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A unified inverse scattering transformation for the local and nonlocal nonautonomous Gross-Pitaevskii equations

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We present the inverse scattering transformation for a nonisospectral AKNS hierarchy in which the spectral parameter is determined by an ordinary differential equation with polynomial nonlinearity, and thus, we give a unified treatment for the local and nonlocal nonautonomous Gross-Pitaevskii equations which possess the parity-time ($\mathcal{PT}$) symmetric invariance. We find that unlike the local case, the $\mathcal{PT}$-symmetry of the nonlocal Gross-Pitaevskii equation allows two different choices of the symmetry relations of the eigenfunctions which guarantee two different kinds of inverse scattering solutions. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4974772]

I. INTRODUCTION

The inverse scattering transformation (IST) is a powerful method to solve integrable nonlinear evolution equations.[1–5] It is heavily based on matrix spectral problems and can be applied to a whole hierarchy of soliton equations. The so-called AKNS-ZS formalism of the IST method was developed in two early seminal articles,[6,7] which aim to deal with isospectral problems. Later, the AKNS-ZS formalism was extended so that it can be used to deal with nonisospectral problems.[8–11] Generally speaking, one needs to solve the Gel'fand-Levitan-Marchenko integral equation in performing the inverse scattering procedure, and the inverse scattering problem can also be written as a Riemann-Hilbert factorization problem.[2]

Recently, Ablowitz and Musslimani[12] presented a nonlocal nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}Q_t(x,t) = Q_{xx}(x,t) \pm 2Q(x,t)^2 Q^*(-x,t), \tag{1}$$

where Q^* denotes the complex conjugate of Q. Like the local NLS, Equation (1) shares the $\mathcal{PT}$-symmetry,[13,14] i.e., it is invariant under the transformation $x \to -x$, $t \to -t$ as well as the conjugate transformation. In the case of classical optics, Equation (1) amounts to the invariance of the so-called self-induced potential[15] $V(x,t) = Q(x,t)Q^*(-x,t)$ under the combined action of parity and time reversal symmetry, and the nonlocality is referred to that the value of the potential $V(x,t)$ at x requires the information on $Q(x,t)$ at x as well as at $-x$.[16] The $\mathcal{PT}$-symmetry breaking within the realm of optics has been observed in experiments.[17,18] In Refs. 12 and 19, the inverse scattering transformation was developed for the nonlocal NLS Equation (1), a novel scheme called left-right Riemann-Hilbert problem was proposed, and some new features were revealed due to the nonlocality and the $\mathcal{PT}$-symmetry. In contrast with the standard NLS equation, the $\mathcal{PT}$-symmetric nonlocal

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NLS model (1) has many different properties. For example, the focusing nonlocal NLS Equation (1) (with sign "+") has both static bright and dark soliton solutions,[16] which is different from the standard NLS equation. For recent progress concerning with (1), one can refer to Refs. 20–24 and the references therein.

Under the situation of nonautonomous soliton[25–27] and soliton management,[9,28,29] in this paper, we consider the following nonautonomous Gross-Pitaevskii (GP) equation:

$$\mathrm{i}Q_t(x,t) + f(x,t)Q_{xx}(x,t) + g(x,t)Q(x,t)^2Q^*(\epsilon x,t) + V(x,t)Q(x,t) + \mathrm{i}\gamma(x,t)Q(x,t) = 0, \quad (2)$$

where $\epsilon = \pm 1$. When $\epsilon = +1$, the equation is local, and we denote it by (GP_+); when $\epsilon = -1$, the equation is nonlocal, and we denote it by (GP_-). In (2), $f(x,t)$ and $g(x,t)$ are the dispersion and nonlinearity management parameters, respectively; $V(x,t)$ denotes the external potential, and $\gamma(x,t)$, the dissipation (loss) $(\gamma > 0)$ or gain $(\gamma < 0)$. (GP_+) appears in the realm of Bose-Einstein condensates,[30,31] nonlinear optics,[32,33] and inhomogeneous Heisenberg spin chain,[29,34,35] and (GP_-) is its nonlocal version with $\mathcal{PT}$-symmetric self-induced potential.

