Complexiton solutions to soliton equations by the Hirota method

Yuan Zhou, and Wen-Xiu Ma

Citation: Journal of Mathematical Physics **58**, 101511 (2017);
View online: https://doi.org/10.1063/1.4996358
View Table of Contents: http://aip.scitation.org/toc/jmp/58/10
Published by the American Institute of Physics

JOURNAL OF MATHEMATICAL PHYSICS **58**, 101511 (2017)

Complexiton solutions to soliton equations by the Hirota method

Yuan Zhou[a)] and Wen-Xiu Ma[b)]
Department of Mathematics and Statistics, University of South Florida, Tampa, Florida 33620-5700, USA

(Received 16 July 2017; accepted 6 October 2017; published online 19 October 2017)

We apply the Hirota direct method to construct complexiton solutions (complexitons). The key is to use Hirota bilinear forms. We prove that taking pairs of conjugate wave variables in the $2N$-soliton solutions generates N-complexion solutions. The general theory is used to construct multi-complexion solutions to the Korteweg–de Vries equation. *Published by AIP Publishing.* https://doi.org/10.1063/1.4996358

I. INTRODUCTION

The aim of this paper is to apply the Hirota direct method[1–3] to find complexiton solutions of partial differential equations (PDEs). The Hirota method is a powerful tool invented by Hirota in 1971 in finding N-soliton solutions (solitons) for PDEs, which does not involve the use of complicated mathematical calculations. The Hirota method has also been successfully extended to solve larger classes of nonlinear PDEs.[4–6]

It is well known that the Korteweg–de Vries (KdV) equation[7] is a mathematical model of shallow water waves on surfaces. The Kadomtsev–Petviashvili (KP) equation[8] is a generalization to two spatial dimensions, x and y, of the KdV equation. Both the KdV equation and the KP equation are notable as the prototypical examples of an exactly solvable model.

It is an important research area to find exact solutions for nonlinear partial differential equations. A lot of tools have been developed, such as the inverse scattering transform, the Hirota method, and the Wronskian technique. The concept of complexiton solutions, which are a combination of exponential waves and trigonometric waves, was first proposed for the KdV equation in Ref. 9 and then for the KdV equation with self-consistent sources in Ref. 10. Various kinds of solutions of the KdV equation, including complexion solutions, were further investigated in Ref. 11. The Wronskian technique[11,12] and the Riccati equation method[13] are two popular methods in searching for exact solutions especially for complexitons. Recently, the authors[14] established linear superposition principles to solitons and complexitons which work for a large class of nonlinear PDEs. In this paper, we use the Hirota direct method to seek complexitons for Hirota bilinear equations satisfying the Hirota condition.

II. THE HIROTA METHOD

In this section, we introduce the Hirota method[1] to obtain a one-soliton solution of the KdV equation. In the following discussion, we will consider the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

We introduce the Hirota derivatives first. Suppose that $M \geq 1$ is an integer and variables $x \in \mathbb{R}^M$. Let $D = (D_1, \ldots, D_M)$, where D_j is the Hirota bilinear derivative with respect to x_j, $1 \leq j \leq M$, which is defined as follows.

a)Electronic mail: zhouy@mail.usf.edu.
b)Electronic mail: mawx@cas.usf.edu. Also at College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, China; College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China; and International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa.

Let f, g denote differentiable functions in $\mathbb{R}^M$ and $x = (x_1, \cdots, x_M)^T, x' = (x'_1, \cdots, x'_M)^T$. Then

$$D_{j_1} D_{j_2} \cdots D_{j_k} f \cdot g = \prod_{l=1}^{k} (\partial_{x_{j_l}} - \partial_{x'_{j_l}}) f(x) g(x')|_{x'=x}, \tag{2}$$

where $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, M\}$ which need not be distinct. Here are some simple examples,

$$D_x f \cdot g = f_x g - f g_x,$$
$$D_{xx} f \cdot g = f_{xx} g - 2 f_x g_x + f g_{xx},$$
$$D_x D_t f \cdot g = D_t D_x f \cdot g = f_{tx} g - f_t g_x - f_x g_t + f g_{tx}.$$

