^2 (R_3^2 + \lambda + \mu - K).$$

Let

$$\phi_j = R_j \cos \theta_j \exp(i\alpha_j) \quad \psi_j = R_j \sin \theta_j \exp(i\beta_j)$$

then the constraint $\langle \Phi_2, \Phi_1 \rangle = 0$ leads to

$$\theta_2 = \sigma \theta_1 + \pi/2 + i\pi \quad \exp(i(\beta_2 - \beta_1 - \alpha_2 + \alpha_1)) = \sigma$$
where \( l \) is an integer and \( \sigma = \pm 1 \). Note that (70) is invariant under the transformation \( \theta_1 \rightarrow -\theta_1, \beta_1 \rightarrow \beta_1 + \pi \). Hence we can always choose \( \sigma = 1 \).

Let \( \delta = \beta_1 - \beta_3 - \alpha_1 + \alpha_3 \) and \( \rho = \cos^2 \theta_1 \). Substituting (70) into the second equation of (68), we obtain

\[
R_2^2 \cos^2 \theta_3 = G - (R_1^2 - R_2^2) \rho. \tag{72}
\]

The third equation of (68) gives

\[
\lambda \cos^2 \theta_1 + \mu \sin^2 \theta_1 + R_1^2 (\cos^2 \theta_1 \cos^2 \theta_3 + \sin^2 \theta_1 \sin^2 \theta_3) + 2R_1^2 \cos \theta_1 \sin \theta_1 \cos \theta_3 \sin \theta_3 \cos \delta = K \tag{73}
\]

from which \( \delta \) can be solved as a function of \( \rho \).

From equation (19), we have

\[
\begin{align*}
\theta_{1,s} &= R_2^2 \cos \theta_3 \sin \theta_3 \sin \delta \\
\theta_{3,s} &= (R_2^2 - R_1^2) \cos \theta_1 \sin \theta_1 \sin \delta \\
\alpha_{1,s} &= \lambda + R_2^2 \cos^2 \theta_3 + R_1^2 \cos \theta_3 \sin \theta_3 \tan \theta_3 \cos \delta \\
\alpha_{2,s} &= \lambda + R_2^2 \cos^2 \theta_3 - R_1^2 \cos \theta_3 \sin \theta_3 \cot \theta_3 \cos \delta \\
\alpha_{3,s} &= R_1^2 \cos^2 \theta_3 + R_2^2 \sin^2 \theta_1 + (R_1^2 - R_2^2) \cos \theta_1 \sin \theta_1 \tan \theta_3 \cos \delta \\
\beta_{1,s} &= \mu + R_2^2 \sin^2 \theta_3 + R_1^2 \cos \theta_3 \sin \theta_3 \cot \theta_3 \cos \delta \\
\beta_{2,s} &= \mu + R_2^2 \sin^2 \theta_3 - R_1^2 \cos \theta_3 \sin \theta_3 \tan \theta_3 \cos \delta \\
\beta_{3,s} &= R_1^2 \sin^2 \theta_1 + R_2^2 \cos^2 \theta_1 + (R_1^2 - R_2^2) \cos \theta_1 \sin \theta_1 \cot \theta_3 \cos \delta.
\end{align*}
\]

The first equation of the above system leads to

\[
\rho_s = -2R_2^2 \cos \theta_3 \sin \theta_3 \cos \theta_3 \sin \theta_3 \sin \delta. \tag{75}
\]

Solving \( \cos \delta \) from (73), we obtain the equation of \( \rho \):

\[
\rho_s = -\sqrt{P(\rho)} \tag{76}
\]

where

\[
P(\rho) = 4b(\lambda - \mu) \rho^3 + (4(\mu - \lambda)G + 4b(\mu - K) - (\lambda - \mu + b - a)^2) \rho^2 \\
+ 4(K - \mu)G + 2(\lambda - \mu + b - a)(K + G - \mu - a) \rho \\
-(K + G - \mu - a)^2 \tag{77}
\]

which is a cubic polynomial,

\[
a = R_2^2 \quad b = R_1^2 - R_2^2. \tag{78}
\]

Suppose \( b > 0, \lambda > \mu \) and \( P \) has three different real roots \( \rho_1 < \rho_2 < \rho_3 \). Moreover, suppose

\[
K + G - \mu - a \neq 0 \quad K - G - \lambda + b \neq 0 \\
\max \left(0, \frac{G-a}{b}\right) < \min \left(1, \frac{G}{b}\right). \tag{79}
\]

Then the solution \( \rho \) can be expressed by elliptic functions of \( x \). Let \( \rho = \rho_1 + (\rho_2 - \rho_1) \omega^2 \), then

\[
\omega_s = \pm \sqrt{(1 - \omega^2)(1 - k^2 \omega^2)} \tag{80}
\]
where
\[ k = \sqrt{(\rho_2 - \rho_1)(\rho_3 - \rho_1)} \quad p = \sqrt{b(\lambda - \mu)(\rho_3 - \rho_1)}. \]  
(81)

Hence \( \omega = \pm \text{sn}(p(x - \tilde{x}_0)) \),
\[ \rho = \rho_1 + (\rho_2 - \rho_1) \text{sn}^2(p(x - \tilde{x}_0)) \]  
(82)

where \( \tilde{x}_0 \) is independent of \( x \), but may depend on \( y \) and \( t \). \( \rho \) is a periodic function of \( x \).

