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Research paper

Lump and lump-soliton solutions to the Hirota–Satsuma–Ito equation

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ABSTRACT

We apply the Hirota direct method to construct lump and interaction solutions to the Hirota–Satsuma–Ito (HSI) equation. We establish a general theory for finding the lump-soliton to (2+1)-dimensional nonlinear PDEs. We generate the corresponding lump and lump-soliton solutions to the HSI equation by the logarithm transformation of the dependent variables.

1. Introduction

The aim of this paper is to find lump and interaction solutions [1–3] to the Hirota–Satsuma–Ito (HSI) equation by the Hirota direct method [4,5], which is a powerful tool for finding N-soliton solutions. The Hirota method can also be used to find other kinds of exact solutions (see, e.g, [6–18]).

The Hirota–Satsuma shallow water wave equation [4]

$$u_t = u_{xxt} + 3uu_t - 3u_x v_t - u_x, \quad v_x = -u \tag{1.1}$$

has a Hirota bilinear form

$$(D_t D_x^3 - D_t D_x - D_x^2) f \cdot f = 0 \tag{1.2}$$

under the logarithm transformation $u = 2(\ln f)_{xx}$. It has an integrable $(2+1)$-dimensional extension called the Hirota–Satsuma–Ito equation [19]:

$$w_t = u_{xxt} + 3uu_t - 3u_x v_t + \alpha u_x, \quad w_x = -u_y, \quad v_x = -u \tag{1.3}$$

with the bilinear form

$$(D_t D_x^3 + D_t D_y + \alpha D_x^2) f \cdot f = 0, \tag{1.4}$$

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where α is a real nonzero constant and D_x, D_y and D_t are all Hirota derivatives [4] which are defined as follows. Let f, g be infinitely differentiable functions in $\mathbb{R}^3$,

$$\begin{aligned}&(D_x{}^k D_y{}^m D_t{}^n f\cdot g)(x,y,t)\\&=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^k\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,y,t)g(x',y',t')\Big|_{x'=x,y'=y,t'=t},\end{aligned}\tag{1.5}$$

where k, m, n are nonnegative integer.

A lump solution is a rational function solution which is real analytic and decays in all directions of space variables. Lump solutions to many $(2+1)$-dimensional equations have been widely discussed. For example, in [20,21], lump solutions to the Kadomtsev–Petviashvili (KP) equations [4,20] are obtained by taking a "long wave" limit of N-solition solutions. In [11], a general theory for finding positive quadratic function solutions to a bilinear equation is presented. The results were extended to generalized bilinear equation [12]. Recently, there have been some discussions on lump-soliton solutions by the method of ansatz [3,22] and the interaction of multi-lump solutions have been intensively studied within the framework of the KP-I equation [23,24]. In this paper, we will establish a general approach to find lump solutions and their interaction with exponential waves. we find lump part first, then we determine soliton part. In Section 2, we give the necessary and sufficient conditions for generating the lump-soliton solutions in Corollary 1. Unlike previous work [3,22], our soliton part could contain a sum of exponential waves.

2. Lump and lump-soliton solutions

Let us consider bilinear equation

$$P(D_x,D_y,D_t)f\cdot f=0,\tag{2.1}$$

where $P(x, y, t)$ is an even polynomial of (x, y, t) with $P(0,0,0)=0$.

In order to find lump-soliton solutions, set

$$f(x,y,t)=g(x,y,t)+\sum_{k=1}^{n}d_k\exp(a_kx+b_ky+c_kt),\tag{2.2}$$

where g is a polynomial. When g is positive and $d_k>0$ for $k=1,2,\cdots n,$ the corresponding $u=2[\ln(f)]_x$ or $u=2[\ln(f)]_{xx}$ is a lump-soliton function.

By the property of the Hirota derivatives, we have

$$P(D_x,D_y,D_t)\exp(\xi_i)\cdot\exp(\xi_j)=P(a_i-a_j,b_i-b_j,c_i-c_j)\exp(\xi_i+\xi_j).\tag{2.3}$$