One of the important properties of Hirota derivative is

$$D_{j_1} D_{j_2} \cdots D_{j_k} f \cdot g = (-1)^k D_{j_1} D_{j_2} \cdots D_{j_k} g \cdot f, \tag{3}$$

where $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, M\}$. As a consequence, if k is an odd number, then

$$D_{j_1} D_{j_2} \cdots D_{j_k} f \cdot f = 0. \tag{4}$$

In order to solve the KdV equation, we apply the transformation $u = 2(\ln f)_{xx}$ from Eq. (1) to get the bilinear KdV equation

$$(D_x D_t + D_x^4) f \cdot f = 0, \tag{5}$$

and we define the polynomial corresponding to the bilinear KdV equation by

$$P_1(x, t) := x^4 + xt. \tag{6}$$

Now we consider travelling wave solutions to the KdV equation (5). For any positive integer j, define the function

$$\eta_j(x, y, t) := k_j x + w_j t + \eta_j^0, \quad k_j, w_j, \eta_j^0 \in \mathbb{R}. \tag{7}$$

By the direct method, we have a one-soliton solution of (1),

$$u = 2(\ln f)_{xx} = \frac{k_1^2}{2} \operatorname{sech}^2 \frac{\eta_1}{2}, \tag{8}$$

where η_1 satisfy the dispersion relation $k_1^3 + w_1 = 0$.

III. GENERAL BILINEAR EQUATIONS

In this section, we apply the Hirota method to general bilinear equations.

Suppose P is a real polynomial of M variables with the properties that $P(0) = 0$ and $P(-x) = P(x)$. Functions f and g are differentiable on $\mathbb{R}^M$, and $\partial := (\partial_1, \ldots, \partial_M)^T$. We consider the following bilinear equation:

$$P(D) f \cdot f = 0. \tag{9}$$

In order to find travelling wave solutions of (9), let complex wave variable

$$\eta_j = \beta_0 + \sum_{k=1}^{M} \beta_k x_k, \tag{10}$$

where $\beta_k \in \mathbb{C}, k = 0, \ldots, M$.

Applying the Hirota method, we consider the expansion

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots. \tag{11}$$

For one-soliton solution, suppose $f_1 = \exp(\eta)$. Substituting the above expansion into (9) and collecting terms of each order of ε, we have

$$\varepsilon : P(D)\{f_1 \cdot 1 + 1 \cdot f_1\} = 0,$$
$$\varepsilon^2 : P(D)\{f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2\} = 0,$$
$$\varepsilon^3 : P(D)\{f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3\} = 0,$$
$$\cdots.$$

From the coefficient of ε, we get

$$P(\partial)f_1 = P(\beta_1, \cdots, \beta_M)f_1 = 0. \tag{12}$$

Therefore, a complex function $f = 1 + \exp(\eta)$ is a solution to (9) if and only if the following nonlinear dispersion relation holds:

$$P(\beta_1, \cdots, \beta_M) = 0. \tag{13}$$

Since P and x are real, we have (the bar denoting complex conjugation)

$$\overline{P(\beta_1, \cdots, \beta_M)} = P(\bar{\beta}_1, \cdots, \bar{\beta}_M).$$

Therefore, the functions η and $\bar{\eta}$ satisfy the same dispersion relation.

By the direct method, we have if $\eta_j = \beta_{j0} + \sum_{k=1}^{M} \beta_{jk}x_k, j = 1, 2,$ satisfy the dispersion relation (13) and $P(\beta_{21} + \beta_{11}, \cdots, \beta_{2M} + \beta_{1M}) \neq 0$, then the (complex) function

$$f = 1 + \exp(\eta_1) + \exp(\eta_2) + a_{12}\exp(\eta_1 + \eta_2), \tag{14}$$

where $a_{12} = -\frac{P(\beta_{21} - \beta_{11}, \cdots, \beta_{2M} - \beta_{1M})}{P(\beta_{21} + \beta_{11}, \cdots, \beta_{2M} + \beta_{1M})}$, is a solution of (9). Taking $\eta_2 = \bar{\eta}_1$, we get

$$\begin{aligned} f &= 1 + \exp(\eta) + \exp(\bar{\eta}) + a_{12}\exp(\eta + \bar{\eta}) \\ &= 1 + 2\exp(\mathrm{Re}(\eta))\cos(\mathrm{Im}(\eta)) + a_{12}\exp(2\mathrm{Re}(\eta)) \in \mathbb{R}, \end{aligned} \tag{15}$$

since P is a real and even polynomial,

$$a_{12} = -\frac{P(2i\mathrm{Im}(\beta_1), \cdots, 2i\mathrm{Im}(\beta_M))}{P(2\mathrm{Re}(\beta_1), \cdots, 2\mathrm{Re}(\beta_M))} \in \mathbb{R}. \tag{16}$$

