Computers and Mathematics with Applications 74 (2017) 1399–1405

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Mixed lump–kink solutions to the KP equation

Hai-qiong Zhao [a], Wen-Xiu Ma [b,c,d,e,f,*]

[a] School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai 201620, PR China
[b] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
[c] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, PR China
[d] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, PR China
[e] College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, PR China
[f] Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

ARTICLE INFO

Article history:
Received 25 April 2017
Received in revised form 12 June 2017
Accepted 17 June 2017
Available online 8 July 2017

Keywords:
Bilinear form
Lump solution
Soliton solution

ABSTRACT

By using the Hirota bilinear form of the KP equation, twelve classes of lump–kink solutions are presented under the help of symbolic computations with Maple. Analyticity is naturally achieved by taking special choices of the involved parameters to guarantee a positive constant term. A key step in generating lump–kink solutions is to combine quadratic functions and the exponential function in the second-order logarithmic derivative transformation.

1. Introduction

Since the Korteweg–de Vries (KdV) equation is solved by the inverse scattering transform [1], there are many studies on other integrable equations [2,3]. Integrable equations possess Hirota bilinear forms [4], which generate solitons describing a kind of nonlinear physical phenomena [5]. Besides the KdV equation, such equations contain the Boussinesq equation, the Kadomtsev–Petviashvili (KP) equation, the B-Kadomtsev–Petviashvili (BKP) equation, the Volterra lattice equation, and Toda lattice equation. From the mathematical point of view, solitons are determined by exponentially localized functions [3], and Plücker relations and Pfaffian identities play a crucial role in formulating solitons in terms of determinants [6], though some intelligent guesswork is often needed [7].

The KP equation

$$P_{KP}(u) := (u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.1}$$

possesses the following class of lump solutions [8]:

$$\begin{aligned} u = 2(\ln f)_{xx}, \quad f &= \left(a_1x + a_2y + \frac{a_1{a_2}^2 - a_1{a_6}^2 + 2\,a_2a_5a_6}{{a_1}^2 + {a_5}^2}t + a_4\right)^2 \\ &+ \left(a_5x + a_6y + \frac{2a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2}{{a_1}^2 + {a_5}^2}t + a_8\right)^2 + \frac{3({a_1}^2 + {a_5}^2)^3}{(a_1a_6 - a_2a_5)^2}, \end{aligned} \tag{1.2}$$

* Corresponding author at: Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA.
E-mail address: mawx@cas.usf.edu (W.X. Ma).

http://dx.doi.org/10.1016/j.camwa.2017.06.034
0898-1221/

where the six parameters a_i's are arbitrary but $a_1a_6 - a_2a_5 \neq 0$, which guarantees that u will present lump solutions. This class contains a subclass of lump solutions presented earlier [9]:

$$u = 4\frac{-[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2}{\{[x + ay + (a^2 - b^2)t]^2 + b^2(y + 2at)^2 + 3/b^2\}^2}, \tag{1.3}$$

involving two free parameters a and b. The BKP equation

$$(u_t + 15u_xu_{xxx} + 15{u_x}^3 - 15u_xu_y + u_{5x})_x - 5u_{xxxy} - 5u_{yy} = 0 \tag{1.4}$$

has the following lump solutions [10]:

$$\begin{aligned} u &= 2(\ln f)_x, \quad f = \left(a_1x + a_2y + \frac{5(a_1{a_2}^2 - a_1{a_6}^2 + 2\,a_2a_5a_6)}{{a_1}^2 + {a_5}^2}\,t + a_4\right)^2 \\ &\quad + \left(a_5x + a_6y + \frac{5(2\,a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2)}{{a_1}^2 + {a_5}^2}\,t + a_8\right)^2 - \frac{3\,(a_1a_2 + a_5a_6)\left({a_1}^2 + {a_5}^2\right)^2}{(a_1a_6 - a_2a_5)^2}, \end{aligned} \tag{1.5}$$

where we impose the positivity condition $a_1a_2 + a_5a_6 < 0$. This class contains a subclass of lump solutions

$$u = 2(\ln f)_x = \frac{4g}{f}, \tag{1.6}$$

with

$$f = g^2 + \left[6\alpha\beta y + 180\,\alpha\beta(\alpha^2 - \beta^2)t + a_8\right]^2 + \frac{\beta^2 - \alpha^2}{4\,\alpha^2\beta^2}, \tag{1.7}$$

$$g = x + 3(\alpha^2 - \beta^2)y + 45(\alpha^4 - 6\,\alpha^2\beta^2 + \beta^4)t + a_4, \tag{1.8}$$

where $|\alpha| < |\beta|$. This is a particular subclass of lump solutions generated from taking long-wave limits of a 2-soliton solution in [11]. The class (1.5) contains another subclass of lump solutions [10]:

