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Lump-soliton interaction solutions to differential-difference mKdV systems in (2+1)-dimensions

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ABSTRACT

Lump-soliton interaction solutions to continuous integrable systems have been pretty well studied, but there are relatively few results in the differential-difference (DΔ) case. In this paper, some (2+1)-dimensional DΔ-mKdV systems are investigated by using Hirota's bilinear operator method. By setting appropriate variable transformations and assuming auxiliary functions as quadratic and exponential functions, lump-soliton interaction solutions are derived. Certain fission\fusion phenomena of the physical quantity, the velocity of the potential, are explored by analyzing dynamical behaviors of the resultant solutions with different values of the involved parameters.

Introduction

With the rapid development of soliton theory, exact solutions of nonlinear evolution equations (NLEEs), especially integrable systems, have aroused great interest among many scientists and engineers [1–20]. By means of exact solutions, one can better understand natural phenomena described by mathematical models and further explore other new potential applications. Many effective methods have been proposed to construct exact solutions of NLEEs such as the inverse scattering theory (IST), the Lie group analysis method, the Painlevé test, the Darboux transformation, the Riemann–Hilbert method, the multi-linear variable separation (MLVS) approach and the Hirota bilinear operator method [1–27].

Lump solutions, soliton solutions and lump-soliton interaction solutions are three important classes of exact solutions, which have been applied to almost all branches of physics such as condense matter physics, quantum field theory, plasma physics, fluid mechanics and nonlinear optics. Lump solutions are rationally localized in all directions in space and the study of lump solutions has a long history. The previous idea to construct lump solutions is to take long wave limit of multi-soliton solutions. Recently, a new direct ansatz method to seek for lump solutions and lump-soliton interaction solutions has been given [15–20]. The crucial step is to take combinations of positive quadratic functions and exponential functions, which solve Hirota's bilinear equations. Then the logarithmic transformations yield lump-soliton interaction solutions to given continuous NLEEs.

Differential-difference (DΔ) equations play a crucial role in modeling of much physical phenomena such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains. Generally speaking, DΔ equations are semi-discrete equations, in which some of the spatial variables are discrete and the time variables are continuous. Therefore, the difficulty of solving DΔ equations is often huge. Lump-soliton interaction solutions of continuous integrable systems have been well studied, but relatively few results are known in the DΔ case. Taking some DΔ-mKdV systems as examples, we will extend the direct ansatz method [15–20] to the DΔ case in this paper. In Section "Lump-soliton interaction solutions and fission\fusion phenomena", lump-soliton solutions of DΔ-mKdV systems are obtained and fission\fusion phenomena are analyzed. Section "Conclusions" contains conclusions.

Lump-soliton interaction solutions and fission\fusion phenomena

We will construct lump-soliton interaction solutions for the following DΔ-mKdV system firstly

$$
\begin{aligned}
&u_t(n) + u_{yyy}(n) + u_y^3(n) + 3u_y(n)v_{yy}(n) + \delta u_y(n) = 0,\\
&v(n+1) - v(n) = u(n+1) + u(n)
\end{aligned}
\tag{1}
$$

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by using Hirota's bilinear operator method. Here $u(n) \equiv u(n,y,t)$ and $v(n) \equiv v(n,y,t)$ are functions of the discrete variable n and the continuous variables $\{y,t\}$. In the study of nonlinear problems, to find the integrable discretizations of continuous integrable systems is a very important topic, because of the rapid development of symbolic computing software. The DΔ-mKdV system (1) is interesting, because its continuous version is the famous asymmetric Nizhnik–Novikov–Veselov (NNV) equation and possesses possible physical applications [24–26]. So, just as mentioned in Ref. [24], here we can also call Eq. (1) the DΔ-asymmetric NNV system. Setting $\bar{u}(n) \equiv \bar{u}(n,y,t) = u_y(n)$, $\bar{v}(n) \equiv \bar{v}(n,y,t) = v_{yy}(n)$, we can rewrite Eqs. (1) as follows,

$$\begin{aligned}&\bar{u}_t(n)+\bar{u}_{yyy}(n)+3\bar{u}^2(n)\bar{u}_y(n)+3\bar{u}_y(n)\bar{v}(n)+3\bar{u}(n)\bar{v}_y(n)+\delta\bar{u}_y(n)=0,\\&\bar{v}(n+1)-\bar{v}(n)=\bar{u}_y(n+1)+\bar{u}_y(n),\end{aligned} \tag{2}$$