Now we consider $N \geq 3$, according to Ref. 1, N-soliton solutions can be written as

$$\sum \exp\left[\sum_{j=1}^{N} \mu_j\eta_j + \sum_{j<k} A_{jk}\mu_j\mu_k\right], \tag{17}$$

where $\mu_j = 0$ or 1 for $j = 1, 2, \cdots, N$ and $e^{A_{jk}} := a_{jk}$ denoted by

$$a_{jk} := -\frac{P(\beta_{k1} - \beta_{j1}, \cdots, \beta_{kM} - \beta_{jM})}{P(\beta_{k1} + \beta_{j1}, \cdots, \beta_{kM} + \beta_{jM})} = a_{kj}, \quad 1 \leq j < k \leq N. \tag{18}$$

However, the polynomial P must satisfy the Hirota condition to have N-soliton solutions,

$$\sum P\Big(\sum_{j=1}^{N} \sigma_j\beta_{j1}, \cdots, \sum_{j=1}^{N} \sigma_j\beta_{jM}\Big) \prod_{k<j} P(\sigma_k\beta_{k1} - \sigma_j\beta_{j1}, \cdots, \sigma_k\beta_{kM} - \sigma_j\beta_{jM})\sigma_k\sigma_j = 0, \tag{19}$$

where the summation over all possible combinations of $\sigma_j = \pm 1, j, k = 1, 2, \cdots, N$.

We have the following result for multi-complexitons.

Theorem 1. *Let P be a real polynomial satisfying $P(0) = 0$, $P(-x) = P(x)$ for $x \in \mathbf{R}^M$, and N be a positive integer. Assume that the complex wave variables $\eta_j = \beta_{j0} + \sum_{l=1}^{M} \beta_{jl}x_l, j = 1, 3, \cdots, 2N-1$, satisfy the dispersion relation (13) and the Hirota condition (19). Suppose $\eta_{2j} = \bar{\eta}_{2j-1}, j = 1, \cdots, N$. Then the function*

$$f = 1 + \sum_{n=1}^{2N} \sum_{\sum_{j=1}^{2N}\mu_j = n} \exp\left(\sum_{j=1}^{2N} \mu_j\eta_j + \sum_{k<j} A_{kj}\mu_k\mu_j\right), \tag{20}$$

where $\mu_j = 0$ or 1 for $j = 1, 2, \cdots, 2N$, and $a_{kj} = e^{A_{kj}}, j, k = 1, 2, \cdots, 2N$, determined by (18) presents a complexiton solution to (9).

Proof. We only need to show that function f given by (20) is real and we use the mathematical induction. We have proved the case of $N = 1$. Suppose $N' \geq 1$ is an integer and we assume for $1 \leq n \leq 2N'$,

$$\sum_{\sum \mu_j = n} \exp\left(\sum_{j=1}^{2N'} \mu_j \eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right) \in \mathbb{R}. \tag{21}$$

When $N = N' + 1$, for any fixed $n : 1 \le n \le 2N$, we want to show

$$\sum_{\sum \mu_j = n} \exp\left(\sum_{j=1}^{2N'+2} \mu_j \eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right) \in \mathbb{R}. \tag{22}$$

For fixed $n \ge 1$, the sum in (22) consists of three parts: $\sum_1^{2N'} \mu_j = n, n-1, n-2$. In the first case $\mu_{2N'+1} = \mu_{2N'+2} = 0$, by induction we know the sum is real.