$$u = 2(\ln f)_x = \frac{12(\alpha^2 - \beta^2)g + 24\,\alpha\beta h}{f}, \tag{1.9}$$

with

$$f = g^2 + h^2 - \frac{81(\alpha^2 - \beta^2)(\alpha^2 + \beta^2)^4}{4\,\alpha^2\beta^2}, \tag{1.10}$$

$$g = 3(\alpha^2 - \beta^2)x + y + \frac{5(\alpha^2 - \beta^2)}{3(\alpha^2 + \beta^2)^2}\,t + a_4, \tag{1.11}$$

$$h = 6\,\alpha\beta x - \frac{10\,\alpha\beta}{3(\alpha^2 + \beta^2)^2}\,t + a_8, \tag{1.12}$$

where $|\alpha| < |\beta|$. Many other integrable equations also possess lump solutions. Such physically significant examples contain the three-dimensional three-wave resonant interaction [12], the BKP equation [10,11], the Davey–Stewartson equation II [13], and the Ishimori-I equation [14]. Moreover, symbolic computations show that various nonintegrable equations possess lump solutions as well, and such examples include $(2+1)$-dimensional generalized KP, BKP and Sawada–Kotera equations [15–17].

The Wronskian formulation, the Casoratian formulation and the Grammian or Pfaffian formulation [6] have been applied to the study of general rational solutions to integrable equations, including the KdV equation, the Boussinesq equation and the nonlinear Schrödinger equation in $(1+1)$-dimensions, the KP and BKP equations in $(2+1)$-dimensions, and the Toda and Ablowitz–Ladik type lattice equations in $(0+1)$-dimensions (see, e.g., [18–22]). Direct searches have been also conducted for general rational solutions to nonlinear partial differential equations, including generalized bilinear differential equations by Maple (see, e.g., [23–29]).

In this paper, we would like to consider a kind of interaction solutions between lumps and kinks of nonlinear partial differential equations. Kinks are traveling wave solutions, expressed by the exponential wave functions which satisfy Hirota bilinear equations. We will take the KP equation as an example, and by Maple symbolic computations, generate twelve classes of its lump–kink solutions, based on the previous studies on lumps in $(2+1)$-dimensions (see, e.g., [8]). The resulting lump–kink solutions supplement existing lumps and kinks in the literature. A key step is to search for positive combined solutions to the corresponding bilinear KP equation within the Hirota bilinear formulation. A few concluding remarks will be given in the final section.

2. Abundant lump–kink solutions

The KP equation (1.1) is a mathematical model describing water waves of long wavelength with weakly nonlinear restoring forces and frequency dispersion. It is among the entire KP hierarchy [30] and linked to a bilinear equation [4]:

$$\begin{aligned} B_{KP}(f) &:= (D_xD_t + D_x^4 - D_y^2)f\cdot f \\ &= 2(f_{xt}f - f_tf_x + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}{}^2 - f_{yy}f + f_y{}^2) = 0, \end{aligned} \tag{2.1}$$

under the second-order logarithmic derivative transformation:

$$u = 2(\ln f)_{xx}. \tag{2.2}$$

This is one of the two characteristic transformations used in Bell polynomial theories on integrable equations (see, e.g., [31,32]), and it is direct to find

$$P_{KP}(u) = \left(\frac{B_{KP}(f)}{f^2}\right)_{xx}. \tag{2.3}$$

Therefore, when f solves the bilinear KP equation (2.1), $u = 2(\ln f)_{xx}$ will present a solution to the KP equation (1.1).

In what follows, we focus on computing lump–kink solutions to the KP equation (1.1) through a careful search for combined solutions of quadratic functions and the exponential function to the bilinear KP equation (2.1) with symbolic computations. This provides innovative ideas to generate new exact solutions to integrable equations, besides soliton solutions and dromion-type solutions, and supplements other solution methods such as the Hirota perturbation technique, the inverse scattering transform, Darboux transformation, and symmetry reductions and constraints (see, e.g., [33–42]).

We adopt an ansatz of combining quadratic functions and the exponential function

$$f = \xi_1{}^2 + \xi_2{}^2 + e^{\xi_3} + a_{13}, \tag{2.4}$$

with three wave variables

$$\begin{cases} \xi_1 = a_1x + a_2y + a_3t + a_4, \\ \xi_2 = a_5x + a_6y + a_7t + a_8, \\ \xi_3 = a_9x + a_{10}y + a_{11}t + a_{12}, \end{cases} \tag{2.5}$$

where the parameters a_i's are to be determined. Plug this function f into the bilinear KP equation (2.1), to obtain a system of nonlinear algebraic equations on the parameters, and try to solve the resulting system for the parameters. The function f will generate positive combined solutions to the bilinear KP equation (2.1) and further analytical lump–kink solutions to the KP equation (1.1) under (2.2). With some classification, careful symbolic computations yield the following twelve sets of solutions for the parameters $a_i,\ 1 \le i \le 13$.

