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Hirota bilinear equations with linear subspaces of hyperbolic and trigonometric function solutions

Hong-Chan Zheng [a,*], Wen-Xiu Ma [b], Xiang Gu [b]

[a] Department of Applied Mathematics, Northwestern Polytechnical University, 710072 Xi'an, China
[b] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA

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ABSTRACT

Linear superposition principles of hyperbolic and trigonometric function solutions are analyzed for Hirota bilinear equations, with an aim to construct a specific sub-class of N-soliton solutions formulated by linear combinations of hyperbolic and trigonometric functions. An algorithm using weights is discussed and a few illustrative application examples are presented.

1. Introduction

It is significantly important in mathematical physics to search for exact solutions to nonlinear differential equations. Exact solutions play a vital role in understanding various qualitative and quantitative features of nonlinear phenomena. There are diverse classes of interesting exact solutions, such as traveling wave solutions and soliton solutions, but it often needs specific mathematical techniques to construct exact solutions due to the nonlinearity present in dynamics [1,2]. Typical integrable equations, such as the KdV equation, the Boussinesq equation and the KP equation, possess multi-soliton solutions, generated from combinations of multiple exponential waves on the basis of their Hirota bilinear forms [3]. Various equations of mathematical physics can be written as Hirota bilinear forms through dependent variable transformations [3,4]. Wronskian solutions, including solitons, positons and complexitons [5–8], and quasi-periodic solutions [9–11] can be presented systematically based on Hirota bilinear forms. Recently, Hirota bilinear operators are generalized and their applications are presented in [12].

Besides soliton solutions, another class of interesting multiple exponential wave solutions is linear combinations of exponential waves, which implies the existence of linear subspaces of solutions. Hirota bilinear equations which possess linear subspaces of exponential traveling wave solutions are discussed and it is shown that a kind of nonlinear equations can possess such a linear superposition principle under some conditions [13,14].

In this paper, we would like to explore when Hirota bilinear equations possess linear subspaces of hyperbolic and trigonometric function solutions, aiming to construct a specific sub-class of N-soliton solutions formulated by linear combinations of hyperbolic and trigonometric functions.

Based on the Hirota bilinear formulation, we will present sufficient and necessary conditions, with an algorithm, to guarantee the applicability of the linear superposition principles to hyperbolic and trigonometric function solutions. A few illustrative examples will be computed.

* Corresponding author.
E-mail address: zhenghc@nwpu.edu.cn (H.-C. Zheng).

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2. Linear superposition principles

We begin with a Hirota bilinear equation

$$P(D_{x_1}, D_{x_2}, \ldots, D_{x_M})f \cdot f = 0, \tag{2.1}$$

where P is a polynomial in the indicated variables, satisfying

$$P(0, 0, \ldots, 0) = 0, \tag{2.2}$$

and $D_{x_i},\ 1 \leqslant i \leqslant M$, are Hirota's differential operators defined by

$$D_y^p f(y)g(y) = \left(\partial_y - \partial_{y'}\right)^p f(y)g(y')|_{y'=y} = \partial_{y'}^p f(y+y')g(y-y')|_{y'=0}, \quad p \geqslant 1.$$

Note that a term of odd degree in P produces zero in the resulting Hirota bilinear equation, and so we assume that P is an even polynomial, i.e.,

$$P(-x_1, -x_2, \ldots, -x_M) = P(x_1, x_2, \ldots, x_M). \tag{2.3}$$

Let us now introduce N wave variables:

$$\eta_i = \mathbf{k}_i \cdot \mathbf{x} = k_{1,i}x_1 + k_{2,i}x_2 + \cdots k_{M,i}x_M, \quad 1 \leqslant i \leqslant N,$$

and N exponential wave functions

$$f_i = e^{\eta_i} = e^{\mathbf{k}_i \cdot \mathbf{x}} = e^{k_{1,i}x_1 + k_{2,i}x_2 + \cdots k_{M,i}x_M}, \quad 1 \leqslant i \leqslant N,$$

where the $k_{j,i}$'s are real constants to construct a wave related vector $\mathbf{k}_i$ and $\mathbf{x}$ denotes the dependent variable vector, namely

$$\mathbf{k}_i = (k_{1,i}, k_{2,i}, \ldots, k_{M,i}), \quad \mathbf{x} = (x_1, x_2, \ldots, x_M), \quad 1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant M.$$

Now consider a linear combination:

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{\eta_i},$$

where $\varepsilon_i,\ 1 \leqslant i \leqslant N$, are arbitrary constants. It is known from [14] that any linear combination f of the N exponential wave solves the Hirota bilinear equation (2.1) if and only if the following condition

$$P(\mathbf{k}_i - \mathbf{k}_j) = P(k_{1,i} - k_{1,j}, \ldots, k_{M,i} - k_{M,j}) = 0, \quad 1 \leqslant i < j \leqslant N$$

is satisfied.