In the second case $\mu_{2N'+1} = 1, \mu_{2N'+2} = 0$ or $\mu_{2N'+1} = 0, \mu_{2N'+2} = 1$. Since we take all the possible sums, this part of sum equals

$$\sum_{\sum_{j=1}^{2N'} \mu_j = n-1} \exp\left(\sum_{j=1}^{2N'} \mu_j \eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right) \sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+m}\right). \tag{23}$$

By (18), we have for $1 \le j < k \le N' + 1$,

$$a_{2j-1,2k-1} = \bar{a}_{2j,2k}, \qquad a_{2j,2k-1} = \bar{a}_{2j-1,2k}, \tag{24}$$

and

$$a_{2k-1,2k} := -\frac{P(2i\mathrm{Im}(\beta_{2k-1,1}), \cdots, 2i\mathrm{Im}(\beta_{2k-1,M}))}{P(2\mathrm{Re}(\beta_{2k-1,1}), \cdots, 2\mathrm{Re}(\beta_{2k-1,M}))} \in \mathbb{R}. \tag{25}$$

We introduce a map $* : \mathbb{N} \to \mathbb{N}$

$$(2j-1)^* = 2j, \quad (2j)^* = 2j-1, \qquad \forall j \in \mathbb{N}.$$

This map has the property

$$(j^*)^* = j, \quad \forall j \in \mathbb{N}$$

and

$$\sum_{j=1}^{2N} \mu_j = \sum_{j^*=1}^{2N} \mu_{j^*}, \quad \forall N \in \mathbb{N}.$$

Case I. If $(\mu_{1^*}, \mu_{2^*}, \ldots, \mu_{(2N'-1)^*}, \mu_{(2N')^*}) = (\mu_1, \mu_2, \ldots, \mu_{2N'-1}, \mu_{2N'})$, then $\mu_{2j-1} = \mu_{2j}$ for $1 \le j \le N'$,

$$\mu_{2j-1}\eta_{2j-1} + \mu_{2j}\eta_{2j} = \mu_{2j-1}(\eta_{2j-1} + \bar{\eta}_{2j-1}) \in \mathbb{R}. \tag{26}$$

When $\mu_{2j-1} = \mu_{2k-1} = 1$, we get

$$a_{2j-1,2k-1} + a_{2j-1,2k} + a_{2j,2k-1} + a_{2j,2k} = 2\mathrm{Re}(a_{2j-1,2k-1} + a_{2j-1,2k}). \tag{27}$$

Therefore

$$\exp\left(\sum_{j=1}^{2N'} \mu_j \eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right) \in \mathbb{R}. \tag{28}$$

On the other hand,

$$\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+m} = \eta_{2N'+m} + \sum_{j=1}^{N'} \mu_{2j-1}(A_{2j-1,2N'+m} + A_{2j,2N'+m}) \tag{29}$$

implies

$$\exp\left(\eta_{2N'+1} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+1}\right) = \overline{\exp\left(\eta_{2N'+2} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+2}\right)} \tag{30}$$

and this concludes

$$\sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_j A_{j,2N'+m}\right) \in \mathbb{R}$$

and hence (23) is real.

Case II. If $(\mu_{1^*}, \mu_{2^*}, \cdots, \mu_{(2N'-1)^*}, \mu_{(2N')^*}) \neq (\mu_1, \mu_2, \cdots, \mu_{2N'-1}, \mu_{2N'})$, then $\mu_{2j-1} \neq \mu_{2j}$ for some $1 \leq j \leq N'$. Because we have $\mu_{j*}\eta_{j*} = \mu_j\bar{\eta}_j$. Suppose $\mu_m = \mu_j = 1$ and $m < j, j \neq m^*$, then by (24) and (25) we have

$$a_{j,j^*} = a_{j^*,j} \in \mathbb{R}, \quad a_{m,j} = \bar{a}_{m^*,j^*}, \quad a_{m,j^*} = \bar{a}_{m^*,j}. \tag{31}$$

Therefore

$$\exp\left(\sum_{j^*=1}^{2N'} \mu_{j^*}\eta_{j^*} + \sum_{m^*<j^*} A_{m^*,j^*}\mu_{m^*}\mu_{j^*}\right) = \exp\left(\sum_{j=1}^{2N'} \mu_j\bar{\eta}_j + \sum_{m<j} \bar{A}_{mj}\mu_m\mu_j\right). \tag{32}$$