Class 1. Solution 1 by solving for all the parameters:

$$\left\{a_1 = 0,\ a_2 = b\,a_5a_9,\ a_3 = 2\,b\,a_6a_9,\ a_7 = -\frac{3\,a_9{}^2a_5{}^2 - a_6{}^2}{a_5}, a_{10} = \frac{a_6a_9}{a_5},\ a_{11} = -\frac{a_9\left(a_9{}^2a_5{}^2 - a_6{}^2\right)}{a_5{}^2},\ a_{13} = \frac{a_5{}^2}{a_9{}^2}\right\},$$

where b solves $b^2 - 3 = 0$.

Class 2. Solution 2 by solving for all the parameters:

$$\begin{aligned} &\left\{a_1 = \frac{b\,(a_5a_{10} - a_6a_9)}{a_9{}^2},\ a_2 = \frac{1}{3}\,\frac{3\,a_5a_9{}^4 + a_5a_{10}{}^2 - a_6a_9a_{10}}{b\,a_9{}^3},\right. \\ &\quad a_3 = \frac{1}{3}\,\frac{3\,a_5a_9{}^4a_{10} + 3\,a_6a_9{}^5 + a_5a_{10}{}^3 - a_6a_9a_{10}{}^2}{b\,a_9{}^4},\ a_7 = -\frac{3\,a_5a_9{}^4 + a_5a_{10}{}^2 - 2\,a_6a_9a_{10}}{a_9{}^2}, \\ &\quad \left. a_{11} = -\frac{a_9{}^4 - a_{10}{}^2}{a_9},\ a_{13} = \frac{1}{3}\,\frac{3\,a_5{}^2a_9{}^4 + a_5{}^2a_{10}{}^2 - 2\,a_5a_6a_9a_{10} + a_6{}^2a_9{}^2}{a_9{}^6}\right\}, \end{aligned}$$

where b solves $3\,b^2 - 1 = 0$.

Class 3. In terms of a_1 and a_5:

$$\begin{aligned} &\left\{a_2 = \frac{b}{a_9},\ a_3 = \frac{-3\,a_1a_9{}^4 + 2\,b\,a_{10} - a_1a_{10}{}^2}{a_9{}^2},\ a_6 = -\frac{b\,a_1 - a_1{}^2a_{10} - a_5{}^2a_{10}}{a_5a_9},\right. \\ &\quad \left. a_7 = -\frac{3\,a_5{}^2a_9{}^4 + 2\,b\,a_1a_{10} - 2\,a_1{}^2a_{10}{}^2 - a_5{}^2a_{10}{}^2}{a_5a_9{}^2}, a_{11} = -\frac{a_9{}^4 - a_{10}{}^2}{a_9},\ a_{13} = \frac{a_1{}^2 + a_5{}^2}{a_9{}^2}\right\}, \end{aligned}$$

where b solves $b^2 - 2\,a_1a_{10}b - 3\,a_5{}^2a_9{}^4 + a_1{}^2a_{10}{}^2 = 0$.

Class 4. In terms of a_3 and a_7:

$$\Big\{a_1 = \frac{a_2^2a_3 + 2\,a_2a_6a_7 - a_3a_6^2}{a_3^2 + a_7^2},\ a_5 = -\frac{a_2^2a_7 - 2\,a_2a_3a_6 - a_7a_6^2}{a_3^2 + a_7^2},$$
$$a_9 = \frac{b\,(a_2a_7 - a_3a_6)}{a_2^2 + a_6^2},\ a_{10} = \frac{b\,(a_2a_7 - a_3a_6)(a_2a_3 + a_6a_7)}{\left(a_2^2 + a_6^2\right)^2},$$
$$a_{11} = \frac{1}{3}\,\frac{b\,(a_2a_7 - a_3a_6)(3\,a_2^2a_3^2 - a_2^2a_7^2 + 8\,a_7a_3a_6a_2 - a_3^2a_6^2 + 3\,a_6^2a_7^2)}{\left(a_2^2 + a_6^2\right)^3},$$
$$a_{13} = \frac{3\,(a_2^8 + 4\,a_2^6a_6^2 + 6\,a_2^4a_6^4 + 4\,a_2^2a_6^6 + a_6^8)}{(a_2a_7 - a_3a_6)^2(a_3^2 + a_7^2)}\Big\},$$

where b solves $3\,b^2 - 1 = 0$.

