IL NUOVO CIMENTO Vol. 115 B, N. 12 Dicembre 2000

Algebro-geometric solutions of the (2+1)-dimensional Gardner equation

Ruguang Zhou(1)(*) and Wen-Xiu Ma(2)(**)

(1) *Department of Mathematics, Xuzhou Normal University Xuzhou 221009, P. R. China*
(2) *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong*

(ricevuto il 21 Settembre 2000; approvato il 5 Dicembre 2000)

Summary. — It is shown that any solutions of the first and second members in the (1+1)-dimensional coupled AKNS-Kaup-Newell equation hierarchy may give rise to the solutions of the (2+1)-dimensional Gardner equation. Furthermore, the coupled AKNS-Kaup-Newell equation hierarchy is reduced to solvable ordinary differential equations. The Abel-Jacobi coordinates are introduced to straighten the flows, from which the explicit algebro-geometric solution of the (2+1)-dimensional Gardner equation is obtained in terms of the Riemann theta-functions.

PACS 02.90 – Other topics in mathematical methods in physics.
PACS 46.05 – General theory of continuum mechanics of solids.

1. – Introduction

Algebro-geometric solution is a remarkable class of exact solutions to soliton equations and there are various approaches for it. As early as in the late of 1970s an approach, analogue of inverse scattering transformation, has been developed by Dubrovin, Its and Matveev *et al.*, cf., *e.g.*, [1-4], and the monograph [5]. Recently, an alternative approach, based on the nonlinearization technique of Lax pairs or the restricted flow technique [6-9], has been proposed in [10] and an essentially similar approach presented in [11]. From this approach, algebro-geometric solutions for many-soliton equations in one spatial and one temporal (*i.e.*, 1+1) dimensions have been obtained. Very recently, this approach has been generalized to the study of soliton equations in two spatial and one temporal (*i.e.*, 2+1) dimensions. The explicit solutions of several (2+1)-dimensional soliton equations such as the Kadomtsev-Petviashvili (KP) equation, the modified KP(mKP) equation and the coupled modified KP equation, have been constructed [12-14].

(*) E-mail: rgzhou@public.xz.js.cn
(**) E-mail: mawx@math.cityu.edu.hk

In this paper, we extend this method to the (2+1)-dimensional Gardner equation and obtain its algebro-geometric solutions. It is well-known that the (2+1)-dimensional Gardner equation is closely connected with the KP equation [15] and its rational solutions, including the decaying and plane lumps, solutions with functional parameters and plane solitons, have been constructed explicitly by the $\overline{\partial}$-dressing method in [16]. To construct its algebro-geometric solution, we first establish a relation between it and the (1+1)-dimensional coupled AKNS-Kaup-Newell soliton hierarchy [17, 18]. It is shown that any solutions of the first two members in the (1+1)-dimensional coupled AKNS-Kaup-Newell soliton hierarchy may give rise to a solution of (2+1)-dimensional Gardner equation. Then we reduce the solutions of the equations in the coupled AKNS-Kaup-Newell hierarchy to solving systems of ordinary differential equations. We find that, in the Abel-Jacobi coordinates, these flows are linear and thus can be integrated. Finally the explicit theta-function solutions of the (2+1)-dimensional Gardner equation are obtained through the Abel-Jacobi inversion.

2. – Relation with the coupled AKNS-Kaup-Newell hierarchy

The aim of this section is to establish a relation between the (2+1)-dimensional Gardner equation and the coupled AKNS-Kaup-Newell soliton hierarchy. Let us first recall some useful facts on the coupled AKNS-Kaup-Newell soliton hierarchy presented in [18].

The coupled AKNS-Kaup-Newell soliton hierarchy is connected with the spectral problem

$$\phi_x = U\phi, \quad U = \begin{pmatrix} \lambda & q \\ (\alpha+\beta\lambda)r & -\lambda \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{1}$$

where λ is a spectral parameter, and α and β are arbitrary constants. To deduce the soliton hierarchy, we solve the stationary zero-curvature equation $V_x = [U, V]$ first. Suppose that

$$V = \begin{pmatrix} \lambda a & b \\ (\alpha+\beta\lambda)c & -\lambda a \end{pmatrix} = \sum_{i\geq 0} \begin{pmatrix} a_i & b_i \\ (\alpha+\beta\lambda)c_i & -a_i \end{pmatrix} \lambda^{-i}, \tag{2}$$

and then the stationary zero-curvature equation becomes

$$\begin{cases} \lambda a_x = (\alpha+\beta\lambda)(qc - rb), \\ b_x = 2\lambda b - 2qa\lambda, \\ c_x = 2ra\lambda - 2\lambda c. \end{cases} \tag{3}$$

which yields

$$a_{0,x} = 0, \quad b_0 = a_0 q, \quad c_0 = a_0 r,$$

and

$$KS_j = JS_{j+1}, \quad S_j = (c_j, b_j, a_j)^T, \tag{4}$$

where

$$K=\begin{pmatrix} 0 & \partial & 0 \\ \partial & 0 & 0 \\ \alpha q & -\alpha r & 0 \end{pmatrix},\quad J=\begin{pmatrix} 0 & 2 & -2q \\ -2 & 0 & 2r \\ \beta q & -\beta r & \partial \end{pmatrix}. \tag{5}$$