In the same way, it is easy to see

$$\sum_{m=1}^{2} \exp\left(\eta_{(2N'+m)^*} + \sum_{j^*=1}^{2N'} \mu_{j^*}A_{j^*,(2N'+m)^*}\right) = \sum_{m=1}^{2} \exp\left(\bar{\eta}_{2N'+m} + \sum_{j=1}^{2N'} \mu_j\bar{A}_{j,2N'+m}\right),$$

which means

$$\exp\left(\sum_{j^*=1}^{2N'} \mu_{j^*}\eta_{j^*} + \sum_{m^*<j^*} A_{m^*j^*}\mu_{m^*}\mu_{j^*}\right)\sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j^*=1}^{2N'} \mu_{j^*}A_{j^*,2N'+m}\right)$$
$$= \overline{\exp\left(\sum_{j=1}^{2N'} \mu_j\eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right)\sum_{m=1}^{2} \exp\left(\eta_{2N'+m} + \sum_{j=1}^{2N'} \mu_jA_{j,2N'+m}\right)}. \tag{33}$$

So we know that (23) is real.

In the third case $\mu_{2N'+1} = \mu_{2N'+2} = 1$. Let

$$C_0 := \exp\left(\eta_{2N'+1} + \eta_{2N'+2} + A_{2N'+1,2N'+2}\right) = a_{2N'+1,2N'+2}\exp(2\mathrm{Re}(\eta_{2N'+1})) \in \mathbb{R}.$$

And we also have

$$\exp\sum_{m^*=1}^{2N'} \mu_{m^*}(A_{m^*,2N'+1} + A_{m^*,2N'}) = \overline{\exp\sum_{m=1}^{2N'} \mu_m(A_{m,2N'+1} + A_{m,2N'})}.$$

Therefore

$$\exp\left(\sum_{j^*=1}^{2N'} \mu_{j^*}\eta_{j^*} + \sum_{m^*<j^*} A_{m^*,j^*}\mu_{m^*}\mu_{j^*}\right)\exp\left(\sum_{j^*=1}^{2N'} \mu_{j^*}(A_{j^*,2N'+1} + A_{j^*,2N'+2})\right)$$
$$= \overline{\exp\left(\sum_{j=1}^{2N'} \mu_j\eta_j + \sum_{m<j} A_{m,j}\mu_m\mu_j\right)\exp\left(\sum_{j=1}^{2N'} \mu_j(A_{j,2N'+1} + A_{j,2N'+2})\right)}. \tag{34}$$

This tells us that

$$\sum_{\sum_{j=1}^{2N'}\mu_j = n-2} \exp\left(\sum_{j=1}^{2N'} \mu_j\eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right)\exp\left(\sum_{m=1}^{2} \eta_{2N'+m}\right.$$
$$\left. + A_{2N'+1,2N'+2} + \sum_{m=1}^{2}\sum_{j=1}^{2N'} \mu_jA_{j,2N'+m}\right)$$
$$= C_0 \sum_{\sum_{j=1}^{2N'}\mu_j = n-2} \exp\left(\sum_{j=1}^{2N'} \mu_j\eta_j + \sum_{m<j} A_{mj}\mu_m\mu_j\right)\exp\left(\sum_{m=1}^{2}\sum_{j=1}^{2N'} \mu_jA_{j,2N'+m}\right)$$

is real.

Combining with the above proofs, we get (22) that is real for any $n \geq 1$, which concludes the function f is real for $N = N' + 1$. This completes the proof by the induction. □

IV. THE BILINEAR KdV EQUATION

In this section, we will apply Theorem 1 to multi-complexiton solutions to the bilinear KdV equation. By Refs. 1 and 3, we know that the bilinear KdV equation satisfies the Hirota condition.