Class 5. Solution 1 in terms of a_1, a_2 and a_5:

$$\Big\{a_3 = -\frac{3\,a_1^2a_9^2 + 3\,a_5^2a_9^2 - a_2^2}{a_1},\ a_6 = \frac{a_5(b\,a_2 - a_2^2 + 3\,a_1^2a_9^2 + 3\,a_5^2a_9^2)}{a_1(b - a_2)},$$
$$a_7 = \frac{2\,b\,a_1^2a_2 + 2\,b\,a_2a_5^2 - 2\,a_1^2a_2^2 - a_2^2a_5^2 + 3\,a_1^2a_5^2a_9^2 + 3\,a_5^4a_9^2}{a_1^2a_5},$$
$$a_8 = \frac{a_4a_5(b\,a_2 - a_2^2 + 3\,a_1^2a_9^2 + 3\,a_5^2a_9^2)}{a_1a_2(b - a_2)},\ a_{10} = \frac{b\,a_9}{a_1},$$
$$a_{11} = \frac{a_9(2\,b\,a_2 - a_2^2 - a_1^2a_9^2 + 3\,a_5^2a_9^2)}{a_1^2},\ a_{13} = \frac{a_1^2 + a_5^2}{a_9^2}\Big\},$$

where b solves $b^2 - 2\,a_2b + a_2^2 - 3\,a_5^2a_9^2 = 0$.

Class 6. Solution 2 in terms of a_1, a_2 and a_5:

$$\Big\{a_3 = -\frac{3\,a_1^2a_9^2 + 3\,a_5^2a_9^2 - a_2^2}{a_1},\ a_6 = \frac{b\,a_1^2 + b\,a_5^2 - a_1^2a_2}{a_1a_5},$$
$$a_7 = \frac{2\,b\,a_1^2a_2 + 2\,b\,a_2a_5^2 - 2\,a_1^2a_2^2 - a_2^2a_5^2 + 3\,a_1^2a_5^2a_9^2 + 3\,a_5^4a_9^2}{a_1^2a_5},$$
$$a_{10} = \frac{b\,a_9}{a_1},\ a_{11} = \frac{a_9(2\,b\,a_2 - a_2^2 - a_1^2a_9^2 + 3\,a_5^2a_9^2)}{a_1^2},\ a_{13} = \frac{a_1^2 + a_5^2}{a_9^2}\Big\},$$

where b solves $b^2 - 2\,a_2b + a_2^2 - 3\,a_5^2a_9^2 = 0$.

Class 7. In terms of a_2, a_3 and a_7:

$$\Big\{a_1 = \frac{a_2^2a_3 + 2\,a_2a_6a_7 - a_3a_6^2}{a_3^2 + a_7^2},\ a_5 = -\frac{a_2^2a_7 - 2\,a_2a_3a_6 - a_7a_6^2}{a_3^2 + a_7^2},$$
$$a_9 = \frac{b\,(a_2a_7 - a_3a_6)}{a_2^2 + a_6^2},\ a_{10} = \frac{b\,(a_2a_7 - a_3a_6)(a_2a_3 + a_6a_7)}{\left(a_2^2 + a_6^2\right)^2},$$
$$a_{11} = \frac{1}{3}\,\frac{b\,(a_2a_7 - a_3a_6)(3\,a_2^2a_3^2 - a_2^2a_7^2 + 8\,a_7a_3a_6a_2 - a_3^2a_6^2 + 3\,a_6^2a_7^2)}{(a_2^2 + a_6^2)^3},$$
$$a_{13} = \frac{3\,(a_2^8 + 4\,a_2^6a_6^2 + 6\,a_2^4a_6^4 + 4\,a_2^2a_6^6 + a_6^8)}{(a_2a_7 - a_3a_6)(a_2a_3^2a_7 + a_2a_7^3 - a_3^3a_6 - a_3a_6a_7^2)}\Big\},$$

where b solves $3\,b^2 - 1 = 0$.

Class 8. In terms of a_2, a_3, a_9 and a_{10}:

$$\Big\{a_1 = \frac{a_9\,(2\,a_2a_{10} - a_3a_9)}{3\,a_9^4 + a_{10}^2},\ a_5 = -\frac{1}{3}\,\frac{3\,a_2a_9^4 - a_2a_{10}^2 + a_3a_9a_{10}}{b\,a_9(3\,a_9^4 + a_{10}^2)},$$
$$a_6 = \frac{b\,(a_2a_{10} - a_3a_9)}{a_9^2},\ a_7 = \frac{1}{3}\,\frac{3\,a_2a_9^4 + a_2a_{10}^2 - a_3a_9a_{10}}{b\,a_9^3},$$
$$a_{11} = -\frac{a_9^4 - a_{10}^2}{a_9},\ a_{13} = \frac{1}{3}\,\frac{3\,a_2^2a_9^4 + a_2^2a_{10}^2 - 2\,a_2a_3a_9a_{10} + a_3^2a_9^2}{\left(3\,a_9^4 + a_{10}^2\right)a_9^4}\Big\},$$

where b solves $3\,b^2 - 1 = 0$.