It is known from Painlevé test[27] that the integrability condition for GP_+ is $f(x,t) = f(t)$, $g(x,t) = g(t)$, $\gamma(x,t) = \gamma(t)$, $V(x,t) = V_0(t) + V_1(t)x + V_2(t)x^2$, where $V_0(t)$ and $V_1(t)$ are arbitrary real functions, $f(t)$, $g(t)$, $\gamma(t)$, $V_2(t)$ are real functions satisfying the constrained relation

$$(4f^2gg_t - 2ff_tg^2)\gamma - 4f^2g^2\gamma^2 - 2f^2g^2\gamma_t - g^2ff_{tt} + f^2gg_{tt} - 2f^2g_t^2 + f_t^2g^2 + f_tgfg_t + 4V_2f^3g^2 = 0, \quad (3)$$

and thus the integrable GP_+ model is

$$\mathrm{i}Q_t(x,t) + f(t)Q_{xx}(x,t) + g(t)Q(x,t)^2Q^*(x,t) + (V_0(t) + V_1(t)x + V_2(t)x^2)Q(x,t) + \mathrm{i}\gamma(t)Q(x,t) = 0. \quad (4)$$

Similarly, by Painlevé test, we confirm that the integrability condition for the GP_- equation is the same as that of the GP_+ equation (4), in addition to that the parameter $V_1(t)$ should be a pure imaginary function and thus we replace $V_1(t)$ with $i\tilde{V}_1(t)$, where $\tilde{V}_1(t)$ is also a real function. Then the integrable GP_- model reads

$$\mathrm{i}Q_t(x,t) + f(t)Q_{xx}(x,t) + g(t)Q(x,t)^2Q^*(-x,t) + (V_0(t) + V_2(t)x^2)Q(x,t) + \mathrm{i}\left(\tilde{V}_1(t)x + \gamma(t)\right)Q(x,t) = 0, \quad (5)$$

where $f(t)$, $g(t)$, $\gamma(t)$, $V_2(t)$ satisfy the constrained relation (3).

We remark that in a special case $f(t) = -1, g(t) = -2, V_0(t) = 0, V_1(t) = 0, \tilde{V}_1(t) = 0$, (4) is reduced to the classical focusing cubic nonlinear Schrödinger equation while (5) is reduced to the focusing nonlocal nonlinear Schrödinger equation (1).

The main purpose of this paper is to establish the inverse scattering transformation for both (4) and (5) in a uniform way. Different from the procedure in Refs. 12 and 19, here we use the Gel'fand-Levitan-Marchenko equation to carry out the method. We start from a novel nonisospectral AKNS hierarchy in which the spectral parameter is determined by an ordinary differential equation with polynomial nonlinearity, and then we investigate its inverse scattering transformation reductions for both local and nonlocal equations. The new condition of our nonisospectral AKNS hierarchy induces many novel integrable equations, and our result is valid for all the induced equations. Especially, we compute exact solutions for the reduced equations (4) and (5). We find that for the GP_+ equation, there is only one choice of the symmetry relations of the eigenfunctions, but for the GP_- equation, there are two different choices: one is similar to that in Ref. 19, and another seems to be a new try. Thus for the GP_- equation, we obtain two different solutions.

The paper is organized as follows: In Section II, we construct a nonisospectral AKNS hierarchy in which the spectral parameter determined by an ordinary differential equation with polynomial nonlinearity and present some reduced integrable equations. In Section III, we develop the inverse scattering transformation for the above hierarchy. In Section IV, in order to construct solutions for the local and nonlocal GP equation, we analyze the symmetry reductions of the inverse scattering data. In Section V, we give some explicit solutions for (4) and (5). Finally in Section VI, we give a brief summary.