2.1. Linear superposition principle of hyperbolic function solutions

Now, suppose $f_i = \text{ch}\eta_i = \frac{1}{2}(e^{\eta_i} + e^{-\eta_i}),\ 1 \leqslant i \leqslant N$, be hyperbolic function solutions to (2.1). Based on [14], we expect to have

$$P(2\mathbf{k}_i) = P(2k_{1,i}, 2k_{2,i}, \ldots, 2k_{M,i}) = 0. \tag{2.4}$$

Set

$$f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \cdots \varepsilon_N f_N = \sum_{i=1}^{N} \varepsilon_i \text{ch}\eta_i = \sum_{i=1}^{N} \varepsilon_i \frac{1}{2}(e^{\eta_i} + e^{-\eta_i}), \tag{2.5}$$

being a general linear combination of hyperbolic function solutions. Naturally, we would like to ask if it will still present a solution to the Hirota bilinear Eq. (2.1) as each f_i does.

The answer is affirmative. We will show that a linear superposition principle of those hyperbolic function solutions will apply to Hirota bilinear equations, under some additional conditions on the hyperbolic function solutions and possibly on the polynomial P as well.

Observing that we have a bilinear identity [3]:

$$P(D_{x_1}, D_{x_2}, \ldots, D_{x_M})e^{\eta_i} \cdot e^{\eta_j} = P(\mathbf{k}_i - \mathbf{k}_j)e^{\eta_i + \eta_j} = P\left(k_{1,i} - k_{1,j}, \ldots, k_{M,i} - k_{M,j}\right)e^{\eta_i + \eta_j}. \tag{2.6}$$

Following (2.6), we can compute that

$$
\begin{aligned}
P(D_{x_1},D_{x_2},\ldots,D_{x_M})f\cdot f &= P(D_{x_1},D_{x_2},\ldots,D_{x_M})\sum_{i=1}^{N}\varepsilon_i \mathrm{ch}\eta_i\cdot\sum_{j=1}^{N}\varepsilon_i \mathrm{ch}\eta_j=\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j P(D_{x_1},D_{x_2},\ldots,D_{x_M})\mathrm{ch}\eta_i\cdot\mathrm{ch}\eta_j\\
&=\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j P(D_{x_1},D_{x_2},\ldots,D_{x_M})\frac{1}{2}(e^{\eta_i}+e^{-\eta_i})\cdot\frac{1}{2}(e^{\eta_j}+e^{-\eta_j})\\
&=\frac{1}{4}\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j P(D_{x_1},D_{x_2},\cdots,D_{x_M})(e^{\eta_i}\cdot e^{\eta_j}+e^{\eta_i}\cdot e^{-\eta_j}+e^{-\eta_i}\cdot e^{\eta_j}+e^{-\eta_i}\cdot e^{-\eta_j})\\
&=\frac{1}{4}\sum_{1\leqslant i<j\leqslant N}^{N}\varepsilon_i\varepsilon_j\left[P(\mathbf{k}_i-\mathbf{k}_j)e^{\eta_i+\eta_j}+P(\mathbf{k}_i+\mathbf{k}_j)e^{\eta_i-\eta_j}+P(-\mathbf{k}_i-\mathbf{k}_j)e^{-\eta_i+\eta_j}+P(-\mathbf{k}_i+\mathbf{k}_j)e^{-\eta_i-\eta_j}\right]\\
&\quad+\frac{1}{4}\sum_{1\leqslant i>j\leqslant N}^{N}\varepsilon_i\varepsilon_j\left[P(\mathbf{k}_i-\mathbf{k}_j)e^{\eta_i+\eta_j}+P(\mathbf{k}_i+\mathbf{k}_j)e^{\eta_i-\eta_j}+P(-\mathbf{k}_i-\mathbf{k}_j)e^{-\eta_i+\eta_j}+P(-\mathbf{k}_i+\mathbf{k}_j)e^{-\eta_i-\eta_j}\right]\\
&\quad+\frac{1}{4}\sum_{1\leqslant i=j\leqslant N}^{N}\varepsilon_i\varepsilon_j\left[P(\mathbf{k}_i-\mathbf{k}_j)e^{\eta_i+\eta_j}+P(\mathbf{k}_i+\mathbf{k}_j)e^{\eta_i-\eta_j}+P(-\mathbf{k}_i-\mathbf{k}_j)e^{-\eta_i+\eta_j}+P(-\mathbf{k}_i+\mathbf{k}_j)e^{-\eta_i-\eta_j}\right].
\end{aligned}
$$