Class 9. Solution 1 in terms of a_2, a_7, a_9 and a_{10}:

$$\Big\{a_1 = -\frac{a_2 a_9(3\,b\,a_9^4 + 3\,a_7 a_9^4 - b\,a_{10}^2 + a_7 a_{10}^2)}{a_{10}(3\,b\,a_9^4 - 3\,a_7 a_9^4 + b\,a_{10}^2 - a_7 a_{10}^2)},\ a_3 = \frac{a_2(3\,b\,a_9^4 + 3\,a_7 a_9^4 + b\,a_{10}^2 - a_7 a_{10}^2)}{a_9(b - a_7)a_{10}},$$
$$a_5 = \frac{b\,a_9^2}{3\,a_9^4 + a_{10}^2},\ a_6 = \frac{(b + a_7)a_9}{2\,a_{10}},\ a_{11} = -\frac{a_9^4 - a_{10}^2}{a_9},$$
$$a_{13} = \frac{c}{(3\,a_2^2 a_9^4 + b\,a_7 a_9^2 + a_2^2 a_{10}^2)a_{10}^2(3\,a_9^4 + a_{10}^2)}\Big\},$$

where b solves $b^2 - 2\,a_7 b + a_7^2 - 12\,a_2^2 a_9^2 = 0$ and

$$c = 9\,a_2^4 a_9^8 + 6\,b\,a_2^2 a_7 a_9^6 + 6\,a_2^4 a_9^4 a_{10}^2 + 12\,a_2^2 a_7^2 a_9^6 + 2\,b\,a_2^2 a_7 a_9^2 a_{10}^2 + 2\,b\,a_7^3 a_9^4 + a_{10}^4 a_2^4 - a_7^4 a_9^4.$$

Class 10. Solution 2 in terms of a_2, a_7, a_9 and a_{10}:

$$\Big\{a_1 = -\frac{(6\,a_2^2 a_9^4 + b\,a_7 a_9^2 - 2\,a_2^2 a_{10}^2 - a_7^2 a_9^2)a_9}{2\,a_2(3\,a_9^4 + a_{10}^2)a_{10}},\ a_3 = \frac{6\,a_2^2 a_9^4 + b\,a_7 a_9^2 + 2\,a_2^2 a_{10}^2 - a_7^2 a_9^2}{2\,a_2 a_9 a_{10}},$$
$$a_5 = \frac{b\,a_9^2}{3\,a_9^4 + a_{10}^2},\ a_6 = \frac{(b + a_7)a_9}{2\,a_{10}},\ a_{11} = -\frac{a_9^4 - a_{10}^2}{a_9},$$
$$a_{13} = \frac{c}{a_{10}^2(9\,a_2^2 a_9^8 + 3\,b\,a_7 a_9^6 + 6\,a_2^2 a_9^4 a_{10}^2 + b\,a_7 a_9^2 a_{10}^2 + a_2^2 a_{10}^4)}\Big\},$$

where b solves $b^2 - 2\,a_7 b + a_7^2 - 12\,a_2^2 a_9^2 = 0$ and

$$c = 9\,a_2^4 a_9^8 + 6\,b\,a_2^2 a_7 a_9^6 + 6\,a_2^4 a_9^4 a_{10}^2 + 12\,a_2^2 a_7^2 a_9^6 + 2\,b\,a_2^2 a_7 a_9^2 a_{10}^2 + 2\,b\,a_7^3 a_9^4 + a_{10}^4 a_2^4 - a_7^4 a_9^4.$$

Class 11. Solution 1 in terms of a_1, a_2, a_9 and a_{10}:

$$\Big\{a_2 = \frac{3\,a_1^2 a_9^4 + a_1^2 a_{10}^2 - a_6^2 a_9^2 + b\,a_6 a_9^2}{a_1 a_9 a_{10}},\ a_3 = \frac{3\,a_1^2 a_9^4 + 2\,b\,a_6 a_9^2 + a_1^2 a_{10}^2 - 2\,a_6^2 a_9^2}{a_1 a_9^2},$$
$$a_5 = \frac{b\,a_9}{a_{10}},\ a_7 = -\frac{3\,b\,a_9^4 + b\,a_{10}^2 - 2\,a_6 a_{10}^2}{a_9 a_{10}},\ a_{11} = -\frac{a_9^4 - a_{10}^2}{a_9},$$
$$a_{13} = \frac{c}{a_9^2 a_{10}^2(3\,a_1^2 a_9^4 + a_1^2 a_{10}^2 - a_6^2 a_9^2 + 2\,b\,a_6 a_9^2)}\Big\},$$