Let us fix initial data

$$a_0=1,\quad b_0=q,\quad c_0=r,$$

and set

$$S_j|_{(q,r)=0}=0,$$

then all the $S_j{}'s$ can be recursively defined. In particular,

$$S_1=\begin{pmatrix} -\frac{1}{2}(r_x+\beta qr^2) \\ \frac{1}{2}(q_x-\beta q^2 r) \\ -\frac{1}{2}\beta qr \end{pmatrix},\quad S_2=\begin{pmatrix} \frac{1}{4}(r_{xx}-2\alpha qr^2+3\beta qrr_x+\frac{3}{2}\beta^2q^2r^3) \\ \frac{1}{4}(q_{xx}-2\alpha q^2r-3\beta qq_xr+\frac{3}{2}\beta^2q^3r^2) \\ -\frac{1}{2}\alpha qr+\frac{1}{4}\beta(qr_x-q_xr)+\frac{3}{8}\beta^2q^2r^2 \end{pmatrix}.$$

Choose an auxiliary problem

$$\phi_{t_n}=V^{(n)}\phi, \tag{6}$$

where

$$V^{(n)}=\sum_{j=0}^{n}\begin{pmatrix} \lambda a_j & b_j \\ (\alpha+\beta\lambda)c_j & -\lambda a_j \end{pmatrix}\lambda^{n-j}+ + \begin{pmatrix} \alpha\partial^{-1}(qc_n-rb_n) & 0 \\ 0 & -\alpha\partial^{-1}(qc_n-rb_n) \end{pmatrix}, \tag{7}$$

then the zero-curvature equation yields the coupled AKNS-Kaup-Newell system

$$(q_{t_n},r_{t_n})=X_n, \tag{8}$$

where

$$X_n=\begin{pmatrix} b_{n,x}+2\alpha q\partial^{-1}(qc_n-rb_n) \\ c_{n,x}-2\alpha q\partial^{-1}(qc_n-rb_n) \end{pmatrix}.$$

The first two nontrivial equations are as follows ($t_1=y, t_2=t$):

$$\begin{cases} q_y=\frac{1}{2}(q_{xx}-2\alpha q^2r-\beta(q^2r)_x), \\ r_y=\frac{1}{2}(-r_{xx}+2\alpha qr^2-\beta(qr^2)_x), \end{cases} \tag{9}$$

$$\begin{cases} q_t=\frac{1}{4}(q_{xxx}-6\alpha qq_xr-3\beta(qq_xr)_x+\frac{3}{2}\beta^2(q^3r^2)_x+3\alpha\beta q^3r^2), \\ r_t=\frac{1}{4}(r_{xxx}-6\alpha qrr_x+3\beta(qrr_x)_x+\frac{3}{2}\beta^2(q^2r^3)_x-3\alpha\beta q^2r^3). \end{cases} \tag{10}$$

To deduce the (1+2)-dimensional integrable systems, we impose the constraint as follows:

$$u(x,y,t) = q(x,y,t)r(x,y,t), \tag{11}$$

where q, r are the solutions of eqs. (9) and (10), then we have

$$\begin{aligned} u_y &= \frac{1}{2}(q_{xx}r - r_{xx}q - 3\beta u u_x),\\ \partial_x^{-1}u_y &= \frac{1}{2}(q_x r - q r_x) - \frac{3}{4}\beta u^2,\\ \partial^{-1}u_{yy} &= \frac{1}{4}[(qr_{xxx} + q_{xxx}r) - (q_{xx}r_x + q_x r_{xx}) + 4\beta u(qr_{xx} - rq_{xx}) +\\ &\quad +3\beta u_x(q_x r - qr_x) - 4\alpha u u_x + 9\beta^2 u^2 u_x],\\ u_t &= \frac{1}{4}[(q_{xxx}r + qr_{xxx}) + 3\beta u(qr_{xx} - q_{xx}r) +\\ &\quad +3\beta u_x(qr_x - q_x r) - 6\alpha u u_x + \frac{15}{2}\beta^2 u^2 u_x]. \end{aligned}$$

These imply

$$4u_t = \frac{1}{4}u_{xxx} - 3\alpha u u_x - \frac{3}{8}\beta^2 u^2 u_x - \frac{3}{2}\beta u_x \partial^{-1}u_y + 3\partial^{-1}u_{yy}. \tag{12}$$

It is easy to see that as $\beta = 0$ this equation reduces to the well-known KP equation and as $\alpha = 0$ it reduces to the mKP equation. It is a mixed KP and mKP equation, called (2+1)-dimensional Gardner equation, which can be transformed to the form in [15] after a simple transformation of variables.