II. THE NONISOSPECTRAL AKNS HIERARCHY

We start from the 2×2 linear eigenvalue problem

$$\phi_x = U\phi, \quad U = \begin{pmatrix} -\mathrm{i}k & q \\ r & \mathrm{i}k \end{pmatrix}, \tag{6}$$

where $q = q(x,t)$ and $r = r(x,t)$ are potentials, and $\phi = \phi(x,t,k(t))$ is a two component column vector representing the eigenfunction (we have suppressed the variables for convenience). We consider the time-dependent case of the spectral parameter $k = k(t)$,

$$k_t = \sum_{j=0}^{n} f_j k^{n-j}, \tag{7}$$

where $f_j = f_j(t)$ are smooth complex functions of t. The time evolution of the eigenfunction reads

$$\phi_t = V\Phi, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{8}$$

where A, B, C are polynomials of the spectral parameter k, which will be determined later. The zero curvature equation of the systems (6) and (8), i.e., $U_t - V_x + [U,V] = 0$, yields

$$\begin{cases} A = \partial^{-1}(qC - rB) - \mathrm{i}k_t x + A_0, \\ q_t = B_x + 2\mathrm{i}kB + 2qA, \\ r_t = C_x - 2\mathrm{i}kC - 2rA, \end{cases} \tag{9}$$

where $A_0 = A_0(t)$ is the integration constant of A with respect to x, and we assume

$$A_0 = \sum_{j=0}^{n} g_j k^{n-j}, \tag{10}$$

where $g_j = g_j(t)$ are smooth complex functions of t. Moreover, set

$$B = \sum_{j=1}^{n} b_j k^{n-j}, \qquad C = \sum_{j=1}^{n} c_j k^{n-j}, \qquad n = 1,2,\ldots, \tag{11}$$

where $b_j = b_j(x,t), c_j = c_j(x,t), j = 1,\ldots,n,$ will be determined later. Substituting (7), (10), and (11) into (9), comparing the coefficients of k, and denoting

$$\mathcal{F}_j = \mathcal{F}(f_j, g_j) = (xf_j + \mathrm{i}g_j)\begin{pmatrix} q \\ r \end{pmatrix}, \qquad j = 0,\ldots n, \tag{12}$$

we derive the recursion relations

$$\begin{pmatrix} b_1 \\ c_1 \end{pmatrix} = \mathcal{F}_0, \tag{13}$$

$$\begin{pmatrix} b_j \\ c_j \end{pmatrix} = L\begin{pmatrix} b_{j-1} \\ c_{j-1} \end{pmatrix} + \mathcal{F}_{j-1} = \sum_{l=0}^{j-1} L^l \mathcal{F}_{j-1-l}, \qquad j = 2,\ldots,n, \tag{14}$$

$$\begin{pmatrix} -q \\ r \end{pmatrix}_t = 2\mathrm{i}L\begin{pmatrix} b_n \\ c_n \end{pmatrix} + 2\mathrm{i}\mathcal{F}_n = 2\mathrm{i}\sum_{l=0}^{n} L^l \mathcal{F}_{n-l}, \qquad n = 1,2,\ldots, \tag{15}$$

where the operator L is defined by

$$L = \begin{pmatrix} \frac{\mathrm{i}}{2}\partial - \mathrm{i}q\partial^{-1}r & \mathrm{i}q\partial^{-1}q \\ -\mathrm{i}r\partial^{-1}r & -\frac{\mathrm{i}}{2}\partial + \mathrm{i}r\partial^{-1}q \end{pmatrix}.$$

Substituting (14) into (11), we obtain

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{n} \sum_{l=0}^{j-1} L^l \mathcal{F}_{j-1-l} k^{n-j}, \quad n = 1,2,\ldots. \tag{16}$$

(15) gives a nonisospectral AKNS hierarchy, from which we obtain some novel integrable equations. The first few sets are as follows:
$n = 1$:

$$\begin{cases} q_t = f_0(qx)_x + \mathrm{i}g_0 q_x - 2\mathrm{i}f_1 qx + 2g_1 q, \\ r_t = f_0(rx)_x + \mathrm{i}g_0 r_x + 2\mathrm{i}f_1 rx - 2g_1 r, \end{cases} \tag{17}$$