which is studied in Ref. [24] by using the MLVS approach. By means of

$$\begin{aligned}&\bar{u}(n,y,t)=\frac{1}{2}\epsilon U(\epsilon n,y,t)=\frac{1}{2}\epsilon U(x,y,t),\\&\bar{v}(n,y,t)=V(\epsilon n,y,t)=V(x,y,t),\end{aligned}$$

we have

$$\begin{aligned}\bar{u}(n+1,y,t)&=\frac{1}{2}\epsilon U(\epsilon n+\epsilon,y,t)=\frac{1}{2}\epsilon U(x+\epsilon,y,t)\\&=\frac{1}{2}\epsilon U(x,y,t)+o(\epsilon^2),\\\bar{v}(n+1,y,t)&=V(\epsilon n+\epsilon,y,t)=V(x+\epsilon,y,t)\\&=V(x,y,t)+\epsilon V_x(x,y,t)+o(\epsilon^2).\end{aligned}$$

Substituting these expressions into Eqs. (2) and neglecting the higher order terms of ϵ, we have

$$\begin{aligned}&U_t+U_{yyy}+3U_yV+3UV_y+\delta U_y=0,\\&V_x=U_y,\end{aligned}$$

being just the well-known asymmetric NNV equation with $\delta = 0$ in Ref. [24]. The MLVS approach has applied to the DΔ-mKdV system (1) by using the following bi-logarithmic transformation

$$\begin{aligned}&u(n)=\ln\left[\frac{f(n+1)}{f(n)}\right],\\&v(n)=\ln[f(n)f(n+1)]+v_0(y,t),\end{aligned}$$

with a suitable MLVS ansatz

$$f(n)\equiv f(n,y,t)=a_0+a_1p(n,t)+a_2q(y,t)+a_3p(n,t)q(y,t). \tag{3}$$

and abundant semi-discrete localized coherent structures are constructed by appropriately selecting the arbitrary functions [24–26].

Next, we use a combination of positive quadratic functions and exponential functions to construct lump-soliton interaction solutions. The key step here is to replace Eq. (3) with a new form. Firstly, we adopt the following bi-logarithmic transformation

$$\begin{aligned}&u(n)=\ln\left[\frac{f(n+1)}{f(n)}\right],\\&v(n)=\ln[f(n+1)f(n)]\end{aligned} \tag{4}$$

to bilinearize Eq. (1). Actually, in this transformation (4) we take the seed solution to zero. Substituting the above transformation into Eq. (1), we have

$$(D_t+D_y^3+\delta D_y)f(n+1)\cdot f(n)=0. \tag{5}$$

Here Hirota's bilinear differential operator $D_y^mD_t^k$ and difference operator $\exp(D_n)$ are defined by

$$\begin{aligned}&D_y^mD_t^ka\cdot b=\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k a(y,t)b(y',t')\Big|_{y=y',t=t'},\\&\exp(\alpha D_n)a(n)\cdot b(n)=\exp\left[\alpha\left(\frac{\partial}{\partial n}-\frac{\partial}{\partial n'}\right)\right]a(n)b(n)\Big|_{n=n'}\\&\qquad=a(n+\alpha)b(n-\alpha).\end{aligned} \tag{6}$$

To search for lump solutions, we begin with a quadratic function solution as

$$\begin{aligned}f(n)&=g^2(n)+h^2(n)+a_9\\&=(a_1n+a_2y+a_3t+a_4)^2+(a_5n+a_6y+a_7t+a_8)^2+a_9\end{aligned} \tag{7}$$

with constraint

$$\begin{vmatrix}a_1 & a_2\\ a_5 & a_6\end{vmatrix}\neq 0, \tag{8}$$

where a_i $(i=1,2,\ldots,9)$ are all real parameters to be determined.