3. – Associated ordinary differential equations

Since the algebro-geometric solutions of the KP and mKP equation have been obtained in much literature [5, 12, 13], in the rest of this paper we always assume that $\alpha\beta \neq 0$. In what follows, we shall reduce the solutions of the (2+1)-dimensional Gardner equation to solve three systems of ordinary differential equations.

Let $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$ be the basic solutions of eqs. (1) and (6). Define a matrix W of three functions f, g, h by

$$W = \frac{1}{2}(\phi\psi^T + \psi\phi^T)\sigma = \begin{pmatrix} f & g \\ h & -f \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

A direct calculation shows that

$$W_x = [U, W], \quad W_{t_n} = [V^{(n)}, W], \tag{14}$$

which implies that the function $\det W$ is a constant of motion for both x-flow and t_n flow. Equation (14) can be written as

$$f_x = qh - g(\alpha + \beta\lambda)r, \quad g_x = 2\alpha g - 2qf, \quad h_x = 2(\alpha + \beta\lambda)rf - 2\lambda h, \tag{15}$$

and

$$f_{t_n} = hV_{12}^{(n)} - gV_{21}^{(n)}, \quad g_{t_n} = 2gV_{11}^{(n)} - 2fV_{12}^{(n)}, \quad h_{t_n} = 2fV_{21}^{(n)} - 2hV_{11}^{(n)}. \tag{16}$$

To construct the algebro-geometric solution of (12), we suppose that the functions f, g, and h are finite-degree polynomials in λ:

$$f = \sum_{j=0}^{N} f_j \lambda^{N+1-j}, \quad g = \sum_{j=0}^{N} g_j \lambda^{N-j}, \quad h = \sum_{j=0}^{N} (\alpha + \beta\lambda) h_j \lambda^{N-j}. \tag{17}$$

Substituting (17) into (15) yields

$$KG_j = JG_{j+1}, \quad JG_0 = 0, \quad KG_N = 0, \quad G_j = (h_j, g_j, f_j)^T. \tag{18}$$

It is easy to see that equation $JG_0 = 0$ has the general solution

$$G_0 = \alpha_0 S_0, \tag{19}$$

where α_0 is a constant. We can determine G_j recursively by the relation (18). In fact, noticing that $\ker J = \{cS_0 | c \text{ is a constant}\}$ and acting with the operator $(J^{-1}K)^{k+1}$ upon (19), we obtain from (18) that

$$G_k = \sum_{j=0}^{k} \alpha_j S_{k-j}, \quad 0 \le k \le N, \tag{20}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k$ are integral constants. Substituting (20) into $KG_N = 0$ yields a stationary equation:

$$\alpha_0 X_N + \alpha_1 X_{N-1} + \ldots + \alpha_N X_0 = 0.$$

This means that (q, r) is a finite-band solution of the coupled AKNS-Kaup-Newell hierarchy.

Without loss of generality, let $\alpha_0 = 1$. Thus, from (19), we obtain using the recursion relation (18):

$$\begin{aligned}
&f_0 = 1,\ g_0 = q,\ h_0 = r,\\
&f_1 = -\tfrac{1}{2}\beta qr + \alpha_1,\ g_1 = \tfrac{1}{2}(q_x - \beta q^2 r) + \alpha_1 q,\ h_1 = \tfrac{1}{2}(-r_x - \beta q r^2) + \alpha_1 r,\\
&f_2 = -\tfrac{1}{2}\alpha qr + \tfrac{1}{4}\beta(q r_x - q_x r) + \tfrac{3}{8}\beta^2 q^2 r^2 - \tfrac{1}{2}\alpha_1 \beta qr + \alpha_2,\\
&g_2 = \tfrac{1}{4}(q_{xx} - 2\alpha q^2 r - 3\beta q q_x r + \tfrac{3}{2}\beta^2 q^3 r^2) + \tfrac{1}{2}\alpha_1(q_x - \beta q^2 r) + \alpha_2 q,\\
&h_2 = \tfrac{1}{4}(r_{xx} - 2\alpha q r^2 + 3\beta q r r_x + \tfrac{3}{2}\beta^2 q^2 r^3) + \tfrac{1}{2}\alpha_1(-r_x - \beta q r^2) + \alpha_2 r,
\end{aligned} \tag{21}$$

where α_1, α_2 are integral constants.