$n = 2$:

$$\begin{cases} q_t = \mathrm{i}f_0\zeta_1 - g_0\xi_1 + f_1(qx)_x + \mathrm{i}g_1 q_x - 2\mathrm{i}f_2 qx + 2g_2 q, \\ r_t = -\mathrm{i}f_0\tilde{\zeta}_1 + g_0\tilde{\xi}_1 + f_1(rx)_x + \mathrm{i}g_1 r_x + 2\mathrm{i}f_2 rx - 2g_2 r, \end{cases} \tag{18}$$

where

$$\zeta_1 = \frac{1}{2}q_{xx}x - q^2 rx - q\int qr dx + q_x, \quad \xi_1 = \frac{1}{2}q_{xx} - q^2 r,$$
$$\tilde{\zeta}_1 = \frac{1}{2}r_{xx}x - r^2 qx - r\int qr dx + r_x, \quad \tilde{\xi}_1 = \frac{1}{2}r_{xx} - r^2 q,$$

$n = 3$:

$$\begin{cases} q_t = f_0\zeta_2 + \mathrm{i}g_0\xi_2 + \mathrm{i}f_1\zeta_1 - g_1\xi_1 + f_2(qx)_x + \mathrm{i}g_2 q_x - 2\mathrm{i}f_3 qx + 2g_3 q, \\ r_t = f_0\tilde{\zeta}_2 + \mathrm{i}g_0\tilde{\xi}_2 - \mathrm{i}f_1\tilde{\zeta}_1 + g_1\tilde{\xi}_1 + f_2(rx)_x + \mathrm{i}g_2 r_x + 2\mathrm{i}f_3 rx - 2g_3 r, \end{cases} \tag{19}$$

where

$$\zeta_2 = \frac{1}{2}q\int (q_{xx}r - qr_{xx})x dx + 2q\int q_x r dx + \frac{1}{2}q_x\int qr dx - \frac{3}{4}q_{xx} + (\frac{1}{2}q^2 r_x + qrq_x$$
$$-\frac{1}{4}q_{xxx})x,$$
$$\xi_2 = \frac{1}{2}q\int (q_{xx}r - qr_{xx})x dx + qrq_x + \frac{1}{2}q^2 r_x - \frac{1}{4}q_{xxx},$$
$$\tilde{\zeta}_2 = \frac{1}{2}r\int (qr_{xx} - q_{xx}r)x dx - 2r\int q_x r dx + \frac{1}{2}r_x\int qr dx - \frac{3}{4}r_{xx} + (\frac{1}{2}r^2 q_x + qrr_x$$
$$-\frac{1}{4}r_{xxx})x + 2r^2 q,$$
$$\tilde{\xi}_2 = \frac{1}{2}r\int (qr_{xx} - q_{xx}r)x dx + qrr_x + \frac{1}{2}r^2 q_x - \frac{1}{4}r_{xxx}.$$

We consider the reduction $r(x,t) = -q^*(\epsilon x,t)$, and in order to make the two equations of (15) compatible under this reduction, one has to impose extra symmetry conditions on the parameters f_j and g_j, $j = 0,\ldots,n$. Precisely, substituting $r(x,t) = -q^*(\epsilon x,t)$ into (15), we have the following two cases.

(i) When $\epsilon = +1$, f_j and g_j should satisfy

$$f_j^* = f_j, \qquad g_j^* = -g_j, \qquad j = 0,\ldots,n, \tag{20}$$

which implies that f_j are real functions of t and g_j are imaginary functions of t. In this case, the system (15) induces local integrable equations.

(ii) When $\epsilon = -1$, f_j and g_j should satisfy

$$\begin{cases} f_{n-2k}^*(t) = -f_{n-2k}(t), \qquad f_{n-(2k+1)}^*(t) = f_{n-(2k+1)}(t), \\ g_{n-2k}^*(t) = -g_{n-2k}(t), \qquad g_{n-(2k+1)}^*(t) = g_{n-(2k+1)}(t), \qquad k = 0,\ldots,[\frac{n}{2}], \end{cases} \tag{21}$$

where [.] denotes the greatest integer function. (21) implies that f_k and g_k are imaginary functions when n and k have the same parity, and real functions when n and k have a different parity. In this case, the system (15) induces nonlocal integrable equations.