From (2.2) and (2.3) we know that

$$
\begin{aligned}
P(D_{x_1},D_{x_2},\ldots,D_{x_M})f\cdot f &= \frac{1}{4}\sum_{1\leqslant i<j\leqslant N}^{N}\varepsilon_i\varepsilon_j\left[2P(\mathbf{k}_i-\mathbf{k}_j)e^{\eta_i+\eta_j}+2P(\mathbf{k}_i+\mathbf{k}_j)e^{\eta_i-\eta_j}+2P(\mathbf{k}_i+\mathbf{k}_j)e^{-\eta_i+\eta_j}+2P(\mathbf{k}_i-\mathbf{k}_j)e^{-\eta_i-\eta_j}\right]\\
&\quad+\frac{1}{4}\sum_{1\leqslant i\leqslant N}^{N}\varepsilon_i^2[P(\mathbf{k}_i-\mathbf{k}_i)e^{\eta_i+\eta_i}+P(\mathbf{k}_i+\mathbf{k}_i)e^{\eta_i-\eta_i}+P(-\mathbf{k}_i-\mathbf{k}_i)e^{-\eta_i+\eta_i}+P(-\mathbf{k}_i+\mathbf{k}_i)e^{-\eta_i-\eta_i}]\\
&=\frac{1}{2}\sum_{1\leqslant i<j\leqslant N}^{N}\varepsilon_i\varepsilon_j\left[P(\mathbf{k}_i-\mathbf{k}_j)(e^{\eta_i+\eta_j}+e^{-\eta_i-\eta_j})+P(\mathbf{k}_i+\mathbf{k}_j)(e^{\eta_i-\eta_j}+e^{-\eta_i+\eta_j})\right]+\frac{1}{2}\sum_{1\leqslant i\leqslant N}^{N}\varepsilon_i^2P(2\mathbf{k}_i).
\end{aligned}
\tag{2.7}
$$

It now follows directly from (2.7) and (2.4) that a linear combination function f of the N hyperbolic function solutions $f_i=\mathrm{ch}\eta_i=\frac{1}{2}(e^{\eta_i}+e^{-\eta_i}),\ 1\leqslant i\leqslant N$, solves the Hirota bilinear Eq. (2.1) if and only if the following conditions

$$P(\mathbf{k}_i-\mathbf{k}_j)=P(k_{1,i}-k_{1,j},\ldots,k_{M,i}-k_{M,j})=0,\quad 1\leqslant i\leqslant j\leqslant N, \tag{2.8}$$

$$P(\mathbf{k}_i+\mathbf{k}_j)=P(k_{1,i}+k_{1,j},\ldots,k_{M,i}+k_{M,j})=0,\quad 1\leqslant i\leqslant j\leqslant N, \tag{2.9}$$

are satisfied.

We conclude the above analysis in the following theorem.

Theorem 1 (*Linear superposition principle of hyperbolic function solutions*). *Let $P(x_1,x_2,\ldots,x_M)$ be an even polynomial satisfying $P(0,0,\ldots,0)=0$, and the N wave variables defined by $\eta_i=\mathbf{k}_i\cdot\mathbf{x}=k_{1,i}x_1+k_{2,i}x_2+\cdots k_{M,i}x_M,\ 1\leqslant i\leqslant N$, where $k_{j,i}$'s are real constants. Then any linear combination of the hyperbolic function solutions $f_i=\mathrm{ch}\,\eta_i=\frac{1}{2}(e^{\eta_i}+e^{-\eta_i}),\ 1\leqslant i\leqslant N$, solves the Hirota bilinear* Eq. (2.1) *if and only if the system*

$$P(\mathbf{k}_i\pm\mathbf{k}_j)=0,\quad 1\leqslant i\leqslant j\leqslant N, \tag{2.10}$$

defined by conditions (2.8) and (2.9), is satisfied.

Theorem 1 shows a linear superposition principle of hyperbolic function solutions that applies to Hirota bilinear equations, and paves a way of constructing N-wave solutions from linear combinations of hyperbolic function solutions to Hirota bilinear equations. The system (2.10) is a key condition we need to handle. Once we get a solution of the wave related numbers $k_{j,i}$'s by solving the system, we can present an N-wave solution, formed by (2.5), to the considered bilinear equation.