Substituting Eq. (7) into Eq. (5) yields the algebraic system of determining equations

$$\begin{aligned}&(a_2\delta+a_3)a_1+(a_6\delta+a_7)a_5=0,\\&a_2a_3\delta+a_6a_7\delta+a_3^2+a_7^2=0,\\&(a_2^2+a_6^2)\delta+a_2a_3+a_6a_7=0,\\&a_1a_3^2+[a_5a_7+\delta(a_1a_2-a_5a_6)]a_3^2+[a_1a_7^2+2\delta(a_1a_6+a_2a_5)a_7]a_3\\&-[\delta(a_1a_2-a_5a_6)-a_5a_7]a_7^2=0,\\&[a_1^2+2a_1a_4+a_5(a_5+2a_8)](a_3^2+a_7^2)+\{(a_2a_3+a_6a_7)a_1^2\\&+[(2a_2a_4+2a_6a_8)a_3-2a_7(a_2a_8-a_4a_6)]a_1\\&+[(a_2a_5+2a_2a_8-2a_4a_6)a_3+2a_7(a_2a_4+\frac{1}{2}a_5a_6+a_6a_8)]a_5\}\delta=0,\\&a_1a_2^3\delta+(a_5a_6\delta+a_1a_3-a_5a_7)a_2^2+(a_1a_6\delta+2a_1a_7+2a_3a_5)a_2a_6\\&-(a_1a_3-a_5a_7-a_5a_6\delta)a_6^2=0,\\&(a_2^2\delta+a_2a_3+a_6^2\delta+a_6a_7)a_1^2+[2a_2^2a_4\delta+(2a_3a_4+2a_7a_8)a_2\\&-2a_6(a_3a_8-a_4a_7-a_4a_6\delta)]a_1\\&+[(a_5+2a_8)a_2^2\delta+(a_3a_5+2a_3a_8-2a_4a_7)a_2+(a_5a_6\delta+a_5a_7+2a_6a_8\delta\\&+2a_3a_4+2a_7a_8)a_6]a_5=0,\\&[(a_2a_4+a_6a_8)\delta+a_3a_4+a_7a_8]a_1^2+[(a_2a_4^2+2a_4a_6a_8-a_2a_8^2-a_2a_9)\delta\\&-6a_2^3-6a_2a_6^2+a_3a_4^2-a_3a_8^2+2a_4a_7a_8-a_3a_9]a_1\\&+[(a_2a_4\delta+a_6a_8\delta+a_3a_4+a_7a_8)a_5+(2a_2a_4a_8-a_6a_4^2+a_6a_8^2-a_6a_9)\delta\\&-6a_6a_2^2+2a_3a_4a_8-a_7a_4^2-6a_6^3+a_7a_8^2-a_7a_9]a_5=0.\end{aligned}$$

By solving this system with symbolic computation software, we have

$$a_1=-\frac{a_5a_6}{a_2},\qquad a_3=-a_2\delta,\qquad a_7=-a_6\delta. \tag{9}$$

From the above calculation result, the linear term $\delta u_y(n)$ in Eq. (1) is necessary, otherwise there will be no suitable solution to Eq. (1). Thus, we have obtained a new lump solution for the DΔ-mKdV system (1),

$$\begin{aligned}u(n)&=\ln\left[\frac{f(n+1)}{f(n)}\right]\\&=\ln\left[\frac{\left(-\frac{a_5a_6}{a_2}(n+1)+a_2y-a_2\delta t+a_4\right)^2+\left(a_5(n+1)+a_6y-a_6\delta t+a_8\right)^2+a_9}{\left(-\frac{a_5a_6}{a_2}n+a_2y-a_2\delta t+a_4\right)^2+\left(a_5n+a_6y-a_6\delta t+a_8\right)^2+a_9}\right],\\v(n)&=\ln[f(n+1)f(n)]\\&=\ln\left[\left(\left(-\frac{a_5a_6}{a_2}(n+1)+a_2y-a_2\delta t+a_4\right)^2\right.\right.\\&\qquad\left.+\left(a_5(n+1)+a_6y-a_6\delta t+a_8\right)^2+a_9\right)\\&\qquad\left.\times\left(\left(-\frac{a_5a_6}{a_2}n+a_2y-a_2\delta t+a_4\right)^2+\left(a_5n+a_6y-a_6\delta t+a_8\right)^2+a_9\right)\right].\end{aligned} \tag{10}$$