Especially, for the case $n = 2$, set

$$f_0 = 0, \qquad f_1 = (\ln \frac{g}{f})_t - 2\gamma, \qquad g_0 = -2\mathrm{i} f, \qquad g_1 = 0, \qquad g_2 = \frac{1}{2}\mathrm{i}V_0, \tag{22}$$

$$f_2 = \begin{cases} -\frac{1}{2}V_1, & \text{when } \epsilon = 1, \\ -\frac{1}{2}\mathrm{i}\tilde{V}_1, & \text{when } \epsilon = -1, \end{cases} \tag{23}$$

where we have omitted the variable t for convenience. When $\epsilon = +1$, (18) is reduced to

$$\begin{cases} \mathrm{i}q_t + fq_{xx} - 2fq^2r - \mathrm{i}x((\ln \frac{g}{f})_t - 2\gamma)q_x + V_0 q + \mathrm{i}(V_1 x - (\ln \frac{g}{f})_t + 2\gamma)q = 0, \\ \mathrm{i}r_t - fr_{xx} + 2fr^2q - \mathrm{i}x((\ln \frac{g}{f})_t - 2\gamma)r_x - V_0 r - \mathrm{i}(V_1 x + (\ln \frac{g}{f})_t - 2\gamma)r = 0. \end{cases} \tag{24}$$

When $\epsilon = -1$, (18) is reduced to

$$\begin{cases} \mathrm{i}q_t + fq_{xx} - 2fq^2r - \mathrm{i}x((\ln \frac{g}{f})_t - 2\gamma)q_x + (V_0 - \tilde{V}_1 x)q - \mathrm{i}((\ln \frac{g}{f})_t - 2\gamma)q = 0, \\ \mathrm{i}r_t - fr_{xx} + 2fr^2q - \mathrm{i}x((\ln \frac{g}{f})_t - 2\gamma)r_x - (V_0 - \tilde{V}_1 x)r - \mathrm{i}((\ln \frac{g}{f})_t - 2\gamma)r = 0. \end{cases} \tag{25}$$

With the transformation

$$q(x,t) = \frac{1}{\sqrt{2}}\sqrt{\frac{g}{f}}Q(x,t)\,\mathrm{e}^{\frac{\mathrm{i}}{2}\theta(t)x^2}, \qquad r(x,t) = -q^*(\epsilon x,t), \tag{26}$$

where

$$\theta(t) = \frac{1}{2f}(\ln \frac{g}{f})_t - \frac{\gamma}{f},$$

(24) is transformed to (4) and (25) is transformed to (5).

For the case $n = 3$, upon setting

$$\begin{cases} f_0 = 0, & f_1 = 0, & f_2 = 0, & f_3 = 0, \\ g_0 = 4\mathrm{i}, & g_1 = 0, & g_2 = 0, & g_3 = 0, \end{cases}$$

(19) is reduced to

$$\begin{cases} q_t - q_{xxx} + 6qrq_x = 0, \\ r_t - r_{xxx} + 6qrr_x = 0. \end{cases} \tag{27}$$

When we taking $r(x,t) = -1$, (27) is the standard KdV equation, and when we taking $r(x,t) = q(-x,t)$, (27) is reduced to a nonlocal KdV-like equation,

$$q_t(x,t) - q_{xxx}(x,t) + 6q(x,t)q(-x,t)q_x(x,t) = 0.$$

III. THE INVERSE SCATTERING TRANSFORMATION

In this section, we establish the IST for the nonisospectral AKNS hierarchy (15) with the spectral parameter satisfying (7), which gives a generalization to that in Ref. 11, where the spectral parameter is given by $\eta_t = \frac{1}{2}(2\eta)^n$. In addition, by using the Gel'fand-Levitan-Marchenko equation, we obtain the potentials $q(x,t)$ and $r(x,t)$ for the local and nonlocal equations in a unified way, and we find that the $\mathcal{PT}$-symmetry of the nonlinear evolution equations can be directly used to obtain the potentials $q(x,t)$ and $r(x,t)$.