**Remark 2.** Since \( \rho_j \) is a root of \( P_j \), (77) leads to
\[ 4\rho_j(1 - \rho_j)(G - b\rho_j)(a - G + b\rho_j) \]
\[ = (K - \lambda \rho_j - \mu(1 - \rho_j) - \rho_j(G - b\rho_j) - (1 - \rho_j)(a - G + b\rho_j))^2 \geq 0. \]  
(83)

(This is actually equivalent to (73).) Hence, \( \max(0, \frac{C(x, t)}{4}) \leq \rho_1 < \rho_2 \leq \min(1, \frac{C(x, t)}{4}) \) holds if there is a solution locally, since \( 0 \leq \cos^2 \theta_1 < 1 \) and \( 0 \leq \cos^2 \theta_3 < 1 \) should be satisfied. This also guarantees that the solution is global because \( \rho_1 \leq \rho \leq \rho_2 \). Moreover, under the assumption \( K + G - \mu - a \neq 0 \) and \( K - G - \lambda + b \neq 0 \), \( P(0) \neq 0 \), \( P(1) \neq 0 \). Hence \( 0 < \rho_1 < \rho_2 < 1 \) and \( 0 < \rho < 1 \).

**Remark 3.** Using the formulae
\[ 1 - k^2 \text{sn}^2(\xi) = \text{dn}^2(\xi) = \frac{d^2}{d\xi^2} \ln \Theta(\xi) + \tilde{C} \]  
(84)

where \( \tilde{C} \) is a certain constant, the previous solution \( \rho \) can be expressed as a \( \Theta \) function.

In order to compute the \( y \) and \( t \) equations, we first write down the expressions for \( u \), \( f \) and \( g \). They are
\[ u = \frac{2iR_1R_2}{R_1^2 - R_2^2} \exp(i(\alpha_1 - \alpha_2)) \left( (K - \lambda - R_3^2 \cos^2 \theta_1) \cot \theta_1 - R_3^2 \cos \theta_3 \sin \theta_3 \exp(i\delta) \right) \]
\[ = \frac{2iR_1R_2}{R_1^2 - R_2^2} \exp(i(\alpha_1 - \alpha_2)) \left( (\mu + R_3^2 \sin^2 \theta_3 - K) \tan \theta_1 \right. \]
\[ + \left. R_3^2 \cos \theta_3 \sin \theta_3 \exp(-i\delta) \right) \]
\[ |u|^2 = \frac{4R_1^2R_2^2}{(R_1^2 - R_2^2)^2} \left( K(R_3^2 + \lambda + \mu - K) - \lambda \mu - \lambda R_3^2 \sin^2 \theta_3 - \mu R_3^2 \cos^2 \theta_3 \right) \]  
(85)

\[ f = R_1R_3 \exp(i(\alpha_1 - \alpha_3))(\cos \theta_3 \cos \theta_1 + \sin \theta_3 \sin \theta_1 \exp(i\delta)) \]
\[ g = R_2R_3 \exp(i(\alpha_2 - \alpha_3))(-\cos \theta_3 \sin \theta_1 + \sin \theta_3 \cos \theta_1 \exp(i\delta)). \]

With the help of MAPLE, equations (20) and (21) are reduced to the following simple equations:
\[ \theta_{1,y} = \gamma_1 \theta_{1,x} \quad \theta_{3,y} = \gamma_1 \theta_{3,x} \quad \theta_{1,t} = \gamma_2 \theta_{1,x} \quad \theta_{3,t} = \gamma_2 \theta_{3,x} \]
\[ \alpha_{1,y} = \gamma_1 \alpha_{1,x} - \frac{2R_2^2K}{R_1^2 - R_2^2} \alpha_{2,y} = \gamma_1 \alpha_{2,x} - \frac{2R_1^2R_2^2 + \lambda + \mu - K}{R_1^2 - R_2^2} \]
\[ \alpha_{3,y} = \gamma_1 \alpha_{3,x} - \frac{2R_2^2R_3^2}{R_1^2 - R_2^2} \alpha_{3,t} = \gamma_2 \alpha_{3,x} + C_3 \]
\[ \alpha_{1,t} = \gamma_1 \alpha_{1,x} + C_{12} \quad \alpha_{2,t} = \gamma_2 \alpha_{2,x} - C_{21} \quad \alpha_{3,t} = \gamma_2 \alpha_{3,x} + C_3 \]
\[ (\beta_j - \alpha_j)_y = \gamma_1(\beta_j - \alpha_j)_x \quad (\beta_j - \alpha_j)_t = \gamma_2(\beta_j - \alpha_j)_x + C_0 \]  
\( (j = 1, 2, 3) \)
where the constants $\gamma_1$, $\gamma_2$, $C_0$, $C_{12}$, $C_{21}$ and $C_3$ are given by