We use the notation $i := \sqrt{-1}$. Suppose the function

$$\eta(x,y,t) = kx + wt + \eta^0 = \eta_1 + i\eta_2, \tag{35}$$

where k, w, η^0 are constants and $\eta_1 = \mathrm{Re}(\eta), \eta_2 = \mathrm{Im}(\eta)$. Then $f := 1 + e^{\eta}$ is a complex valued solution to (5) if and only if the following dispersion relation holds:

$$P_1(k,w) = P_1(\bar{k},\bar{w}) = 0. \tag{36}$$

Let $k_1 := \mathrm{Re}(k), k_2 := \mathrm{Im}(k), w_1 := \mathrm{Re}(w), w_2 := \mathrm{Im}(w)$. Equation (36) is equivalent to

$$\begin{cases} w_1 = -k_1^3 + 3k_1k_2^2, \\ w_2 = k_2^3 - 3k_1^2k_2. \end{cases} \tag{37}$$

Now suppose η satisfies (36) or (37). By the two-soliton formulation and the above discussion, we get a one complexiton solution to (5),

$$f = 1 + e^{\eta} + e^{\bar{\eta}} + a_{12}e^{\eta+\bar{\eta}} = 1 + 2e^{\eta_1}\cos(\eta_2) + a_{12}e^{2\eta_1}, \tag{38}$$

where $a_{12} = -\dfrac{P_1(2ik_2, 2iw_2)}{P_1(2k_1, 2w_1)} = -\dfrac{k_2^2}{k_1^2} \in \mathbb{R}$. The solution for the KdV equation reads

$$\begin{aligned} u &= 2(\ln f)_{xx} \\ &= -\frac{2[k_2^2\cos(\eta_2) + k_1^2\sinh(\eta_1) - (k_1^2 - k_2^2)\exp(\eta_1)]}{\cos(\eta_2) - \sinh(\eta_1) + (1 - k_2^2/k_1^2)\exp(\eta_1)} \\ &\quad - \frac{2[k_2\sin(\eta_2) - k_1\cosh(\eta_1) + (1 - k_2^2/k_1^2)k_1\exp(\eta_1)]^2}{[\cos(\eta_2) - \sinh(\eta_1) + (1 - k_2^2/k_1^2)\exp(\eta_1)]^2}, \end{aligned} \tag{39}$$

where $\eta_1 := k_1x + (3k_1k_2^2 - k_1^3)t + \eta_1^0, \eta_2 := k_2x + (k_2^3 - 3k_1^2k_2)t + \eta_2^0$, $k_1 \neq 0, k_2, \eta_1^0, \eta_2^0$, are all real constants. Due to the part $\exp(\eta_1)$ in the denominator of solution u, our solutions differ from the result in Ref. 11. Figure 1 shows the 3d plot of complexitons.

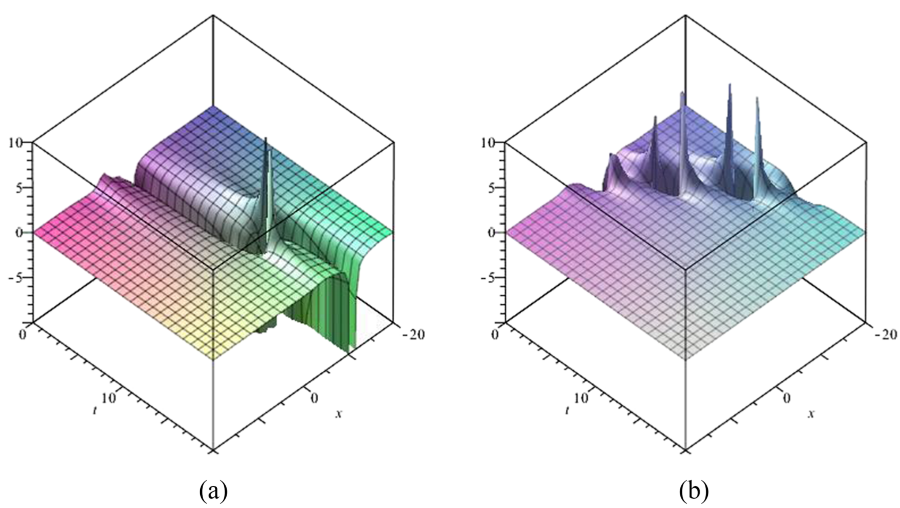

FIG. 1. Plots of complexitons with (a) $k_1 = 0.4, k_2 = 0.5$, (b) $k_1 = 0.5, k_2 = -0.8$.