In particular,

$$P(D_x,D_y,D_t)\exp(\xi_i)\cdot\exp(\xi_i)=P(0,0,0)\exp(2\xi_i)=0.\tag{2.4}$$

Theorem 1. *Let $\xi_k=a_kx+b_ky+c_kt\neq 0,\xi_k+\xi_j\neq 0$ and $\xi_j,\xi_k+\xi_l$ are distinct for $j,k,l=1,2,\cdots,n$. Assume $g(x, y, t)$ is a polynomial and $h(x,y,t)=\sum_{k=1}^{n}d_k\exp(\xi_k)$, $d_k\neq 0,k=1,\cdots,n$. Then $f=g+h$ is a solution of (2.1) if and only if g (solution of (2.1)) generates a rational solution and*

$$P(D_x,D_y,D_t)g\cdot\exp(\xi_k)=0,\quad k=1,2,\cdots,n,\tag{2.5}$$

$$P(D_x,D_y,D_t)\exp(\xi_k)\cdot\exp(\xi_j)=0,k,j=1,2,\cdots,n,k\neq j.\tag{2.6}$$

Proof. By direct computation

$$P(D_x,D_y,D_t)f\cdot f=P(D_x,D_y,D_t)g\cdot g+2P(D_x,D_y,D_t)g\cdot h+P(D_x,D_y,D_t)h\cdot h.\tag{2.7}$$

Notice that $P(D_x, D_y, D_t)g\cdot g$ is a polynomial, $P(D_x, D_y, D_t)g\cdot\exp(\xi_k)$ is a polynomial of x, y and t multiplying $\exp(\xi_k)$ and $P(D_x, D_y, D_t)\exp(\xi_k)\cdot\exp(\xi_j)$ is a multiple of $\exp(\xi_k+\xi_j)$. By the conditions of the theorem, their exponential parts are different. Then the conclusion follows. □

In the case of $n=1$, (2.6) is true by the property (2.4). Therefore, we get the following corollary.

Corollary 1. *Suppose function g is a polynomial solution of (2.1) and g generates a lump solution ($u=2[\ln(g)]_x$ or $u=2[\ln(g)]_{xx}$ is a lump solution) and $h(x,y,t)=d\exp(ax+by+ct)$ with $d>0$ and $ax+by+ct\neq 0$. Then $f=g+h$ generates a lump-soliton solution if and only if*

$$P(D_x,D_y,D_t)g\cdot h=0.\tag{2.8}$$

3. Application to the HSI equation

In this part, we will apply the results of Section 2 to the HSI equation.

We first consider positive quadratic solutions of Eq. (1.4). By the result in [10,11], the largest class of quadratic functions which generate lump solutions to a (2+1)-dimensional nonlinear PDEs can be expressed as:

$$g(x,y,t)=(a_1x+a_2y+a_3t+c_1)^2+(a_4x+a_5y+a_6t+c_2)^2+a_7. \tag{3.1}$$

By using Maple, we can determine the parameters a_2, a_5 and a_7

$$a_2=-\frac{\alpha[(a_1^2-a_4^2)a_3+2a_1a_4a_6]}{a_3^2+a_6^2},\quad a_5=\frac{\alpha(a_1^2a_6-2a_1a_3a_4-a_4^2a_6)}{a_3^2+a_6^2},$$
$$a_7=-\frac{3(a_3^2+a_6^2)(a_1^2+a_4^2)(a_1a_3+a_4a_6)}{\alpha(a_1a_6-a_3a_4)^2},$$

where α is a nonzero real number, c_1, c_2 are arbitrary real constants. Parameters a_1, a_3, a_4, a_6 are all real constants satisfying $a_1a_6\neq a_3a_4$ and $\alpha(a_1a_3+a_4a_6)<0$. It is not difficult to show that $a_7>0$.

We define

$$g_1(x,y,t):=a_1x-\frac{\alpha[(a_1^2-a_4^2)a_3+2a_1a_4a_6]}{a_3^2+a_6^2}y+a_3t+c_1, \tag{3.2}$$

$$g_2(x,y,t):=a_4x+\frac{\alpha(a_1^2a_6-2a_1a_3a_4-a_4^2a_6)}{a_3^2+a_6^2}y+a_6t+c_2, \tag{3.3}$$

$$g(x,y,t):=g_1^2+g_2^2-\frac{3(a_3^2+a_6^2)(a_1^2+a_4^2)(a_1a_3+a_4a_6)}{\alpha(a_1a_6-a_3a_4)^2}, \tag{3.4}$$

Note the condition $a_1a_6\neq a_3a_4$ implies $a_3^2+a_6^2\neq 0$ and $a_1^2+a_4^2\neq 0$ and so g is well defined. The property that functions g_1 and g_2 are linearly independent is guaranteed by $a_1a_5-a_2a_4=\frac{\alpha(a_1^2+a_4^2)(a_1a_6-a_3a_4)}{a_3^2+a_6^2}\neq 0$, which make

$$\begin{aligned}u&=2(\ln g)_{xx}\\&=\frac{4\{(a_1^2+a_4^2)g-2(a_1g_1+a_4g_2)^2\}}{g^2}\end{aligned}$$

decay in all directions, so that the corresponding solution u of (1.3) is a lump solution.