To construct a lump-one soliton interaction solution consisting of a lump and a line soliton wave, we use the following assumption for the function $f(n)$,

$$\begin{aligned}f(n)&=g^2(n)+h^2(n)+a_9+k_5e^{\eta}\\&=(a_1n+a_2y+a_3t+a_4)^2+(a_5n+a_6y+a_7t+a_8)^2+a_9\\&\quad+k_5e^{k_1n+k_2y+k_3t+k_4},\end{aligned} \tag{11}$$

where $a_j (j = 1, 2, \ldots, 9)$ and $k_j (j = 1, 2, \ldots, 5)$ are real parameters need to be determined. Substituting Eqs. (4) and (11) into Eq. (5), and balancing the different powers of n, y and t, we have

$$a_1 = -\frac{a_5 a_6}{a_2}, \qquad a_3 = -a_2\delta, \qquad a_7 = -a_6\delta,$$
$$k_1 = 0, \qquad k_3 = -k_2^3 - \delta k_2. \tag{12}$$

Thus we have obtained a lump-one soliton interaction solution (4) with (11)–(12) of the DΔ-mKdV system (1).

To construct a lump-two soliton interaction solution, we use the following assumption for the function $f(n)$,

$$\begin{aligned} f(n) &= g^2(n) + h^2(n) + a_9 + k_5 e^{\eta_1} + k_{10} e^{\eta_2} \\ &= (a_1 n + a_2 y + a_3 t + a_4)^2 + (a_5 n + a_6 y + a_7 t + a_8)^2 + a_9 \\ &\quad + k_5 e^{k_1 n + k_2 y + k_3 t + k_4} + k_{10} e^{k_6 n + k_7 y + k_8 t + k_9}, \end{aligned} \tag{13}$$

where $a_j (j = 1, 2, \ldots, 9)$ and $k_j (j = 1, 2, \ldots, 10)$ are real parameters need to be determined. Substituting Eqs. (4) and (13) into Eq. (5), and balancing the different powers of n, y and t, we have

$$a_1 = -\frac{a_5 a_6}{a_2}, \qquad a_3 = -a_2\delta, \qquad a_7 = -a_6\delta,$$
$$k_1 = 0, \qquad k_3 = -k_2^3 - \delta k_2,$$
$$k_6 = 0, \qquad k_8 = -k_7^3 - \delta k_7. \tag{14}$$

Thus we have obtained a lump-two soliton interaction solution (4) with (13)–(14) of the DΔ-mKdV system (1).

Remark 1. For different choices of dependent variable transformations, Hirota's bilinear equation can be transformed into completely different types of NLEEs. From the Taylor expansion of the following fundamental formula

$$\exp(\alpha D_y) f(n+1)\cdot f(n) = \exp\left\{\sinh(\alpha\partial_y)\ln\left[\frac{f(n+1)}{f(n)}\right] + \cosh(\alpha\partial_y)\ln[f(n+1)f(n)]\right\},$$

we have

$$\frac{1}{f(n+1)f(n)} D_y f(n+1)\cdot f(n) = u_y(n),$$
$$\frac{1}{f(n+1)f(n)} D_y^2 f(n+1)\cdot f(n) = v_{yy}(n) + u_y^2(n),$$
$$\frac{1}{f(n+1)f(n)} D_y^3 f(n+1)\cdot f(n) = u_{yyy}(n) + 3u_y(n)v_{yy}(n) + u_y^3(n),$$
$$\frac{1}{f(n+1)f(n)} D_y^4 f(n+1)\cdot f(n) = v_{yyyy}(n) + 4u_y(n)v_{yyy}(n) + 3v_{yy}^2(n) + 6u_y^2(n)v_{yy}(n) + u_y^4(n),$$
$$\begin{aligned}\frac{1}{f(n+1)f(n)} D_y^5 f(n+1)\cdot f(n) &= u_{yyyyy}(n) + 10u_{yyy}(n)v_{yy}(n) \\ &\quad + 5u_y(n)v_{yyy}(n) + 15u_t(n)v_{yy}^2(n) \\ &\quad + 10u_y^3(n)v_{yy}(n) + 10u_y^2(n)u_{yyy}(n) + u_y^5(n),\end{aligned}$$
$$\vdots$$

by defining the bi-logarithmic transformation (4). In this way, we can construct higher-order DΔ-mKdV systems such as Eq. (1) and

$$\begin{aligned} &u_t(n) + u_{yyyyy}(n) + 10u_{yyy}(n)v_{yy}(n) + 5u_y(n)v_{yyy}(n) + 15u_t(n)v_{yy}^2(n) \\ &\quad + 10u_y^3(n)v_{yy}(n) + 10u_y^2(n)u_{yyy}(n) + u_y^5(n) = 0, \\ &v(n+1) - v(n) = u(n+1) + u(n). \end{aligned} \tag{15}$$

For the above DΔ-mKdV hierarchy, how to give a unified decision rule to judge whether it has a lump-soliton interaction solution is an open problem.