A. Direct scattering problem

First, let us define the scattering data of the spectral problems (6) and (8). We assume that $q(x,t)$ and $r(x,t)$ decay rapidly at infinity, and then the scattering problem (6) is a homogeneous equation and has four eigenfunctions which satisfy the following boundary conditions:

$$\lim_{x\to-\infty}\phi(x,k)=\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-\mathrm{i}kx},\qquad \lim_{x\to-\infty}\bar{\phi}(x,k)=\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{\mathrm{i}kx},\tag{28}$$

$$\lim_{x\to+\infty}\psi(x,k)=\begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{\mathrm{i}kx},\qquad \lim_{x\to+\infty}\bar{\psi}(x,k)=\begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-\mathrm{i}kx},\tag{29}$$

where $\phi(x,k),\bar{\phi}(x,k),\psi(x,k),\bar{\psi}(x,k)$ are all two component column vectors, and denoted by

$$\phi(x,k)=(\phi_1(x,k),\phi_2(x,k))^T,\qquad \bar{\phi}(x,k)=(\bar{\phi}_1(x,k),\bar{\phi}_2(x,k))^T,$$
$$\psi(x,k)=(\psi_1(x,k),\psi_2(x,k))^T,\qquad \bar{\psi}(x,k)=(\bar{\psi}_1(x,k),\bar{\psi}_2(x,k))^T.$$

In addition, $\phi(x,k)$ and $\psi(x,k)$ are analytic for k on $\Im k>0$ ($\Im$ denoting the imaginary part of a complex number or function), and $\bar{\phi}(x,k)$ and $\bar{\psi}(x,k)$ are analytic for k on $\Im k<0$.[2,11] Moreover, by using (6), a simple computation shows that

$$\frac{\partial}{\partial x}W(\phi(x,k),\bar{\phi}(x,k))=0,\tag{30}$$

where the Wronskian is $W(\phi,\bar{\phi})=\phi_1\bar{\phi}_2-\phi_2\bar{\phi}_1$. Thus ϕ and $\bar{\phi}$ are linearly independent for all k satisfying $a(k)\neq 0$. Similarly, the solutions ψ and $\bar{\psi}$ are linearly independent for all k satisfying $b(k)\neq 0$. Additionally, since (6) is a second order ordinary differential equation, the two bases $\{\phi,\bar{\phi}\}$ and $\{\psi,\bar{\psi}\}$ are linearly dependent and one can express one set in terms of the other,

$$\begin{cases}\phi(x,k)=a(k)\bar{\psi}(x,k)+b(k)\psi(x,k),\\ \bar{\phi}(x,k)=\bar{a}(k)\psi(x,k)+\bar{b}(k)\bar{\psi}(x,k),\end{cases}\tag{31}$$

where $a(k),b(k),\bar{a}(k),\bar{b}(k)$ are scattering coefficients. If we denote $\Phi(x,k)=(\phi(x,k),\bar{\phi}(x,k))^T$, $\Psi(x,k)=(\bar{\psi}(x,k),\psi(x,k))^T$, (31) can be rewritten as

$$\Phi(x,k)=S(k)\Psi(x,k),\qquad S(k)=\begin{pmatrix}a(k)&b(k)\\ \bar{b}(k)&\bar{a}(k)\end{pmatrix}.\tag{32}$$

From (32), we obtain the scattering matrix $S(k)=\Psi(x,k)^{-1}\Phi(x,k)$, i.e.,

$$\begin{cases}a(k)=W(\phi(x,k),\psi(x,k)), & \bar{a}(k)=W(\bar{\psi}(x,k),\bar{\phi}(x,k)),\\ b(k)=W(\bar{\psi}(x,k),\phi(x,k)), & \bar{b}(k)=W(\bar{\phi}(x,k),\psi(x,k)).\end{cases}\tag{33}$$