Taking into account (17), we set

$$g = q \prod_{j=1}^{N} (\lambda - \mu_j), \quad h = (\alpha + \beta\lambda) r \prod_{j=1}^{N} (\lambda - \nu_j). \tag{22}$$

Thus we have

$$g_1 = -q \sum_{i,j=1}^{N} \mu_j, \quad g_2 = q \sum_{\substack{i,j=1 \\ i<j}}^{N} \mu_i\mu_j, \quad h_1 = -r \sum_{j=1}^{N} \nu_j, \quad h_2 = r \sum_{\substack{i,j=1 \\ i<j}}^{N} \nu_i\nu_j. \tag{23}$$

Only for convenience, let us denote by $\{\sigma_k\}_{k=1}^N$ and $\{\tilde{\sigma}_k\}_{k=1}^N$ the k-th order symmetric products of the $\mu_1, \ldots, \mu_N$ and $\nu_1, \ldots, \nu_N$, respectively. In particular,

$$\sigma_1 = \sum_{j=1}^{N} \mu_j, \quad \sigma_2 = \sum_{\substack{i,j=1 \\ i<j}}^{N} \mu_i\mu_j, \quad \tilde{\sigma}_1 = \sum_{j=1}^{N} \nu_j, \quad \tilde{\sigma}_2 = \sum_{\substack{i,j=1 \\ i<j}}^{N} \nu_i\nu_j.$$

Thus from (21) and (23), after a simple calculation, we obtain

$$\begin{aligned} \partial \ln q - \beta qr &= -2(\sigma_1 + \alpha_1), \\ \partial \ln r + \beta qr &= 2(\tilde{\sigma}_1 + \alpha_1), \end{aligned} \tag{24}$$

and

$$\begin{aligned} \partial_y \ln q + \tfrac{1}{2}\beta qr(\partial \ln r - \partial \ln q + \tfrac{3}{2}\beta qr) &= 2(\sigma_2 + \alpha_1\sigma_1 + \alpha_1^2 - \alpha_2), \\ \partial_y \ln r - \tfrac{1}{2}\beta qr(\partial \ln r - \partial \ln q + \tfrac{3}{2}\beta qr) &= -2(\tilde{\sigma}_2 + \alpha_1\tilde{\sigma}_1 + \alpha_1^2 - \alpha_2), \end{aligned} \tag{25}$$

consequently

$$\begin{aligned} &\partial \ln qr = 2(-\sigma_1 + \tilde{\sigma}_1), \\ &\partial_y \ln qr = 2(\sigma_2 - \tilde{\sigma}_2) + 2\alpha_1(\sigma_1 - \tilde{\sigma}_1). \end{aligned} \tag{26}$$

On the other hand, from (9),(24) and the first expression of (26), we get

$$\begin{aligned} \partial_y \ln qr &= \frac{1}{2}\left[\partial^2(\ln q - \ln r) + (\partial \ln q)^2 - (\partial \ln r)^2 - 3\beta qr\partial \ln qr\right] \\ &= 2(\sigma_1^2 - \tilde{\sigma}_1^2) - \partial(\sigma_1 + \tilde{\sigma}_1) + (4\alpha_1 - \beta qr)(\sigma_1 - \tilde{\sigma}_1). \end{aligned} \tag{27}$$

This, together with the second expression of (26), yields

$$u = qr = \frac{1}{\beta}\left[2\alpha_1 + \sigma_1 + \tilde{\sigma}_1 + \frac{\sum_{j=1}^N \mu_j^2 - \sum_{j=1}^N \nu_j^2 - \partial(\sigma_1 + \tilde{\sigma}_1)}{\sigma_1 - \tilde{\sigma}_1}\right], \tag{28}$$

here we have used the equality

$$2\sigma_2 = \sigma_1^2 - \sum_{j=1}^{N} \mu_j^2, \quad 2\tilde{\sigma}_2 = \tilde{\sigma}_1^2 - \sum_{j=1}^{N} \nu_j^2.$$

As mentioned above, the function $\det W$ is a constant independent of x and t_n. From (17), we have

$$-\det W = f^2 + gh = \prod_{j=1}^{2N+2} (\lambda - \lambda_j) = R(\lambda). \tag{29}$$

Substituting (17) into (29) and comparing the coefficients of λ^{2N+1} and λ^{2N} yield

$$
\begin{aligned}
&2f_0f_1+\beta g_0h_0=-\sum_{j=1}^{2N+2}\lambda_j,\\
&2f_0f_2+f_1^2+\beta h_0g_1+(\alpha h_0+\beta h_1)g_0=\sum_{i<j}^{2N+2}\lambda_i\lambda_j,
\end{aligned}
\tag{30}
$$

which together with (21) lead to

$$
\alpha_1=-\frac{1}{2}\sum_{j=1}^{2N+2}\lambda_j,\quad \alpha_2=\frac{1}{2}\sum_{i<j}^{2N+2}\lambda_i\lambda_j-\frac{1}{8}(\sum_{j=1}^{2N+2}\lambda_j)^2.
\tag{31}
$$