2.2. Linear superposition principle of trigonometric function solutions

Now suppose $f_i=\cos\eta_i=\frac{1}{2}\left(e^{I\eta_i}+e^{-I\eta_i}\right),\ 1\leqslant i\leqslant N$, where $\eta_i=\mathbf{k}_i\cdot\mathbf{x},\ 1\leqslant i\leqslant N,\ I=\sqrt{-1}$, be trigonometric function solution to (2.1). Based on [14], we expect to have

$$P(2I\mathbf{k}_i)=P(2Ik_{1,i},2Ik_{2,i},\ldots,2Ik_{M,i})=0. \tag{2.11}$$

Set

$$f=\varepsilon_1f_1+\varepsilon_2f_2+\cdots\varepsilon_Nf_N=\sum_{i=1}^{N}\varepsilon_i\cos\,\eta_i=\sum_{i=1}^{N}\varepsilon_i\frac{1}{2}\left(e^{I\eta_i}+e^{-I\eta_i}\right) \tag{2.12}$$

being a general linear combination of trigonometric function solutions. Similarly, we would like to ask if it will still present a solution to the Hirota bilinear Eq. (2.1) as each f_i does. The answer is still affirmative. Similarly, combining with (2.11) we know that a linear combination function f of the N trigonometric function solutions $f_i = \cos\eta_i = \frac{1}{2}(e^{I\eta_i} + e^{-I\eta_i}),\ 1 \leqslant i \leqslant N$, solves the Hirota bilinear Eq. (2.1) if and only if the following conditions

$$P(I\mathbf{k}_i - I\mathbf{k}_j) = P(Ik_{1,i} - Ik_{1,j}, \ldots, Ik_{M,i} - Ik_{M,j}) = 0, \quad 1 \leqslant i \leqslant j \leqslant N, \tag{2.13}$$

$$P(I\mathbf{k}_i + I\mathbf{k}_j) = P(Ik_{1,i} + Ik_{1,j}, \ldots, Ik_{M,i} + Ik_{M,j}) = 0, \quad 1 \leqslant i \leqslant j \leqslant N \tag{2.14}$$

are satisfied. We conclude the results in the following theorem.

Theorem 2 (*Linear superposition principle of trigonometric function solutions*). *Let $P(x_1, x_2, \ldots, x_M)$ be an even polynomial satisfying $P(0,0,0,0) = 0$, and the N wave variables defined by $\eta_i = \mathbf{k}_i \cdot \mathbf{x} = k_{1,i}x_1 + k_{2,i}x_2 + \cdots k_{M,i}x_M,\ 1 \leqslant i \leqslant N$, where $k_{j,i}$'s are real constants. Then any linear combination of the trigonometric function solutions $f_i = \cos\eta_i = \frac{1}{2}(e^{I\eta_i} + e^{-I\eta_i}),\ 1 \leqslant i \leqslant N,\ I = \sqrt{-1}$, solves the Hirota bilinear Eq. (2.1) if and only if the system*

$$P(I\mathbf{k}_i \pm I\mathbf{k}_j) = 0, \quad 1 \leqslant i \leqslant j \leqslant N, \tag{2.15}$$

defined by conditions (2.13) and (2.14), is satisfied.

Similarly, Theorem 2 shows a linear superposition principle of trigonometric function solutions that applies to Hirota bilinear equations, and paves a way of constructing N-wave solutions from linear combinations of trigonometric function solutions to Hirota bilinear equations. The system (2.15) is a key condition we need to handle. Once we get a solution of the wave related numbers $k_{j,i}$'s by solving the system, we can present an N-wave solution, formed by (2.12), to the considered bilinear equation.

Below we show two concrete examples to shed light on the usage of the linear superposition principles, Theorems 1 and 2, in constructing N-wave solutions.