\begin{align}
\gamma_1 &= \frac{R_1^2 + R_2^2}{R_1^2 - R_2^2} \\
\gamma_2 &= \frac{2(R_1^4 + R_2^4 - (R_1^2 + R_2^2)(R_1^2 + \lambda + \mu))}{R_1^2 - R_2^2} \\
C_0 &= \frac{2(\lambda - \mu)(R_1^4 + R_2^4)}{R_1^2 - R_2^2} \\
C_{ij} &= \frac{2}{(R_1^2 - R_2^2)^2} \left( (R_1^4 + 2R_2^2R_1^2 - R_2^4) \left( \lambda - \mu \right) \left( G - \frac{R_2^2 - R_1^2}{2} \right) \\
&\quad + (R_2^4 - R_1^4) \left( \frac{\lambda + \mu}{2} - K_i \right) - \lambda \mu - \lambda R_2^2 \right) + 4R_1^2R_2^2(K(R_3^2 + \lambda + \mu - K)) \\
&\quad - \lambda(R_2^2 - R_1^2)(R_1^2 + R_2^2) - 2R_1^4K_i^2 \right) \\
K_1 &= K \\
K_2 &= R_3^2 + \lambda + \mu - K \\
C_3 &= \frac{2}{(R_1^2 - R_2^2)^2} \left( R_2^2(R_1^4 + 3R_2^4 - R_2^2R_2^2)(\lambda + R_3^2) + R_2^2(R_1^4 - R_2^4 + R_1^4R_2^2)\mu \\
&\quad - 2(R_1^2 + R_2^2)(R_1^4 + R_2^4)K - 2R_1^2R_2^2(R_1^4 - R_2^4) \right) \\
\end{align}

Hence

$$\rho = \rho_1 + (\rho_2 - \rho_1) \text{sn}^2(p(x + \gamma_1y + \gamma_2t - x_0))$$

where $x_0$ is an arbitrary constant, $p$ is given by (81) and the parameter of the function $\text{sn}$ is $k$ given by (81).

The solutions of the DSI equation are

\begin{align}
u &= \pm \frac{iR_1R_2}{R_1^2 - R_2^2} \frac{1}{\rho(\xi)\sqrt{1 - \rho(\xi)}} \\
&\quad \times \left( 2K - \lambda - \mu - a + b \right) \rho(\xi) + (\mu + a - K - G) - i\sqrt{P(\rho(\xi))} \right) \\
&\quad \times \exp \left( i \int Q(\rho(\xi)) d\xi + i\alpha(x - \gamma_1y + \gamma_2t) + i(C_{12} + C_{21})t \right) \\
\end{align}

and

\begin{align}
v_1 &= 2R_2^2((\mu - \lambda)\rho(\xi) + K - \mu) \\
v_2 &= 2R_1^2((\lambda - \mu)\rho(\xi) + K - \mu + a) \\
\end{align}

where

\begin{align}
\xi &= x + \gamma_1y + \gamma_2t - x_0 \\
\alpha &= \frac{R_2^2K - R_1^2(R_1^2 + \lambda + \mu - K)}{R_1^2 + R_2^2} \\
Q(\rho) &= \frac{2b\rho^2 + (\mu - b - 2G)\rho + (K + G - \mu - a)}{2\rho(1 - \rho)} \\
\rho(\xi) &= \rho_1 + (\rho_2 - \rho_1) \text{sn}^2(p\xi) \\
\end{align}

and the parameter $k$ of the function $\text{sn}$ is given by (81).
\( u \) has no singularity when (79) holds because in this case \( 0 < \rho < 1 \).

Suppose the minimal positive period of the function \( \text{sn} \) with parameter \( k \) is \( T(k) \) and

\[
A = \frac{p}{T(k)} \int_0^{T(k)/p} Q(\rho(\xi)) \, d\xi. \tag{93}
\]

Then we have the following properties of the solutions:

1. \( u \) is a double periodic function on the \((x, y)\)-plane. The period for \( x + \gamma_1 y \) is \( T(k)/p \), while the period for \( (A + \alpha)x + (A - \alpha)\gamma_1 y \) is \( 2\pi \).
2. \( u \) is periodic with respect to \( t \) if and only if

\[
\frac{2\pi p}{(C_{12} + C_{21} + A\gamma_2 + \alpha\gamma_2)T(k)}
\]

is a rational number.
3. \(|u|^2, v_1 \) and \( v_2 \) are periodic functions of \( x + \gamma_1 y + \gamma_2 t \) only, and they extend constantly in a transversal direction on the \((x, y)\)-plane.
4. The phase of \( u \) depends not only on the linear functions of \( x, y \) and \( t \), but also on an \( \text{sn} \) function of \( x + \gamma_1 y + \gamma_2 t \). This can be obtained from (90) and

\[
(\arg u)_x = \text{Re} \frac{1}{iu} \frac{u_x}{u} = \text{Re} \frac{2f\hat{g}}{iu} \neq \text{constant} \tag{94}
\]

by using (85) and a tedious computation.

It is still interesting to solve more general periodic solutions using this method.

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