In addition, from (29) we arrive at

$$
f|_{\lambda=\mu_k}=\sqrt{R(\mu_k)},\quad f|_{\lambda=\nu_k}=\sqrt{R(\nu_k)}.
\tag{32}
$$

From (22), we get

$$
\begin{aligned}
&g_x|_{\lambda=\mu_k}=-q\mu_{k,x}\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i),\\
&h_x|_{\lambda=\nu_k}=-(\alpha+\beta\nu_k)r\nu_{k,x}\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i).
\end{aligned}
\tag{33}
$$

On the other hand, from (15) it follws that

$$
g_x|_{\lambda=\mu_k}=-2qf|_{\lambda=\mu_k}=-2q\sqrt{R(\mu_k)},
\tag{34}
$$

$$
h_x|_{\lambda=\nu_k}=2(\alpha+\beta\nu_k)rf|_{\lambda=\nu_k}=2(\alpha+\beta\nu_k)r\sqrt{R(\nu_k)}.
\tag{35}
$$

Therefore,

$$
\begin{cases}
\mu_{k,x}=\dfrac{2\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)},\\
\nu_{k,x}=-\dfrac{2\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)}.
\end{cases}
\quad 1\leq k\leq N,
\tag{36}
$$

From (7) and (22), we have

$$
V_{12}^{(1)}|_{\lambda=\mu_k}=q(\mu_k-\sigma_1-\alpha_1),
\tag{37}
$$

$$
V_{12}^{(2)}|_{\lambda=\mu_k}=q[\mu_k^2-\mu_k\sigma_1+\sigma_2+\alpha_1(\sigma_1-\mu_k)+\alpha_1^2-\alpha_2],
\tag{38}
$$

$$
V_{21}^{(1)}|_{\lambda=\nu_k}=(\alpha+\beta\nu_k)r(\nu_k-\tilde{\sigma}_1-\alpha_1),
\tag{39}
$$

$$
V_{21}^{(2)}|_{\lambda=\nu_k}=(\alpha+\beta\nu_k)r[\nu_k^2-\nu_k\tilde{\sigma}_1+\tilde{\sigma}_2+\alpha_1(\tilde{\sigma}_1-\nu_k)+\alpha_1^2-\alpha_2].
\tag{40}
$$

In a similar way to the calculation of (36), we get

$$
\begin{cases} \mu_{k,t_n} = \dfrac{2\sqrt{R(\mu_k)}V_{12}^{(n)}(\mu_k)}{q\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)}, \\ \nu_{k,t_n} = -\dfrac{2\sqrt{R(\nu_k)}V_{21}^{(n)}(\nu_k)}{(\alpha+\beta\nu_k)r\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)}. \end{cases} \quad 1 \le k \le N, \tag{41}
$$

Therefore, if the parameters $\lambda_1, \dots, \lambda_{2N+2}$ are given, and let $\mu(x,t_n)$ and $\nu(x,t_n)$ be distinct solutions of ordinary differential equation (36) and (41), then (q,r) determined by (24–26) is a solution of eqs. (9) and (10). Consequently, $u = qr$ is a solution of the (2+1)-dimensional Gardner equation (12).

4. – Algebro-geometric solution

We coordinatize a point on the hyperelliptic Riemann surface Γ:

$$
\Gamma: \quad \xi^2 = R(\lambda), \; R(\lambda) = \prod_{j=1}^{2N+2}(\lambda - \lambda_j),
$$

by the ordered pair $(\lambda, \pm\xi)$. For the same λ, there are two points on different sheets of Γ:

$$
(\lambda, \sqrt{R(\lambda)}\ ), \quad (\lambda, -\sqrt{R(\lambda)}\ ).
$$

Since $R(\lambda)$ is a polynomial of order $2N+2$ in terms of λ, ∞ is not a branch point of Γ. Thus there are two infinite points: ∞_1, ∞_2 on Γ. In the local coordinate $z = \lambda^{-1}$, they are expressed as

$$
\infty_1 = (0,1), \quad \infty_2 = (0,-1),
$$

respectively. On Γ we fix a set of regular cycle paths: $a_1, a_2, \dots, a_N; b_1, b_2, \dots, b_N$, which are independent and have the intersection numbers as follows:

$$
a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{i,j}, \quad i,j = 1, \dots, N. \tag{42}
$$

It is well known that

$$
\tilde{\omega}_l = \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \quad 1 \le l \le N,
$$

are N linearly independent homomorphic differentials on Γ.