The first example is the following even polynomial:

$$P(x, y, t) = x^2 - 3yt + t^2,$$

in 2 + 1 dimensions with

$$\eta_i = \mathbf{k}_i \cdot \mathbf{x} = k_i x + l_i y + \omega_i t,\ 1 \leqslant i \leqslant N.$$

The corresponding Hirota bilinear equation is

$$\left(D_x^2 - 3D_yD_t + D_t^2\right) f \cdot f = 0,$$

which is equivalent to

$$(f_{xx} - 3f_{yt} + f_{tt})f - f_x^2 + 3f_yf_t - f_t^2 = 0. \tag{2.16}$$

By inspection, a solution to the corresponding system (2.10) or (2.15) is

$$l_i = ak_i, \quad \omega_i = bk_i,\ 1 \leqslant i \leqslant N,$$

where $1 - 3ab + b^2 = 0$. Therefore by the linear superposition principles in Theorems 1 and 2, the Hirota bilinear Eq. (2.16) corresponding to the polynomial $P(x,y,z,t) = x^2 - 3yt + t^2$ has the following N-wave solution

$$f = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\eta_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}(k_i x + ak_i y + bk_i t)$$

and

$$f = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos(k_i x + ak_i y + bk_i t),$$

where a, b satisfy $1 - 3ab + b^2 = 0$, and the ε_i's and k_i's are arbitrary constants.

The second example is the following even polynomial:

$$P(x, y, z, t) = -x^4 + 2xz + y^4 + yt,$$

in 3 + 1 dimensions with

$$\eta_i = \mathbf{k}_i \cdot \mathbf{x} = k_i x + l_i y + m_i z + \omega_i t,\ 1 \leqslant i \leqslant N.$$

The corresponding Hirota bilinear equation is

$$\left(-D_x^4 + 2D_xD_z + D_y^4 + D_yD_t\right) f \cdot f = 0,$$

which is equivalent to

$$(-f_{xxxx}+2f_{xz}+f_{yt}+f_{yyyy})f-2f_xf_z+4f_{xxx}f_x-3f_{xx}^2+3f_{yy}^2-f_yf_t-4f_{yyy}f_y=0. \quad (2.17)$$

By inspection, a solution to the corresponding system (2.10) or (2.15) is

$$l_i=k_i,\quad m_i=ak_i^3,\quad \omega_i=-2ak_i^3,\ 1\leqslant i\leqslant N,$$

where a is an arbitrary constant. Therefore by the linear superposition principles in Theorems 1 and 2, the Hirota bilinear Eq. (2.17) corresponding to the polynomial $P(x,y,z,t)=-x^4+2xz+y^4+yt$ has the following N-wave solution

$$f=\sum_{i=1}^{N}\varepsilon_i\mathrm{ch}\eta_i=\sum_{i=1}^{N}\varepsilon_i\mathrm{ch}\left(k_ix+k_iy+ak_i^3z-2ak_i^3t\right),$$

or

$$f=\sum_{i=1}^{N}\varepsilon_i\cos\eta_i=\sum_{i=1}^{N}\varepsilon_i\cos\left(k_ix+k_iy+ak_i^3z-2ak_i^3t\right),$$

where a, and the ε_i's and k_i's are arbitrary constants.

3. Applications

We would like to propose an opposite procedure for conversely constructing Hirota bilinear equations that possess N-wave solutions formulated by linear combinations of hyperbolic or trigonometric functions. This is an opposite question on applying the linear superposition principles in Theorems 1 and 2. The problem can be reduced to how to construct an even multivariate polynomial $P(x_1,x_2,\ldots,x_M)$ satisfying the system (2.10) or (2.15). Based on the idea in [12–14], an algorithm using the concept of weights can be given to find a proper polynomial$P(x_1,x_2,\ldots,x_M)$.

Step 1. Define the weights of independent variables:

$$(w(x_1),w(x_2),\ldots,w(x_M))=(n_1,n_2,\ldots,n_M),$$

where each weight $w(x_i)=n_i,\ 1\leqslant i\leqslant N$, is an integer, and then form an even homogeneous polynomial $P(x_1,x_2,\ldots,x_M)$ in some weight to check if it will satisfy the system (2.10) or (2.15). A nice idea to start our checking is to assume that the wave variables η_i's involve arbitrary constants.

Step 2. Parameterize the vector $k_i=(k_{1,i},k_{2,i},\ldots,k_{M,i})$ by

$$k_{j,i}=b_jk_i^{n_i},\quad 1\leqslant i\leqslant N,\ 1\leqslant j\leqslant M,$$

where k_i are parameters and b_j's are constants to be determined to balance the system (2.10) or (2.15).

Step 3. Plug the parameterized results into the system (2.10) or (2.15) and collect terms by powers of the parameters k_i's. Setting the coefficient of each power to zero to obtain algebraic equations on the constants b_j's and the coefficients of the polynomial $P(x_1,x_2,\ldots,x_M)$.

Step 4. Solve the resulting algebraic equations to determine the polynomial $P(x_1,x_2,\ldots,x_M)$ and the parameterization.