Example 1. For a group of parameters:

$$\alpha=-1,a_1=1,a_3=-0.8,a_4=1.25,a_6=2,c_1=c_2=0,$$

we have

$$g(x,y,t)=(-0.8t+x+1.17y)^2+(2t+1.25x-0.19y)^2+6.74,$$

and

$$u(x,y,t)=\frac{24.44t^2+(-34.85x-52.53y)t-26.27x^2-19.25xy+7.45y^2+69.06}{[(-0.8t+x+1.17y)^2+(2t+1.25x-0.19y)^2+6.74]^2}.$$

The plot of surface and contour of function u when $t=0$ are depicted in Fig. 1.

In order to find lump-soliton solutions to HSI equation, we assume $f=g+h$ for g is given by (3.1) and

$$h(x,y,t)=c_3\exp(a_8x+a_9y+a_{10}t).$$

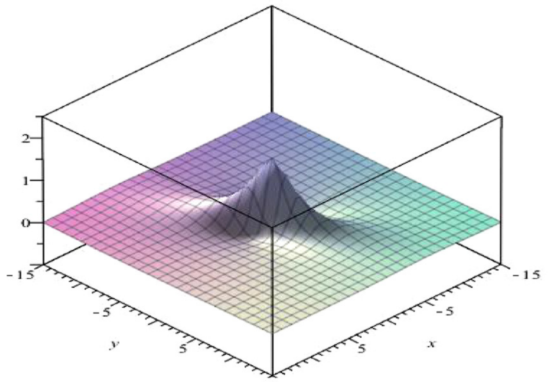
The corresponding lump-soliton solutions are given by

$$u=2(\ln(g+h))_{xx}=\frac{2(g_{xx}+h_{xx})(g+h)-2(g_x+h_x)^2}{(g+h)^2}.$$

By Corollary 1 and symbolic computation, we get two classes of solutions.

Case I: We have solution

$$a_1=-\frac{3a_6a_8^2}{2\alpha},\quad a_2=\frac{3a_4a_8^2}{2},\quad a_3=\frac{2\alpha a_4}{3a_8^2},$$
$$a_5=\frac{9a_6a_8^4}{4\alpha},\quad a_7=0,\quad a_9=\frac{a_8^3}{2},\quad a_{10}=-\frac{2\alpha}{3a_8}.$$



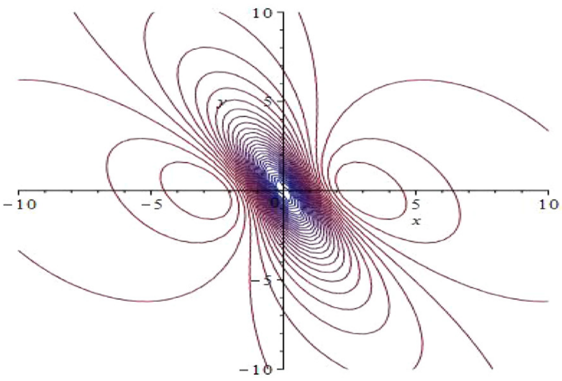


Fig. 1. Plots of a lump solution for $\alpha=-1, a_1=1, a_3=-0.8, a_4=1.25, a_6=2, c_1=c_2=0$ and $t=0$. (a) 3d plot, (b) contour.

Assume that $\alpha\neq 0$, c_1, c_2 are arbitrary real constants, $c_3>0$ and parameters a_4, a_6, a_8 are all real constants satisfying $a_8\neq 0$. Then

$$\begin{cases} g(x,y,t)=\Big(-\dfrac{3a_6a_8^2}{2\alpha}x+\dfrac{3a_4a_8^2}{2}y+\dfrac{2\alpha a_4}{3a_8^2}t+c_1\Big)^2+\Big(a_4x+\dfrac{9a_6a_8^4}{4\alpha}y+a_6t+c_2\Big)^2,\\ h(x,y,t)=c_3\exp\Big(a_8x+\dfrac{a_8^3}{2}y-\dfrac{2\alpha}{3a_8}t\Big). \end{cases} \tag{3.5}$$

We need lump condition $a_1a_5-a_2a_4=-\frac{27a_6^2a_8^6+3\alpha^2a_4^2a_8^2}{8\alpha^2}\neq 0$, which is guaranteed if a_4 and a_6 are not all zeros.