Let

$$
A_{ij} = \int_{a_j} \tilde{\omega}_i, \quad B_{ij} = \int_{b_j} \tilde{\omega}_i, \quad 1 \le i,j \le N.
$$

Then matrix $A = (A_{ij})$ is invertible. We define the matrices C and τ by

$$
C = A^{-1}, \quad \tau = A^{-1}B.
$$

Let us normalize $\tilde{\omega}_j$ into ω_j:

$$\omega_j = \sum_{l=0}^{N-1} C_{jl}\tilde{\omega}_l, \quad j = 1, \dots, N, \tag{43}$$

then

$$\int_{a_i} \omega_j = \delta_{ji}, \quad \int_{b_i} \omega_j = \tau_{ji}. \tag{44}$$

For a fixed point p_0, the Abel-Jacobi coordinates are defined as $(1 \le j \le N)$:

$$\begin{aligned} \rho_j^{(1)}(x,y,t) &= \sum_{k=1}^{N} \int_{p_0}^{\mu_k(x,y,t)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} \int_{p_0}^{\mu_k} C_{jl} \frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}}, \\ \rho_j^{(2)}(x,y,t) &= \sum_{k=1}^{N} \int_{p_0}^{\nu_k(x,y,t)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} \int_{p_0}^{\nu_k} C_{jl} \frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}}. \end{aligned} \tag{45}$$

Therefore

$$\partial\rho_j^{(1)} = \sum_{l=1}^{N}\sum_{k=1}^{N} C_{jl} \frac{\mu^{l-1}\mu_{k,x}}{\sqrt{R(\mu)}} = 2\sum_{l=1}^{N}\sum_{k=1}^{N} C_{jl} \frac{\mu^{l-1}\mu_{k,x}}{\prod_{i=1,i\neq k}^{N}(\mu_k - \mu_i)}, \quad j = 1, \dots, N.$$

Noting the following relations [19]:

$$I_s := \sum_{k=1}^{N} \frac{\mu_k^s}{\prod_{i\neq k}^{N}(\mu_k - \mu_i)} = \delta_{s,N-1}, \quad 1 \le j \le N-1, \tag{46}$$

and

$$I_N = \sigma_1 I_{N-1}, \quad I_{N+1} = \sigma_1 I_N - \sigma_2 I_{N-1}, \tag{47}$$

we have

$$\partial\rho_j^{(1)} = \Omega_j^{(0)}, \quad 1 \le j \le N, \tag{48}$$

where

$$\Omega_j^{(0)} = 2C_{jN}.$$

In the same way, we have

$$\partial_y\rho_j^{(1)} = \Omega_j^{(1)}, \quad \partial_t\rho_j^{(1)} = \Omega_j^{(2)}, \quad 1 \le j \le N, \tag{49}$$

and

$$\partial\rho_j^{(2)} = -\Omega_j^{(0)}, \quad \partial_y\rho_j^{(2)} = -\Omega_j^{(1)}, \quad \partial_t\rho_j^{(2)} = -\Omega_j^{(2)}, \quad 1 \le j \le N, \tag{50}$$

where

$$\Omega_j^{(1)} = 2(C_{j,N-1} - \alpha_1 C_{jN}), \quad \Omega_j^{(2)} = 2[C_{j,N-2} - \alpha_1 C_{j,N-1} + (\alpha_1^2 - \alpha_2)C_{jN}].$$

These relations imply that

$$\begin{aligned}\rho_j^{(1)} &= \Omega_j^{(0)}x + \Omega_j^{(1)}y + \Omega_j^{(2)}t + \gamma_j^{(1)},\ \rho_j^{(2)} \\ &= -\Omega_j^{(0)}x - \Omega_j^{(1)}y - \Omega_j^{(2)}t + \gamma_j^{(2)}, \qquad 1 \le j \le N,\end{aligned} \tag{51}$$

where $\gamma_j^{(1)}$ and $\gamma_j^{(2)}$ are some constants defined as

$$\gamma_j^{(1)} = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(0,0,0)} \omega_j, \qquad \gamma_j^{(2)} = \sum_{k=1}^{N} \int_{p_0}^{\nu_k(0,0,0)} \omega_j, \qquad 1 \le j \le N.$$

Let T be the lattice generated by $2N$ vectors $\{\delta_i, \tau_j\}$. Thus we can obtain the Jacobi variety $J(T) = C^N/T$. An Abel map $\mathcal{A}$ is defined by

$$\begin{aligned}&\mathcal{A} : \mathrm{Div}(\Gamma) \to J(T), \\ &\mathcal{A}(p) = (\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_N),\end{aligned}$$

where p_0 is a fixed point of Riemann surface Γ and $(c_0, \mu(c_0))$ is not a branch point; p is an arbitrary point. Moreover $\mathcal{A}(p)$ can be linearly extended into divisor:

$$\mathcal{A}(\sum n_k p_k) = \sum n_k \mathcal{A}(p_k).$$

Riemann θ-functions on Γ are defined as follows [20]: for any $\zeta \in C^N$

$$\begin{aligned}\theta(\zeta) &= \sum_{z \in \mathbf{Z}^N} \exp\{\pi i(Bz, z) + 2\pi i(\zeta, z)\}, \\ (Bz, z) &= \sum_{i,j=1}^{N} B_{ij} z_i z_j, \quad (\zeta, z) = \sum_{i=1}^{N} z_i \zeta_i.\end{aligned} \tag{52}$$