The resulting parameterization tells that the obtained Hirota bilinear equation corresponding to the polynomial $P(x_1,x_2,\ldots,x_M)$ possesses the linear subspace of hyperbolic function solutions defined by

$$f=\sum_{i=1}^{N}\varepsilon_i\mathrm{ch}\,\eta_i=\sum_{i=1}^{N}\varepsilon_i\,\mathrm{ch}(b_1k_i^{n_1}x_1+b_2k_i^{n_2}x_2+\cdots+b_Mk_i^{n_M}x_M),\ N\geqslant 1,$$

or the linear subspace of trigonometric function solutions defined by

$$f=\sum_{i=1}^{N}\varepsilon_i\cos\eta_i=\sum_{i=1}^{N}\varepsilon_i\cos\left(b_1k_i^{n_1}x_1+b_2k_i^{n_2}x_2+\cdots+b_Mk_i^{n_M}x_M\right),\ N\geqslant 1.$$

We show some examples which apply this algorithm using weights as follows.

First we show a few examples with even polynomial being homogeneous in weight 2.

Example 1. Weights $(w(x),w(y),w(z),w(t))=(1,2,2,1)$:

Let us introduce the weights of independent variables:

$$(w(x),w(y),w(z),w(t))=(1,2,2,1).$$

Then a general even polynomial being homogeneous in weight 2 reads:

$$P(x,t) = c_1x^2 + c_2xt + c_3t^2.$$

Assume that the 1 + 1 dimensional wave variables are

$$\eta_i = k_ix + b_3k_it,\ 1 \leqslant i \geqslant N,$$

where $k_i,\ 1 \leqslant i \geqslant N$, are arbitrary constants, but b_3 are constants to be determined.

This way, a direct computation tells that the corresponding Hirota bilinear equation $P(D_x,D_t)f \cdot f = 0$ possesses the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i \text{ch}\,\eta_i = \sum_{i=1}^{N} \varepsilon_i \text{ch}(k_ix + b_3k_it),$$

or

$$f = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos(k_ix + b_3k_it),$$

where the ε_i's and k_i's are arbitrary, but b_3 satisfy

$$c_1 + c_2b_3 + c_3b_3^2 = 0.$$

Example 2. Weights $(w(x),w(y),w(z),w(t)) = (1,1,2,1)$:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 1, 2, 1).$$

Then a general even polynomial being homogeneous in weight 2 reads:

$$P(x,y,t) = c_1x^2 + c_2xy + c_3xt + c_4y^2 + c_5yt + c_6t^2.$$

A special case of which is the polynomial shown in Section 2. Assume that the 2 + 1 dimensional wave variables are

$$\eta_i = k_ix + b_1k_iy + b_3k_it,\ 1 \leqslant i \geqslant N,$$

where $k_i,\ 1 \leqslant i \geqslant N$, are arbitrary constants, but b_1 and b_3 are constants to be determined.

This way, a direct computation tells that the corresponding Hirota bilinear equation $P(D_x,D_y,D_t)f \cdot f = 0$ possesses the linear subspace of N-wave solutions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i \text{ch}\eta_i = \sum_{i=1}^{N} \varepsilon_i \text{ch}(k_ix + b_1k_iy + b_3k_it),$$

or

$$f = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos(k_ix + b_1k_iy + b_3k_it),$$

where the ε_i's and k_i's are arbitrary, but b_1 and b_3 satisfy

$$c_1 + c_2b_1 + c_3b_3 + c_4b_1^2 + c_5b_1b_3 + c_6b_3^2 = 0.$$

Example 3. Weights $(w(x),w(y),w(z),w(t)) = (1,1,1,1)$:

Let us introduce the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 1, 1, 1).$$

Then a general even polynomial being homogeneous in weight 2 reads:

$$P(x,y,z,t) = c_1x^2 + c_2xy + c_3xz + c_4xt + c_5y^2 + c_6yz + c_7yt + c_8z^2 + c_9zt + c_{10}t^2.$$

Assume that the 3 + 1 dimensional wave variables are

$$\eta_i = k_ix + b_1k_iy + b_2k_iz + b_3k_it,\ 1 \leqslant i \geqslant N,$$

where $k_i,\ 1 \leqslant i \geqslant N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