where b solves $b^2 - 2\,a_6 b + a_6^2 - 3\,a_1^2 a_9^2 = 0$ and

$$\begin{aligned} c = {} & 9\,a_1^4 a_9^8 + 6\,a_1^4 a_9^4 a_{10}^2 + 6\,a_1^2 a_6^2 a_9^6 + 12\,b\,a_1^2 a_6 a_9^6 + a_1^4 a_{10}^4 \\ & - 2\,a_1^2 a_6^2 a_9^2 a_{10}^2 + 4\,b\,a_1^2 a_6 a_9^2 a_{10}^2 - 3\,a_6^4 a_9^4 + 4\,b\,a_6^3 a_9^4. \end{aligned}$$

Class 12. Solution 2 in terms of a_1, a_2, a_9 and a_{10}:

$$\Big\{a_2 = \frac{a_1(9\,a_1^2 a_9^6 + 6\,b\,a_6 a_9^4 + 3\,a_1^2 a_9^2 a_{10}^2 - 3\,a_6^2 a_9^4 + b\,a_6 a_{10}^2 - a_6^2 a_{10}^2)}{a_9 a_{10}(3\,a_1^2 a_9^2 + b\,a_6 - a_6^2)},$$
$$a_3 = \frac{a_1(9\,a_1^2 a_9^6 + 9\,b\,a_6 a_9^4 + 3\,a_1^2 a_9^2 a_{10}^2 - 3\,a_6^2 a_9^4 + b\,a_6 a_{10}^2 - a_6^2 a_{10}^2)}{a_9^2(3\,a_1^2 a_9^2 + b\,a_6 - a_6^2)},$$
$$a_5 = \frac{b\,a_9}{a_{10}},\ a_7 = -\frac{3\,b\,a_9^4 + b\,a_{10}^2 - 2\,a_6 a_{10}^2}{a_9 a_{10}},\ a_{11} = -\frac{a_9^4 - a_{10}^2}{a_9},\ a_{13} = \frac{c_1}{c_2}\Big\},$$

where b solves $b^2 - 2\,a_6 b + a_6^2 - 3\,a_1^2 a_9^2 = 0$ and

$$\begin{aligned} c_1 = {} & 27\,a_1^6 a_9^{10} + 54\,b\,a_1^4 a_6 a_9^8 + 18\,a_1^6 a_9^6 a_{10}^2 + 81\,a_1^4 a_6^2 a_9^8 + 24\,b\,a_1^4 a_6 a_9^4 a_{10}^2 \\ & + 60\,b\,a_1^2 a_6^3 a_9^6 + 3\,a_1^6 a_9^2 a_{10}^4 + 12\,a_1^4 a_6^2 a_9^4 a_{10}^2 - 15\,a_1^2 a_6^4 a_9^6 + 2\,b\,a_1^4 a_6 a_{10}^4 \\ & + 8\,b\,a_1^2 a_6^3 a_9^2 a_{10}^2 + 6\,b\,a_6^5 a_9^4 - a_1^4 a_6^2 a_{10}^4 - 6\,a_6^4 a_9^2 a_{10}^2 a_1^2 - 5\,a_6^6 a_9^4, \\ c_2 = {} & a_9^2 a_{10}^2\big(9\,a_1^4 a_9^6 + 12\,b\,a_1^2 a_6 a_9^4 + 3\,a_1^4 a_9^2 a_{10}^2 + 6\,a_1^2 a_6^2 a_9^4 \\ & + 2\,b\,a_1^2 a_6 a_{10}^2 + 4\,b\,a_6^3 a_9^2 - a_6^2 a_1^2 a_{10}^2 - 3\,a_6^4 a_9^2\big). \end{aligned}$$

In each class of solutions for the parameters presented above, the parameters not expressed in the set are arbitrary, provided that all expressions are well defined. It is also evident that all determining equations for b have real solutions whatever we choose for the involved parameters, and so all solutions are sufficiently well defined. To generate more explicit solutions for the parameters, we tried to solve the resulting systems of nonlinear algebraic equations in terms of other selections of the parameters, but did not succeed in getting any other non-trivial solutions.

The above twelve sets of solutions for the parameters generate twelve classes of combined function solutions defined by (2.4), to the bilinear KP equation (2.1); and further the resulting combined solutions present twelve classes of mixed lump–kink solutions, under the transformation (2.2), to the KP equation (1.1).