Remark 2. We just studied lump-two soliton solution for the DΔ-mKdV system (1) by means of symbolic computation. the construction of multi-lump-multi-soliton solution still needs further study. However, for some complex DΔ equations [28], during the calculation process, due to the large amount of computation, it is easy to cause the symbol calculation software to crash. So we may need to discover more Hirota's bilinear identities to simplify the calculation process.

Next, we study the following special Toda lattice,

$$u_{yt}(n) = e^{u(n+1)-u(n)}[u(n+1)+u(n)]_y - e^{u(n)-u(n-1)}[u(n)+u(n-1)]_y + \delta u_{yy}(n), \tag{16}$$

which is first derived in Ref. [29], and its continuous analogue is also equivalent to the asymmetric NNV equation. Therefore, we can also call this equation another form of the DΔ-mKdV equation. In Ref. [30], by means of the following logarithmic transformation $u(n) = \ln\left[f(n+1)/f(n)\right] + u_0(n,t)$ with corresponding MLVS ansatz (3), abundant semi-discrete localized coherent structures of the physical quantity $W \equiv u_y(n)$ have been obtained.

To construct lump-soliton interaction solutions, we first substitute the following variable transformation

$$u(n) = \ln\left[\frac{f(n+1)}{f(n)}\right] \tag{17}$$

into Eq. (16) to get Hirota's bilinear equation

$$D_y D_t f(n)\cdot f(n) - \delta D_y^2 f(n)\cdot f(n) = 0. \tag{18}$$

Considering that the following process of solving the above equation is similar to that of solving Eq. (1), here we just list the corresponding results. Namely, we have a lump solution (17) with Eqs. (7) and

$$a_1 = -\frac{a_5 a_6}{a_2}, \qquad a_3 = \frac{\delta a_2^2 - 2a_5 a_6}{a_2}, \qquad a_7 = \delta a_6 + 2a_5. \tag{19}$$

We have a lump-one soliton interaction solution (17) with Eqs. (11) and

$$a_1 = -\frac{a_5 a_6}{a_2}, \qquad a_3 = \frac{\delta a_2^2 - 2a_5 a_6}{a_2}, \qquad a_7 = a_6\delta + 2a_5,$$
$$k_2 = 0, \qquad k_3 = e^{k_1} - e^{-k_1}. \tag{20}$$

We have a lump-two soliton interaction solution (17) with Eqs. (13) and

$$a_1 = -\frac{a_5 a_6}{a_2}, \qquad a_3 = \frac{a_2^2\delta + a_6^2\delta - a_6 a_7}{a_2}, \qquad a_7 = a_6\delta + 2a_5,$$
$$k_2 = 0, \qquad k_3 = e^{k_1} - e^{-k_1},$$
$$k_7 = 0, \qquad k_8 = e^{k_6} - e^{-k_6}. \tag{21}$$

Usually, interactions between soliton solutions are regarded to be completely elastic because the velocity, shape and amplitude of solitons keep unchanged after the interactions. However, for some cases, completely inelastic interactions may occur when the wave vectors and velocities of the solitons satisfy some special conditions. For example, at a certain time, one soliton may fission to two or more solitons. Contrarily, at some time, two or more solitons may fusion to one [31, 32]. Next, for the physical quantity $W(n,y,t) \equiv u_y(n,y,t)$, we use symbolic computing software to plot the lump by taking values for the parameters as

$$a_2 = \frac{1}{2}, \qquad a_4 = 2, \qquad a_5 = \frac{1}{2}, \qquad a_6 = \frac{1}{2}, \qquad a_8 = 2, \qquad a_9 = 3, \qquad \delta = 1.$$

When $t = 0$, Fig. 1 shows the shape and orientation of $W(n,y,t)$.