This way, a direct computation tells that the corresponding Hirota bilinear equation $P(D_x,D_y,D_z,D_t)f \cdot f = 0$ possesses the linear subspace of N-wave solutions defined by

$$f=\sum_{i=1}^{N}\varepsilon_i \mathrm{ch}\eta_i=\sum_{i=1}^{N}\varepsilon_i \mathrm{ch}(k_ix+b_1k_iy+b_2k_iz+b_3k_it),$$

or

$$f=\sum_{i=1}^{N}\varepsilon_i \cos\eta_i=\sum_{i=1}^{N}\varepsilon_i \cos(k_ix+b_1k_iy+b_2k_iz+b_3k_it),$$

where the ε_i's and k_i's are arbitrary, but b_1, b_2 and b_3 satisfy

$$c_1+c_2b_1+c_3b_2+c_4b_3+c_5b_1^2+c_6b_1b_2+c_7b_1b_3+c_8b_2^2+c_9b_2b_3+c_{10}b_3^2=0.$$

Then we show a few examples with even polynomial being homogeneous in weight 4.

Example 4. Weights $(w(x),w(y),w(z),w(t))=(1,1,3,3)$:
Let us introduce the weights of independent variables:

$$(w(x),w(y),w(z),w(t))=(1,1,3,3).$$

Then a general even polynomial being homogeneous in weight 4 reads:

$$P(x,y,z,t)=c_1x^4+c_2x^3y+c_3x^2y^2+c_4xy^3+c_5xz+c_6xt+c_7y^4+c_8yz+c_9yt.$$

A special case of which is the polynomial shown in Section 2. Assume that the 3 + 1 dimensional wave variables are

$$\eta_i=k_ix+b_1k_iy+b_2k_i^3z+b_3k_i^3t,\ 1\leqslant i\geqslant N,$$

where $k_i,\ 1\leqslant i\geqslant N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

This way, a direct computation tells that the corresponding Hirota bilinear equation $P(D_x,D_y,D_z,D_t)f\cdot f=0$ possesses the linear subspace of N-wave solutions defined by

$$f=\sum_{i=1}^{N}\varepsilon_i \mathrm{ch}\eta_i=\sum_{i=1}^{N}\varepsilon_i \mathrm{ch}\left(k_ix+b_1k_iy+b_2k_i^3z+b_3k_i^3t\right),$$

or

$$f=\sum_{i=1}^{N}\varepsilon_i \cos\eta_i=\sum_{i=1}^{N}\varepsilon_i \cos\left(k_ix+b_1k_iy+b_2k_i^3z+b_3k_i^3t\right),$$

where the ε_i's and k_i's are arbitrary, but b_1, b_2 and b_3 satisfy

$$\begin{cases} c_1+c_2b_1+c_3b_1^2+c_4b_1^3+c_7b_1^4=0,\\ c_5b_2+c_6b_3+c_8b_1b_2+c_9b_1b_3=0.\end{cases}$$

Example 5. Weights $(w(x),w(y),w(z),w(t))=(1,1,2,3)$:
Let us introduce the weights of independent variables:

$$(w(x),w(y),w(z),w(t))=(1,1,2,3).$$

Then a general even polynomial being homogeneous in weight 4 reads:

$$P(x,y,z,t)=c_1x^4+c_2x^3y+c_3x^2y^2+c_4xy^3+c_5xt+c_6y^4+c_7yt+c_8z^2.$$

Assume that the 3 + 1 dimensional wave variables are

$$\eta_i=k_ix+b_1k_iy+b_2k_i^2z+b_3k_i^3t,\ 1\leqslant i\geqslant N,$$

where $k_i,\ 1\leqslant i\geqslant N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

This way, a direct computation tells that we must have $c_8=0$ to keep the non-triviality $b_1b_2b_3\neq 0$, and b_1, b_2 and b_3 need to satisfy

$$\begin{cases} c_1+c_2b_1+c_3b_1^2+c_4b_1^3+c_6b_1^4=0,\\ c_5b_3+c_7b_1b_3=0.\end{cases}\tag{2.18}$$

It follows now that the corresponding 2 + 1 dimensional Hirota bilinear equation reads

$$\left(c_1D_x^4+c_2D_x^3D_y+c_3D_x^2D_y^2+c_4D_xD_y^3+c_5D_xD_t+c_6D_y^4+c_7D_yD_t\right)f\cdot f=0$$

and it possesses an N-wave solution in 3 + 1 dimensions defined by

$$f=\sum_{i=1}^{N}\varepsilon_i\mathrm{ch}\eta_i=\sum_{i=1}^{N}\varepsilon_i\mathrm{ch}\left(k_ix+b_1k_iy+b_2k_i^2z+b_3k_i^3t\right),$$

or

$$f=\sum_{i=1}^{N}\varepsilon_i\cos\eta_i=\sum_{i=1}^{N}\varepsilon_i\cos\left(k_ix+b_1k_iy+b_2k_i^2z+b_3k_i^3t\right),$$

where b_2, the ε_i's and k_i's are arbitrary, but b_1 and b_3 satisfy (2.18).