Example 2. For a group of parameters:

$$\alpha=-1, a_4=-1, a_6=2, a_8=1, c_1=c_2=0, c_3=1,$$

we have

$$f(x,y,t)=\Big(3x-\frac{3}{2}y+\frac{2}{3}t\Big)^2+\Big(x+\frac{9}{2}y-2t\Big)^2+e^{x+\frac{1}{2}y+\frac{2}{3}t},$$

and we obtain a lump-soliton solution

$$\begin{aligned} u(x,y,t)&=\frac{2(20+e^{x+\frac{1}{2}y+\frac{2}{3}t})}{f(x,y,t)}-\frac{2(20x+e^{x+\frac{1}{2}y+\frac{2}{3}t})^2}{f(x,y,t)^2}\\ &=\frac{2[(20+e^{\xi})[(3x-\frac{3}{2}y+\frac{2}{3}t)^2+(x+\frac{9}{2}y-2t)^2]+(20-40x)e^{\xi}-400x^2]}{f(x,y,t)^2}, \end{aligned}$$

where $\xi=x+\frac{1}{2}y+\frac{2}{3}t$. Since the denominator contains $e^{2\xi}$, u will decay exponentially when (x, y) is fixed as t goes to ∞.

The dynamics of the solution u shows that at first one lump and one line soliton are moving, and after interaction the line soliton seems to absorb the lump and keeps moving. The plot of surfaces of function u when $t=-3, 2, 8$ and 20 are depicted in Fig. 2.

Case II: Assume that $\alpha\neq 0$, c_1, c_2 are arbitrary real constants $c_3>0$ and parameters a_1, a_8, a_{10} are all real constants satisfying $a_1\neq 0, 3a_8^2a_{10}^2+2\alpha a_8a_{10}<0$. Since $\alpha a_8a_{10}<0$, $9a_8^2a_{10}^2+12\alpha a_8a_{10}=3(3a_8^2a_{10}^2+2\alpha a_8a_{10}+2\alpha a_8a_{10})<0$. Let $\beta:=1/\sqrt{-(9a_8^2a_{10}^2+12\alpha a_8a_{10})}$. The other parameters are determined by

$$\begin{aligned} &a_2=-\frac{a_1a_8(3a_8a_{10}+2\alpha)}{a_{10}},\quad a_3=0,\quad a_4=\beta a_1(3a_8a_{10}+2\alpha),\\ &a_5=-\frac{a_1a_8\beta(9a_8^2a_{10}^2+12\alpha a_8a_{10}+2\alpha^2)}{a_{10}},\quad a_6=-\frac{2a_1\alpha}{3\beta a_8^2(3a_8a_{10}+4\alpha)},\\ &a_7=-\frac{8\alpha^2a_1^2(3a_8a_{10}+2\alpha)}{3a_8^3a_{10}(3a_8a_{10}+4\alpha)^2},\quad a_9=-\frac{a_8^2(a_8a_{10}+\alpha)}{a_{10}}. \end{aligned}$$

It is easy to see $a_1a_5-a_2a_4=\dfrac{2\alpha^2\beta a_1^2a_8}{a_{10}}\neq 0, a_7>0$. Therefore function $g(x,y,t)=(a_1x+a_2y+c_1)^2+(a_4x+a_5y+a_6t+c_2)^2+a_7$ with coefficients a_2, a_4, a_5, a_6, a_7 defined above will generate a lump solution. The corresponding

$$u(x,y,t)=2[\ln(g(x,y,t)+c_3e^{a_8x-\frac{a_8^2(a_8a_{10}+\alpha)}{a_{10}}y+a_{10}t})]_{xx}$$