Now we consider $N = 4$. Suppose that the dispersion relation (36) is true for $\tilde{\eta}_1 = \eta_1 + i\eta_2$, $\tilde{\eta}_2 = \eta_1 - i\eta_2$ and $\tilde{\eta}_3 = \eta_3 + i\eta_4, \tilde{\eta}_4 = \eta_3 - i\eta_4$, where $\eta_j = \eta_j^0 + k_j x + w_j t, j = 1, \ldots, 4$ are all real.

Let

$$a_{j'j} = -\frac{P_1(\tilde{k}_{j'} - \tilde{k}_j, \tilde{w}_{j'} - \tilde{w}_j)}{P_1(\tilde{k}_{j'} + \tilde{k}_j, \tilde{w}_{j'} + \tilde{w}_j)}, \qquad 1 \le j' < j \le N. \tag{40}$$

Then we have

$$\begin{aligned}
a_{12} &= -k_2^2/k_1^2 \in \mathbb{R}, \\
a_{13} &= -\frac{P_1(k_1 - k_3 + i(k_2 - k_4), w_1 - w_3 + i(w_2 - w_4))}{P_1(k_1 + k_3 + i(k_2 + k_4), w_1 + w_3 + i(w_2 + w_4))}, \\
a_{14} &= -\frac{P_1(k_1 - k_3 + i(k_2 + k_4), w_1 - w_3 + i(w_2 + w_4))}{P_1(k_1 + k_3 + i(k_2 - k_4), w_1 + w_3 + i(w_2 - w_4))}, \\
a_{23} &= -\frac{P_1(k_1 - k_3 - i(k_2 + k_4), w_1 - w_3 - i(w_2 + w_4))}{P_1(k_1 + k_3 - i(k_2 - k_4), w_1 + w_3 - i(w_2 - w_4))} \\
&= \overline{a_{14}}, \\
a_{24} &= -\frac{P_1(k_1 - k_3 - i(k_2 - k_4), w_1 - w_3 - i(w_2 - w_4))}{P_1(k_1 + k_3 - i(k_2 + k_4), w_1 + w_3 - (w_2 + w_4))} \\
&= \overline{a_{13}}, \\
a_{34} &= -k_4^2/k_3^2 \in \mathbb{R}.
\end{aligned} \tag{41}$$

Let the function f be defined by

$$\begin{aligned}
f = 1 &+ e^{\tilde{\eta}_1} + e^{\tilde{\eta}_2} + e^{\tilde{\eta}_3} + e^{\tilde{\eta}_4} + a_{12}e^{\tilde{\eta}_1+\tilde{\eta}_2} + a_{13}e^{\tilde{\eta}_1+\tilde{\eta}_3} + a_{14}e^{\tilde{\eta}_1+\tilde{\eta}_4} \\
&+ a_{23}e^{\tilde{\eta}_2+\tilde{\eta}_3} + a_{24}e^{\tilde{\eta}_2+\tilde{\eta}_4} + a_{34}e^{\tilde{\eta}_3+\tilde{\eta}_4} + a_{123}e^{\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\eta}_3} + a_{124}e^{\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\eta}_4} \\
&+ a_{134}e^{\tilde{\eta}_1+\tilde{\eta}_3+\tilde{\eta}_4} + a_{234}e^{\tilde{\eta}_2+\tilde{\eta}_3+\tilde{\eta}_4} + a_{1234}e^{\tilde{\eta}_1+\tilde{\eta}_2+\tilde{\eta}_3+\tilde{\eta}_4}
\end{aligned} \tag{42}$$

with

$$\begin{aligned}
a_{1234} &= a_{12}a_{13}a_{14}a_{23}a_{24}a_{34} = a_{12}\overline{a_{24}a_{23}}a_{23}a_{24}a_{34} \in \mathbb{R}, \\
a_{123} &= a_{12}a_{13}a_{23} = \overline{a_{12}a_{24}a_{14}} = \bar{a}_{124}, \\
a_{134} &= a_{13}a_{14}a_{34} = \overline{a_{24}a_{23}a_{34}} = \bar{a}_{234}.
\end{aligned} \tag{43}$$

Then it is a two-complexiton solution to (5).