Example 6. Weights $(w(x),w(y),w(z),w(t))=(1,2,2,3)$:
Let us introduce the weights of independent variables:

$$(w(x),w(y),w(z),w(t))=(1,2,2,3).$$

Then a general even polynomial being homogeneous in weight 4 reads:

$$P(x,y,z,t)=c_1x^4+c_2xt+c_3y^2+c_4yz+c_5z^2.$$

Assume that the 3 + 1 dimensional wave variables are

$$\eta_i=k_ix+b_1k_i^2y+b_2k_i^2z+b_3k_i^3t,\ 1\leqslant i\geqslant N,$$

where $k_i,\ 1\leqslant i\geqslant N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

This way, a direct computation tells that we must have $c_1=c_2=0$ to keep the non-triviality $b_1b_2b_3\neq 0$, and b_1, b_2 and b_3 need to satisfy

$$c_3b_1^2+c_4b_1b_2+c_5b_2^2=0.$$

It follows now that the corresponding 1 + 1 dimensional Hirota bilinear equation reads

$$\left(c_3D_y^2+c_4D_yD_z+c_5D_z^2\right)f\cdot f=0$$

and it possesses an N-wave solution in 3 + 1 dimensions defined by

$$f=\sum_{i=1}^{N}\varepsilon_i\mathrm{ch}\,\eta_i=\sum_{i=1}^{N}\varepsilon_i\mathrm{ch}\left(k_ix+b_1k_i^2y+b_2k_i^2z+b_3k_i^3t\right),$$

or

$$f=\sum_{i=1}^{N}\varepsilon_i\cos\eta_i=\sum_{i=1}^{N}\varepsilon_i\cos\left(k_ix+b_1k_i^2y+b_2k_i^2z+b_3k_i^3t\right),$$

where b_3, the ε_i's and k_i's are arbitrary, but b_1 and b_2 satisfy $c_3b_1^2+c_4b_1b_2+c_5b_2^2=0$.

Example 7. Weights $(w(x),w(y),w(z),w(t))=(1,2,3,3)$:
Let us introduce the weights of independent variables:

$$(w(x),w(y),w(z),w(t))=(1,2,3,3).$$

Then a general even polynomial being homogeneous in weight 4 reads:

$$P(x,y,z,t)=c_1x^4+c_2xz+c_3xt+c_4y^2.$$

Assume that the 3 + 1 dimensional wave variables are

$$\eta_i=k_ix+b_1k_i^2y+b_2k_i^3z+b_3k_i^3t,\ 1\leqslant i\geqslant N,$$

where $k_i,\ 1\leqslant i\geqslant N$, are arbitrary constants, but b_1, b_2 and b_3 are constants to be determined.

This way, a direct computation tells that we must have $c_1=c_4=0$ to keep the non-triviality $b_1b_2b_3\neq 0$, and b_1, b_2 and b_3 need to satisfy

$$c_2b_2+c_3b_3=0.$$

It follows now that the corresponding 2 + 1 dimensional Hirota bilinear equation reads

$$(c_2D_xD_z + c_3D_xD_t)f \cdot f = 0$$

and it possesses an N-wave solution in 3 + 1 dimensions defined by

$$f = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\,\eta_i = \sum_{i=1}^{N} \varepsilon_i \mathrm{ch}\left(k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^3t\right),$$

or

$$f = \sum_{i=1}^{N} \varepsilon_i \cos\eta_i = \sum_{i=1}^{N} \varepsilon_i \cos\left(k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^3t\right),$$

where b_1, the ε_i's and k_i's are arbitrary, but b_2 and b_3 satisfy $c_2b_2 + c_3b_3 = 0$.

4. Conclusions

In this paper, we analyzed when Hirota bilinear equations possess the linear superposition principle of hyperbolic or trigonometric function solutions and discussed how to construct multivariable polynomials which generate such Hirota bilinear equations. An algorithm using weights and a few illustrative examples are given.

Future research questions [12–14] include how to achieve other parameterizations of wave numbers and frequencies by using several parameters and how to create multivariate polynomials whose Hirota bilinear equations possess the linear subspaces of other specific solutions, for example, mixed type function solutions like complexiton solutions [5].

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