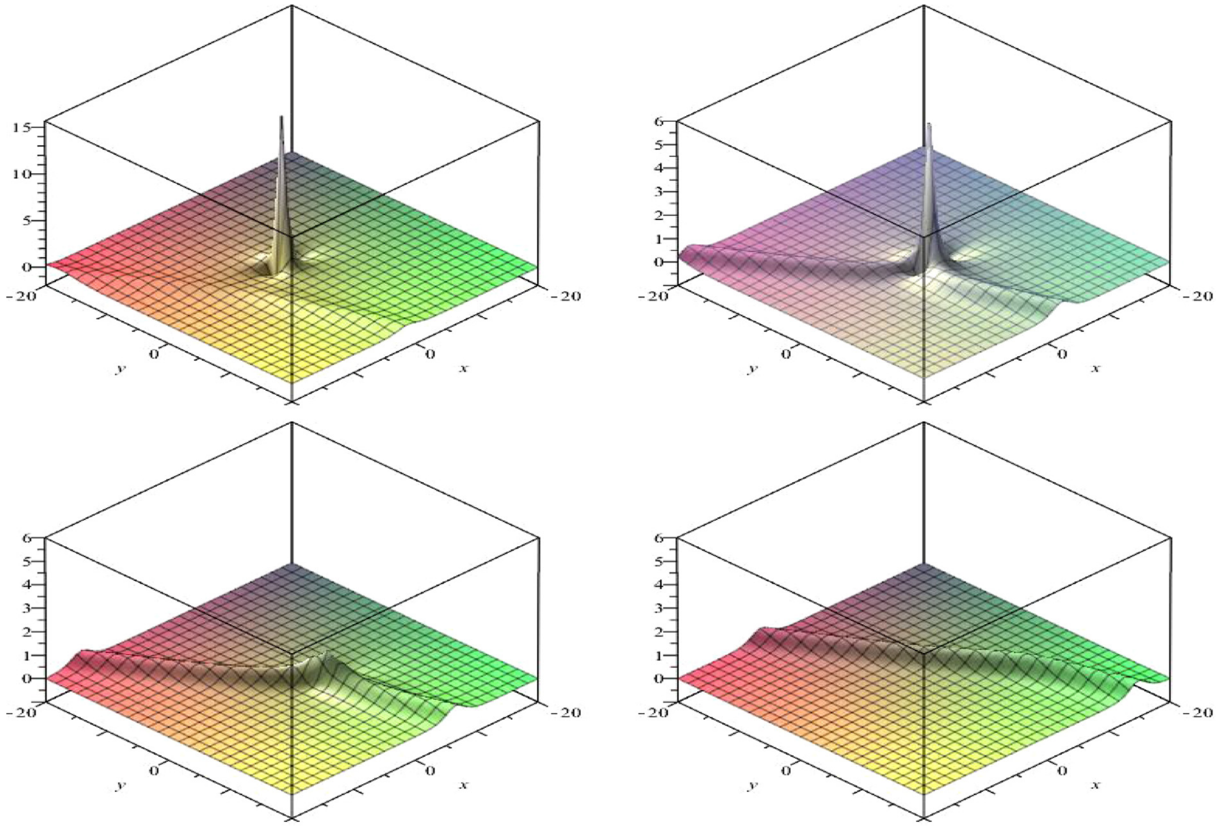


Fig. 2. Plots of a lump-soliton solution in Case I for $\alpha = -1, a_4 = -1, a_6 = 2, a_8 = 1, c_1 = c_2 = 0, c_3 = 1$. (a) t=-3, (b) t=2, (c) t=8, (d) t=20.

is a lump-soliton solution of the HSI equation.

Example 3. For a group of parameters:

$$\alpha = -1, a_1 = 1, a_8 = 1, a_{10} = 0.5, c_1 = c_2 = 0, c_3 = 1,$$

we have

$$f(x, y, t) = (x + y)^2 + \frac{1}{15}(x - 7y + 2t)^2 + \frac{32}{75} + e^{(x+y+t)/2},$$

and we obtain a lump-soliton solution

$$\begin{aligned} u(x, y, t) &= 2(\ln f(x, y, t))_{xx} \\ &= \frac{2[(\frac{32}{15} + \frac{e^{\xi}}{4})f(x, y, t) - (\frac{32x-16y+4t}{15} + \frac{e^{\xi}}{2})^2]}{f(x, y, t)^2}, \end{aligned}$$

where $\xi = (x + y + t)/2$. The dynamics of u are similar to those of Example 2. The plot of surfaces of function u when $t = -20, 0, 20$ and 40 are depicted in Fig. 3.

Note 1 The lump-soliton solutions of Examples 2 and 3 represent the result of instability of the line soliton and dynamical formation of a lump soliton. Such exact solutions were constructed for the KP-I equation in [25,26].