Consider the divisors $\sum_{j=1}^{N} \zeta(\mu_j)$ and $\sum_{j=1}^{N} \zeta(\nu_j)$, from (52) we have

$$A(\sum_{j=1}^{N} \zeta(\mu_j)) = \rho^{(1)}, \quad A(\sum_{j=1}^{N} \zeta(\nu_j)) = \rho^{(2)}. \tag{53}$$

According to the Riemann theorem [20], there exists a constant vector (Riemann constant) $M \in C^N$ determined by Γ itself such that

i) $f^{(1)}(\lambda) = \theta(A(\zeta(\lambda)) - \rho^{(1)} - M)$ has exactly N zeros at $\lambda = \mu_1, \ldots, \mu_N$; and

ii) $f^{(2)}(\lambda) = \theta(A(\zeta(\lambda)) - \rho^{(2)} - M)$ has exactly N zeros at $\lambda = \nu_1, \ldots, \nu_N$.

To make the function single valued, the surface Γ is cut along all a_k, b_k to form a simple connected region, whose boundary is denoted by γ. From [10], we know that the integrals

$$\frac{1}{2\pi\sqrt{-1}}\int_{\gamma}\lambda^k \mathrm{d}\ln f^{(m)}(\lambda) = I_k(\Gamma)$$

are constants independent of $\rho^{(m)}$ with

$$I_k(\Gamma) = \sum_{j=1}^{N}\int_{a_j}\lambda^k\omega_j.$$

Moreover, we have the residue formulas

$$\sum_{l=1}^{N}{\mu_l}^k = I_k(\Gamma) - \sum_{j=1}^{2}\mathrm{Res}_{\lambda=\infty_j}\lambda^k \mathrm{d}\ln f^{(1)}, \tag{54}$$

$$\sum_{l=1}^{N}{\nu_l}^k = I_k(\Gamma) - \sum_{j=1}^{2}\mathrm{Res}_{\lambda=\infty_j}\lambda^k \mathrm{d}\ln f^{(2)}. \tag{55}$$

To get the explicit expression of the solution, we have to compute the residue of the function $\lambda^k \mathrm{d}\ln f(\lambda)$ at points: ∞_1 and ∞_2. In a way similar to calculations in [10], we get

$$\begin{aligned} \mathrm{Res}_{\lambda=\infty_s}\lambda\,\mathrm{d}\ln f^{(m)}(\lambda) &= (-1)^{s+m}\frac{1}{2}\partial\ln\theta_s^{(m)}, \\ \mathrm{Res}_{\lambda=\infty_s}\lambda^2 \mathrm{d}\ln f^{(m)}(\lambda) &= (-1)^{s+m}\frac{1}{2}\partial_y\ln\theta_s^{(m)} + \frac{1}{4}\partial^2\ln\theta_s^{(m)}, \\ &\qquad 1\le m\le 2,\ 1\le s\le 2, \end{aligned} \tag{56}$$

where

$$\theta_s^{(1)} = \theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon^{(s)}), \quad \theta_s^{(2)} = \theta(-\Omega^{(0)}x - \Omega^{(1)}y - \Omega^{(2)}t + \Upsilon^{(s)}),$$

with

$$\begin{aligned} &\Omega^{(j)} = (\Omega_1^{(j)},\dots,\Omega_1^{(j)})^T, \quad \Upsilon^{(j)} = (\Upsilon_1^{(j)},\dots,\Upsilon_1^{(j)})^T, \quad \Lambda^{(j)} = (\Lambda_1^{(j)},\dots,\Lambda_1^{(j)})^T, \\ &\Upsilon_j^{(s)} = \gamma_j^{(1)} + M_j + \textstyle\int_{\infty_s}^{p_0}\omega_j, \quad \Lambda_j^{(s)} = \gamma_j^{(2)} + M_j + \displaystyle\int_{\infty_s}^{p_0}\omega_j, \quad 0\le i\le 2,\ 1\le j\le N. \end{aligned} \tag{57}$$