The corresponding lump-one soliton interaction solution of $W(n,y,t)$ is described in Figs. 2–4 by selecting parameters as

$$a_2 = \frac{1}{3}, \qquad a_4 = 1, \qquad a_5 = \frac{1}{4}, \qquad a_6 = \frac{1}{2}, \qquad a_8 = 2, \qquad a_9 = 3,$$
$$k_1 = \frac{1}{2}, \qquad k_4 = -35, \qquad k_5 = 1, \qquad \delta = 1.$$

Figs. 2–4 vividly show that a fusion occurs, when a lump interacts a line soliton, getting into a locally coherent structure over time.

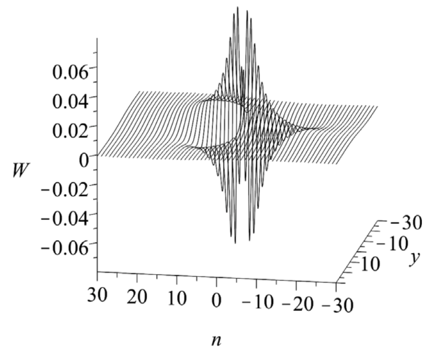

Fig. 1. Lump of W at $t = 0$.

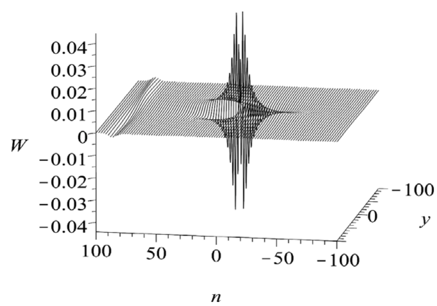

Fig. 2. Lump-one soliton interaction of W at $t = 0$.

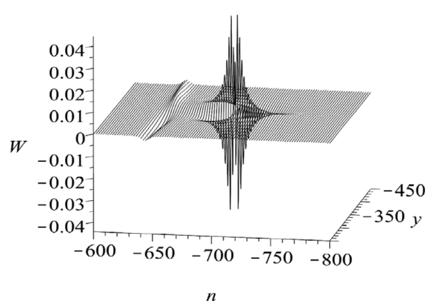

Fig. 3. Lump-one soliton interaction of W at $t = 350$.

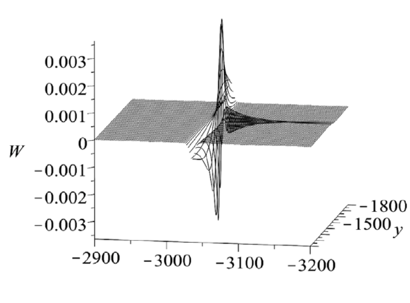

Fig. 4. Lump-one soliton interaction of W at $t = 1500$.

The corresponding lump-two soliton interaction solution of $W(n, y, t)$ is described in Figs. 5–9 by selecting parameters as

$$a_2 = \frac{1}{5}, \quad a_4 = 15, \quad a_5 = -\frac{1}{3}, \quad a_6 = -\frac{1}{5}, \quad a_8 = 2, \quad a_9 = 3,$$
$$k_1 = \frac{1}{2}, \quad k_4 = -35, \quad k_5 = 1, \quad k_6 = -\frac{1}{2}, \quad k_9 = -10, \quad k_{10} = 10,$$
$$\delta = 1. \tag{22}$$

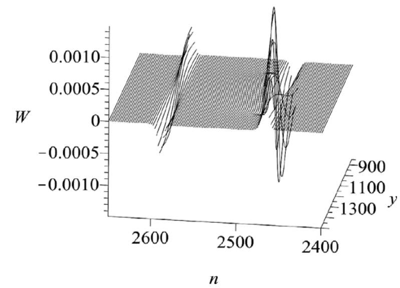

Fig. 5. Lump-two soliton interaction of W at $t = -1200$.

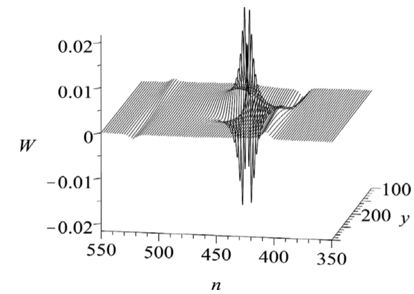

Fig. 6. Lump-two soliton interaction of W at $t = -210$.