The analyticity of those interaction solutions can be naturally achieved, if we choose the parameters guaranteeing $a_{13} > 0$. These lump–kink solutions are reduced to the kinks when the quadratic function disappears, and the lumps when the exponential function disappears. It is also recognized that the resulting lump–kink solutions do not tend to zero in all directions in space due to the existence of a kink wave, and they form a peak at finite times generated by the involved lump wave.

3. Concluding remarks

Through the Hirota formulation and symbolic computations with Maple, we constructed twelve classes of mixed lump–kink solutions to the KP equation explicitly, and the resulting classes of interaction solutions supplement the existing lump and kink solutions in the literature, and inspire us to compute more exact interaction solutions to integrable equations in higher-dimensions.

We point out that if we change the Hirota derivatives in (2.1) into generalized bilinear derivatives [43], the quadratic function part in the solutions presented in the previous section remains true to the generalized bilinear KP equations. It is also interesting to find positive polynomial solutions to other generalized bilinear or even tri-linear differential equations [44,45], formulated in terms of general bilinear derivatives [43], as did for resonant solutions in terms of exponential and trigonometric functions by the linear superposition principle [46–48]. This kind of polynomial solutions will present lump or lump-type solutions, including rogue wave solutions [49–51], to the corresponding nonlinear equations under the transformations $u = 2(\ln f)_x$ and $u = 2(\ln f)_{xx}$.

Note that the KP and BKP equations can be solved by the Wronskian technique [52,53]. Our search for lump–kink solutions stimulates creativity and leads to new ideas to formulate Wronskian solutions with different kinds of entries in form of combined functions. Moreover, if we use generalized bilinear derivatives [43] in (2.1), all solutions computed above will be different. For example, the KP-like equations defined by the generalized bilinear derivatives $D_{p,z}$ with $p = 3, 5$:

$$(D_{3,x}D_{3,t} + D_{3,x}^4 - D_{3,y}^2)f \cdot f = 0,$$
$$(D_{5,x}D_{5,t} + D_{5,x}^4 - D_{5,y}^2)f \cdot f = 0,$$

will have different lump–kink solutions, though lump solutions generated from quadratic functions remain unchanged [27]. Symbolic computations can be also helpful in searching for traveling wave solutions by rational function expansions around solutions to integrable ordinary differential equations [54], particularly various polynomial expansions (see, e.g., [55–57]), and multiple wave solutions to nonlinear wave equations (see, e.g., [58]).

Acknowledgments

The work was supported in part by NNSFC under the grants 11301331, 11371326, 11271008 and 11371086, NSF under the grant DMS-1664561, Natural Science Foundation of Shanghai under the grant 17ZR1411600, and the Distinguished Professorships by Shanghai University of Electric Power and Shanghai Second Polytechnic University.

References

[1] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Phys. Rev. Lett. 19 (1967) 1095–1097.
[2] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons - the Inverse Scattering Method, Consultants Bureau, New York, 1984.
[3] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[4] R. Hirota, Progr. Theoret. Phys. 52 (1974) 1498–1512.
[5] R. Hirota, Phys. Rev. Lett. 27 (1971) 1192–1194.
[6] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, New York, 2004.
[7] P.J. Caudrey, Phil. Trans. R. Soc. A 369 (2011) 1215–1227.
[8] W.X. Ma, Phys. Lett. A 379 (2015) 1975–1978.
[9] S.V. Manakov, V.E. Zakharov, L.A. Bordag, V.B. Matveev, Phys. Lett. A 63 (1977) 205–206.
[10] J.Y. Yang, W.X. Ma, Internat. J. Modern Phys. B 30 (2016) 1640028.
[11] C.R. Gilson, J.J.C. Nimmo, Phys. Lett. A 147 (1990) 472–476.
[12] D.J. Kaup, J. Math. Phys. 22 (1981) 1176–1181.
[13] J. Satsuma, M.J. Ablowitz, J. Math. Phys. 20 (1979) 1496–1503.
[14] K. Imai, Progr. Theoret. Phys. 98 (1997) 1013–1023.
[15] W.X. Ma, Z.Y. Qin, X. Lü, Nonlinear Dynam. 84 (2016) 923–931.
[16] H.Q. Zhang, W.X. Ma, Nonlinear Dynam. 87 (2017) 2305–2310.
[17] J.P. Yu, Y.L. Sun, Nonlinear Dynam. 87 (2017) 2755–2763.
[18] W.X. Ma, Discrete Contin. Dyn. Syst. Suppl. (2009) 506–515.
[19] W.X. Ma, Y. You, Trans. Amer. Math. Soc. 357 (2005) 1753–1778.
[20] W.X. Ma, Y. You, Chaos Solitons Fractals 22 (2004) 395–406.
[21] S. Chakravarty, Y. Kodama, in: W.X. Ma, X.B. Hu, Q.P. Liu (Eds.), Proceedings of Nonlinear and Modern Mathematical Physics, in: AIP Conf. Proc., vol. 1212, Amer. Inst. Phys., Melville, NY, 2010, pp. 312–341.
[22] S.T. Chen, J.B. Zhang, D.Y. Chen, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 2949–2959.
[23] W.X. Ma, M. Chen, Appl. Math. Comput. 215 (2009) 2835–2842.
[24] İ. Aslan, Romanian J. Phys. 58 (2013) 893–903.
[25] Y. Zhang, W.X. Ma, Appl. Math. Comput. 256 (2015) 252–256.