The function f can be simplified as

$$\begin{aligned}
f &= 1 + 2e^{\eta_1}\cos(\eta_2) + 2e^{\eta_3}\cos(\eta_4) + a_{12}e^{2\eta_1} + a_{34}e^{2\eta_3} \\
&+ 2\mathrm{Re}\{a_{13}e^{\eta_1+\eta_3+i(\eta_2+\eta_4)} + a_{14}e^{\eta_1+\eta_3+i(\eta_2-\eta_4)}\} \\
&+ 2\mathrm{Re}\{a_{123}e^{2\eta_1+\eta_3+i\eta_4} + a_{134}e^{\eta_1+2\eta_3+i\eta_2}\} \\
&+ a_{1234}e^{2\eta_1+2\eta_3}.
\end{aligned} \tag{44}$$

In particular, if $\eta_4 = 0$, then $\tilde{\eta}_3 = \tilde{\eta}_4 = \eta_3$. By (40), we get $a_{34} = 0$. It is clear that $a_{14} = a_{13}$, $a_{134} = a_{1234} = 0$. Therefore we have

$$\begin{aligned}
f &= 1 + 2e^{\eta_1}\cos(\eta_2) + 2e^{\eta_3} + a_{12}e^{2\eta_1} + 2\mathrm{Re}\{2a_{13}e^{\eta_1+\eta_3+i\eta_2}\} \\
&+ 2\mathrm{Re}\{a_{123}e^{2\eta_1+\eta_3}\}.
\end{aligned} \tag{45}$$

Then $u = 2(\ln f)_{xx}$ is an interaction of one-soliton and one-complexiton solutions.

V. CONCLUSION

In this paper, we presented a general scheme for constructing multi-complexitons to Hirota bilinear equations satisfying the Hirota condition. The scheme can be used to compute multi-complexions and interaction of complexitons and solitons.

ACKNOWLEDGMENTS

The work is supported in part by the National Natural Science Foundation of China under Grant Nos. 11371326, 11271008, and 11371086 and the Distinguished Professorship by Shanghai University of Electric Power and Shanghai Second Polytechnic University. The authors would also like to thank S. T. Chen, X. Gu, X. Lü, S. Manukure, M. Mcanally, F. D. Wang, Y. Sun, and J. Yu for their valuable discussions.

[1] R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004).
[2] P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, 1989).
[3] J. Hietarinta, "Gauge symmetry and the generalization of Hirota's bilinear method," J. Nonlinear Math. Phys. **3**, 260–265 (1996).
[4] W. X. Ma, "Generalized bilinear differential equations," Stud. Nonlinear Sci. **2**, 140–144 (2011).
[5] W. X. Ma, "Bilinear equations and resonant solutions characterized by Bell polynomials," Rep. Math. Phys. **72**, 41–56 (2013).
[6] W. X. Ma, "Trilinear equations, Bell polynomials, and resonant solutions," Front. Math. China **8**, 1139–1156 (2013).
[7] N. J. Zabusky and M. D. Kruskal, "Interaction of 'solitons' in a collisionless plasma and the recurrence of initial states," Phys. Rev. Lett. **15**, 240–243 (1965).
[8] S. V. Manakov, V. E. Zakhorov, L. A. Bordag, A. R. Its, and V. B. Matveev, "Two-dimensional solitons of the Kadomtsev–Petviashvili equation and their interaction," Phys. Lett. A **63**, 205–206 (1977).
[9] W. X. Ma, "Complexiton solutions to the Korteweg–de Vries equation," Phys. Lett. A **301**, 35–44 (2002).
[10] W. X. Ma, "Complexiton solutions of the Korteweg–de Vries equation with self-consistent sources," Chaos, Solitons Fractals **26**, 1453–1458 (2005).
[11] W. X. Ma and Y. You, "Solving the Korteweg—de Vries equation by its bilinear form: Wronskian solutions," Trans. Am. Math. Soc. **357**, 1753–1778 (2005).
[12] Y. Q. Yao, D. J. Zhang, and D. Y. Chen, "The double Wronskian solution to the Kadomtsev–Petviashvili equation," Mod. Phys. Lett. B **22**, 621–641 (2008).
[13] E. G. Fan, "Extended tanh–function method and its applications to nonlinear equations," Phys. Lett. A **277**, 212–218 (2000).
[14] Y. Zhou and W. X. Ma, "Applications of linear superposition principle to resonant solitons and complexitons," Comput. Math. Appl. **73**(8), 1697–1706 (2017).