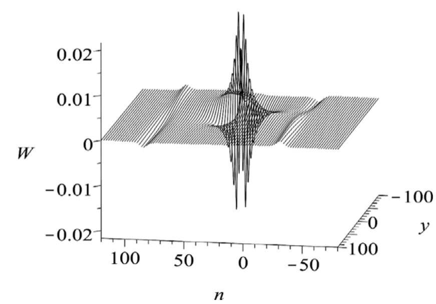

Fig. 7. Lump-two soliton interaction of W at $t = 0$.

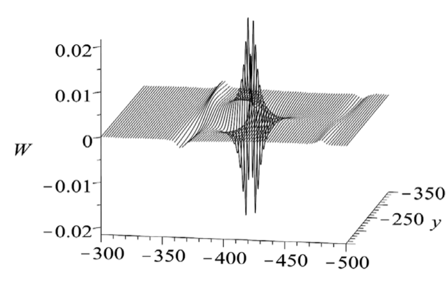

Fig. 8. Lump-two soliton interaction of W at $t = 210$.

Figs. 5–9 show that a lump fission occurs firstly from a one-soliton and then with another one-soliton to a locally coherent structure over time.

Remark 3. In Ref. [33], exact solutions expressed in terms of Pfaffian solutions of the bilinear form of the following symmetric Lotka–Volterra lattice [33]

$$\begin{aligned}
&2u_t(m,n) + e^{u(m,n)+\Delta_m^2\phi(m,n+1)} - e^{-u(m,n)+\Delta_n^2\phi(m,n)} + e^{u(m,n)+\Delta_n^2\phi(m+1,n)} \\
&-e^{-u(m,n)+\Delta_m^2\phi(m,n)} + e^{-u(m,n)+\Delta_n^2\phi(m+1,n-1)} - e^{u(m,n)+\Delta_m^2\phi(m-1,n)} \\
&+e^{-u(m,n)+\Delta_m^2\phi(m-1,n+1)} - e^{u(m,n)+\Delta_n^2\phi(m,n-1)} = 0,
\end{aligned}$$

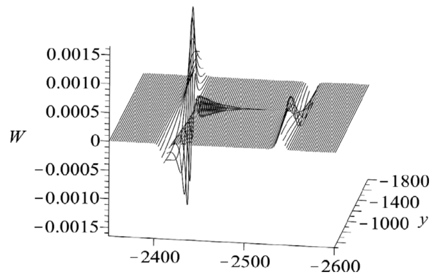

Fig. 9. Lump-two soliton interaction of W at $t = 1200$.

$$u(m,n) = \Delta_m \Delta_n \phi(m,n), \tag{23}$$

are given, where Δ_m and Δ_n are difference operators defined by

$$\Delta_m u(m,n) = u(m+1,n) - u(m,n), \qquad \Delta_n u(m,n) = u(m,n+1) - u(m,n).$$

As a special case of the Pfaffian solutions, the authors have obtained soliton solutions and dromions. An analogous argument can yield a lump solution. However, we cannot construct lump-soliton interaction solutions because of the complexity of this lattice (23). It has also become an open problem.

Conclusions

In this paper, some (2 + 1)-dimensional DΔ-mKdV systems have been studied. Based on Hirota's bilinear operator method, we have generated semi-discrete lump-soliton interaction solutions through computing possible values for the parameters in the combinations of positive quadratic functions and exponential functions. At the same time, by symbolic computing software, we have presented a few three-dimensional plots of the lump-soliton interaction solutions, to show dynamical behaviors of the obtained interaction solutions. We would like to remark that only a few DΔ lump-soliton interaction solutions have been presented, and there should be more complicated interaction solutions [34], which are worth further investigation.

The direct ansatz method adopted in this paper is a computerizable method, and it allows us to work out complicated solutions without making tedious algebraic calculations. In the consideration that the presented general procedure for computing semi-discrete lump-soliton interaction solutions is quite like the one in the continuous case, other types of locally coherent structures in the discrete and semi-discrete cases could be also explored by applying our solution procedure. Another interesting problem is to compute rogue wave and their interaction solutions with solitons to nonlocal integrable equations (see, e.g., [35,36]), and such studies will greatly amend the classical theory of partial differential equations.

CRediT authorship contribution statement

Kai Zhou: Writing – original draft, Conceptualization. **Ya-Nan Hu:** Software, Methodology. **Jun-Da Peng:** Software, Methodology. **Kai-Zhong Shi:** Software, Methodology. **Shou-Feng Shen:** Writing – review & editing. **Wen-Xiu Ma:** Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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