[26] Y.F. Zhang, W.X. Ma, Z. Naturf. a 70 (2015) 263–268.
[27] W.X. Ma, Y. Zhou, R. Dougherty, Internat. J. Modern Phys. B 30 (2016) 1640018.
[28] Y. Zhang, H.H. Dong, X.E. Zhang, H.W. Yang, Comput. Math. Appl. 73 (2017) 246–252.
[29] J.P. Yu, Y.L. Sun, Nonlinear Dynam. 87 (2017) 1405–1412.
[30] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, J. Phys. Soc. Japan 50 (1981) 3813–3818.
[31] C. Gilson, F. Lambert, J. Nimmo, R. Willox, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 452 (1996) 223–234.
[32] W.X. Ma, J. Phys. Conf. Ser. 411 (2013) 012021.
[33] B. Dorizzi, B. Grammaticos, A. Ramani, P. Winternitz, J. Math. Phys. 27 (1986) 2848–2852.
[34] R. Coifman, A.S. Fokas, Important Developments in Soliton Theory, in: Ser. Nonlinear Dynam., Springer, Berlin, 1993, pp. 58–85.
[35] B. Konopelchenko, W. Strampp, Inverse Problems 7 (1991) L17–L24.
[36] Q.L. Zhao, X.Y. Li, F.S. Liu, Appl. Math. Comput. 219 (2013) 5693–5705.
[37] X.X. Xu, Appl. Math. Comput. 251 (2015) 275–283.
[38] X.R. Wang, X.E. Zhang, P.Y. Zhao, Abstr. Appl. Anal. 2014 (2014) Article ID 253102.
[39] X.Y. Li, Q.L. Zhao, Y.X. Li, H.H. Dong, J. Nonlinear Sci. Appl. 8 (2015) 496–506.
[40] K.S. Govinder, B. Abraham-Shrauner, Nonlinear Anal. RWA 10 (2009) 3381–3386.
[41] Q.L. Zhao, X.Y. Li, Anal. Math. Phys. 6 (2016) 237–254.
[42] H.H. Dong, Y. Zhang, X.E. Zhang, Commun. Nonlinear Sci. Numer. Simul. 36 (2016) 354–365.
[43] W.X. Ma, Stud. Nonlinear Sci. 2 (2011) 140–144.
[44] W.X. Ma, Rep. Math. Phys. 72 (2013) 41–56.
[45] W.X. Ma, Front. Math. China 8 (2013) 1139–1156.
[46] W.X. Ma, E.G. Fan, Comput. Math. Appl. 61 (2011) 950–959.
[47] H.C. Zheng, W.X. Ma, X. Gu, Appl. Math. Comput. 220 (2013) 226–234.
[48] Ö. Ünsal, W.X. Ma, Comput. Math. Appl. 71 (2016) 1242–1247.
[49] Z.H. Xu, H.L. Chen, Z.D. Dai, Appl. Math. Lett. 37 (2014) 34–38.
[50] P. Gaillard, J. Math. Phys. 57 (2016) 063505.
[51] H.Q. Zhang, S.S. Yuan, Y. Wang, Modern Phys. Lett. B 30 (2016) 1650208.
[52] N.C. Freeman, J.J.C. Nimmo, Phys. Lett. A 95 (1983) 1–3.
[53] Y.L. Kang, Y. Zhang, L.G. Jin, Appl. Math. Comput. 224 (2013) 250–258.
[54] W.X. Ma, J.-H. Lee, Chaos Solitons Fractals 42 (2009) 1356–1363.
[55] K.A. Gepreel, Comput. Math. Appl. 72 (2016) 2072–2083.
[56] K.A. Gepreel, T.A. Nofal, K.O. Alweil, WSEAS Trans. Math. 15 (2016) 462–476.
[57] K.A. Gepreel, T.A. Nofal, N.S. Al-Sayali, Eng. Lett. 24 (2016) 29–38.
[58] W.X. Ma, T.W. Huang, Y. Zhang, Phys. Script. 82 (2010) 065003.