Note 2 Above two classes of solutions are different. In the first case, a_{10} is determined by a_8 while in the second case, a_8 and a_{10} are independent. Conversely, in the second case, $a_3 = 0$, while in the first case, $a_3 = \frac{2\alpha a_4}{3a_8^2} \neq 0$ if $a_4 \neq 0$. Therefore, neither of these two classes can contain the other as a special case.

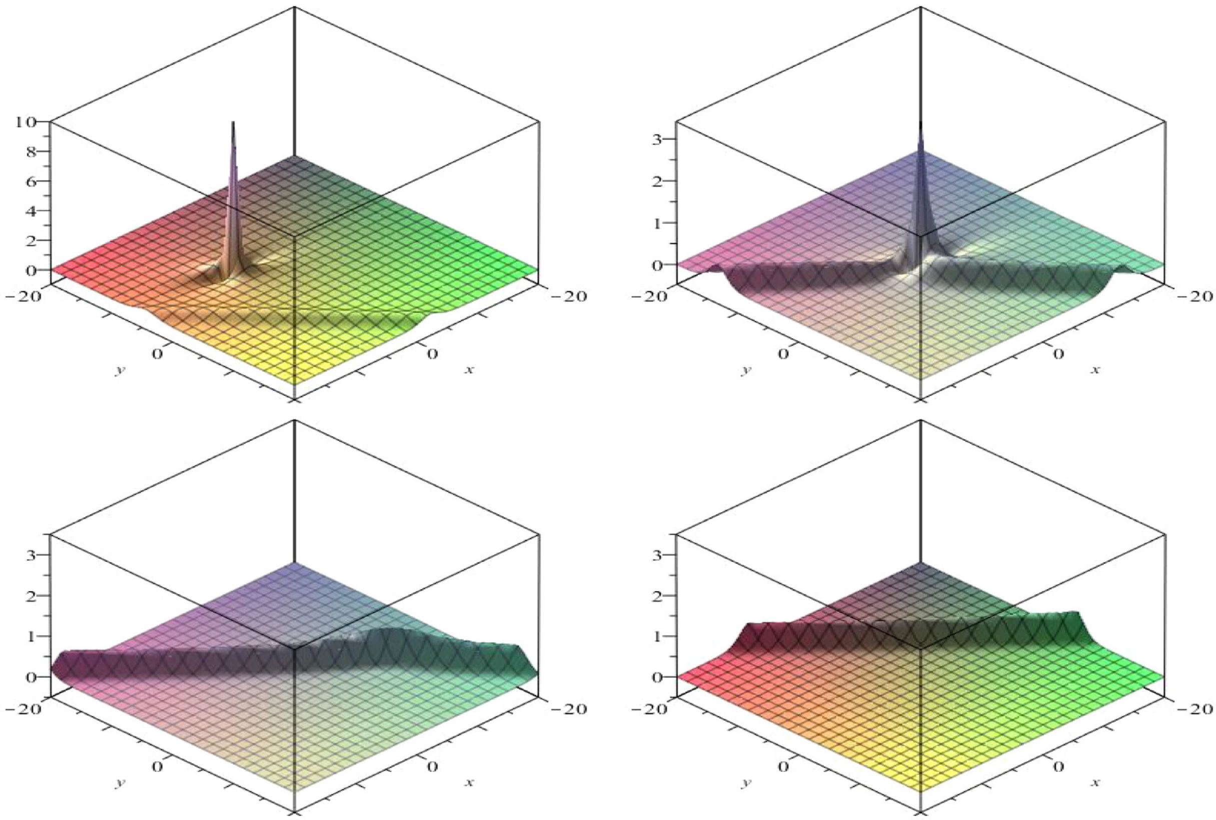


Fig. 3. Plots of a lump-soliton solution in Case II for $\alpha = -1, a_1 = 1, a_8 = 1, a_{10} = 1/2, c_1 = c_2 = 0, c_3 = 1$. (a) t=-20, (b) t=0, (c) t=20, (d) t=40.

4. Conclusion

In this paper, we have studied the lump and lump-soliton solutions of the (2+1)-dimensional Hirota–Satsuma–Ito equation generated by positive quadratic function solutions of bilinear equation (1.4). Our computations are based on Hirota bilinear equations and the symbolic computation software Maple. We know that when $\alpha = -1$ and $y = -x$, the Hirota–Satsuma–Ito equation reduces to the (1+1)-dimensional Hirota–Satsuma equation. However, we are unable to find any lump solution for the Hirota-Satsuma equation. The interaction of the lump solutions involve higher order polynomial solutions of the bilinear equation (1.4) which will be studied in our future research.

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