Thus from eq. (54) and (55) we arrive at

$$\sum_{l=1}^{N}\mu_l = I_1(\Gamma) + \frac{1}{2}\partial\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}, \tag{58}$$

$$\sum_{l=1}^{N}\nu_l = I_1(\Gamma) + \frac{1}{2}\partial\ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}, \tag{59}$$

$$\sum_{l=1}^{N} \mu_l^2 = I_2(\Gamma) + \frac{1}{2}\partial_y \ln \frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \frac{1}{4}\partial^2 \ln \theta_1^{(1)}\theta_2^{(1)}, \tag{60}$$

$$\sum_{l=1}^{N} \nu_l^2 = I_2(\Gamma) + \frac{1}{2}\partial_y \ln \frac{\theta_1^{(2)}}{\theta_2^{(2)}} - \frac{1}{4}\partial^2 \ln \theta_1^{(2)}\theta_2^{(2)}. \tag{61}$$

Substituting (58)–(61) into (28) we obtain algebro-geometric solutions of the (2+1)-dimensional Gardner equation (12):

$$u(x,y,t) = \frac{1}{2\beta}\left[4\alpha_1 + 4I_1(\Gamma) + \partial \ln \frac{\theta_2^{(1)}\theta_1^{(2)}}{\theta_1^{(1)}\theta_2^{(2)}} + \frac{2\partial_y \ln \frac{\theta_2^{(1)}\theta_2^{(2)}}{\theta_1^{(1)}\theta_1^{(2)}} + \partial^2 \ln \frac{\theta_1^{(1)}(\theta_2^{(2)})^3}{\theta_1^{(2)}(\theta_2^{(1)})^3}}{2\partial \ln \frac{\theta_2^{(1)}\theta_2^{(2)}}{\theta_1^{(1)}\theta_1^{(2)}}}\right]. \tag{62}$$

5. – Conclusions

In this paper, we have constructed the algebro-geometric solutions of the (2+1)-dimensional Gardner equation. The essential step is to relate the (2+1)-dimensional Gardner equation to the coupled AKNS-Kaup-Newell soliton hierarchy. To solve the equations in the coupled AKNS-Kaup-Newell hierarchy, we reduce it to solvable systems of ordinary differential equations. The solutions obtained in this way are quasi-periodic.

* * *

This work was in part supported by the City University of Hong Kong (Grant No. 7001041) and the Research Grants Council of Hong Kong (Grant No. 9040466). One of the authors (Zhou) also acknowledges warm hospitality at City University of Hong Kong and the support from the National Science Foundation of China (Project 19801031), the Special Fund of Major State Basic Research Project (No. G2000077301), the Special grant of Excellent PhD Thesis of China (No. 200013).

REFERENCES

[1] Dubrovin B. A., *Funct. Anal. Appl.*, **9** (1975) 41.
[2] Its A. and Matveev V., *Funct. Anal. Appl.*, **9** (1975) 69.
[3] Its A. and Matveev V., *Theor. Math. Phys.*, **23** (1975) 51.
[4] Krichever I. and Novikov S., *Russ. Math. Surv.*, **31** (1980) 55.
[5] Belokolos E. D., Bobenko A. I., Enol'skii V. Z., Its A. R. and Matveev V. B., *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Springer, Berlin) 1994.
[6] Cao C. W., *Sci. China, Ser A*, **33** (1990) 4117.
[7] Cao C. W. and Geng X. G., *Classical integrable systems generated through nonlinearization of eigenvalue problems*, in *Proceedings of the Conference on Nonlinear Physics, Shanghai 1989, Res. Rep. Phys.* (Springer Berlin) 1990 pp. 66-78.
[8] Antonowicz M. and Rauch-Wojciechowski S.,, *Phys. Lett. A*, **147** (1990) 455.
[9] Ragnisco O. and Rauch-Wojciechowski S.,, *Inverse Problems*, **8** (1992) 245.
[10] Zhou R. G., *J. Math. Phys.*, **38** (1997) 2535.
[11] Geng X. G. and Wu Y. T., *J. Math. Phys.*, **40** (1999) 2971.
[12] Cao C. W., Wu Y. T. and Geng X. G., *J. Math. Phys.*, **40** (1999) 3948.
[13] Geng X. G., Wu Y. T. and Cao C. W., *J. Phys. A: Math. Gen.*, **32** (1999) 3733.
[14] Geng X. G. and Cao C. W., *Phys. Lett. A*, **261** (1999) 289.

[15] KONOPELCHENKO B. G., *Introduction to Multidimensional Integrable Equations. The Inverse Spectral Transform in* $2+1$ *Dimensions* (Plenum Press, New York) 1992.
[16] KONOPELCHENKO B. G., *Inverse Problems*, **7** (1991) 739.
[17] ZHANG B. C., *Ann. Diff. Eqns.*, **13** (1997) 408.
[18] MA W. X. and ZHOU R. G., *J. Math. Phys.*, **40** (1999) 4419.
[19] NEWELL A. C., *Solitons in Mathematics and Physics* (SIAM, Philadelphia, PA) 1985.
[20] GRIFFITHS P. and HARRIS J., *Principles of Algebraic Geometry* (Wiley, New York) 1994.