Now from the analyticity properties of the eigenfunctions, we know that $a(k)$ is analytic in the upper half-complex k plane while $\bar{a}(k)$ is analytic in the lower half-complex k plane.[2] $b(k)$ and $\bar{b}(k)$ cannot be extended off the real k axis. Furthermore, $a(k)$ and $\bar{a}(k)$ have a finite number of single roots in their own half-k-plane, denoted by $k_j (j=1,2,\ldots,l)$ and $\bar{k}_j (j=1,2,\ldots,\bar{l})$, respectively. In addition, $\det S(k)=1$, i.e.,

$$a(k)\bar{a}(k)-b(k)\bar{b}(k)=1.$$

Furthermore, denoting $b_j=b(k_j)$, $\bar{b}_j=\bar{b}(\bar{k}_j)$, by the standard procedure,[5,11] we can prove that there exist constants c_j and $\bar{c}_j$ satisfying

$$2\int_{-\infty}^{\infty}{c_j}^2\psi_1(x,k_j)\psi_2(x,k_j)dx=1,\qquad 2\int_{-\infty}^{\infty}{\bar{c}_j}^2\bar{\psi}_1(x,\bar{k}_j)\bar{\psi}_2(x,\bar{k}_j)dx=1,$$

where

$${c_j}^2=\mathrm{i}b_j/\dot{a}(k_j),\qquad {\bar{c}_j}^2=-\mathrm{i}\bar{b}_j/\dot{\bar{a}}(\bar{k}_j).\tag{34}$$

c_j and $\bar{c}_j$ are named the normalization constants for the eigenfunctions $\psi(x,k_j)$ and $\bar{\psi}(x,\bar{k}_j)$, and $c_j\psi(x,k_j),\bar{c}_j\bar{\psi}(x,\bar{k}_j)$ are the corresponding normalization eigenfunctions, respectively.

Similar to Ref. 11, hereafter we call the set

$$\begin{aligned} S(k) = \big[&k(Imk = 0), \quad R(k) = b(k)/a(k), \quad \bar{R}(k) = \bar{b}(k)/\bar{a}(k);\\ &k_j(Imk_j > 0), \quad c_j, \quad j = 1,2,\dots,l,\\ &\bar{k}_j(Im\bar{k}_j < 0), \quad \bar{c}_j, \quad j = 1,2,\dots,\bar{l}\big] \end{aligned} \tag{35}$$

the scattering data of the spectral problem (6), where $R(k)$ and $\bar{R}(k)$ are the reflection coefficients corresponding to the continuous spectral k, while k_j and $\bar{k}_j$ are discrete spectral parameters, with c_j and $\bar{c}_j$ being norming constants.

B. The inverse scattering problem: Recovery of the potentials $q(x)$ and $r(x)$

The potentials and the scattering data have a one-to-one correspondence, that is to say,

$$\{q(x,0),r(x,0)\} \to S(\lambda,0), \quad \{q(x,t),r(x,t)\} \to S(\lambda,t).$$

The direct scattering problem is to map the potentials into the scattering data. The scattering data are determined by the eigenvalues and the behavior of eigenfunctions, and it is associated with the spectral problem (6). Precisely, at time $t = 0$, for a given initial potential $q(x,0)$, solving the scattering problem (6) and deriving the corresponding eigenfunctions, then from (33) and (35), we obtain the initial scattering data. The inverse scattering problem is to reconstruct the potentials $\{q(x,t),r(x,t)\}$ from the scattering data $S(\lambda,t)$, which is the required solutions of the nonlinear evolution equations. Let us recall how to construct exact solutions of the integrable equations by virtue of the Gel'fand-Levitan-Marchenko integral equations. The following result can be proved by an analogous argument to the one in Ref. 11.

Lemma 1. Given the scattering data (35) of the spectral problem (6), by using the Gel'fand-Levitan-Marchenko integral equations, the nonisospectral AKNS hierarchy has the exact solutions,

$$\begin{aligned} q(x,t) &= 2tr(W^{-1}(x,t)\bar{\Lambda}(x,t)\bar{\Lambda}^T(x,t)),\\ q(x,t)r(x,t) &= -2\frac{\partial}{\partial x}tr\big(W^{-1}(x,t)E(x,t)\frac{\partial}{\partial x}E^T(x,t)\big), \end{aligned} \tag{36}$$

where $tr(A)$ denotes the trace of the matrix A, and

$$E(x,t) = \Big(\frac{\mathrm{i}c_j\bar{c}_j}{k_j-\bar{k}_j}\mathrm{e}^{\mathrm{i}(k_j-\bar{k}_j)x}\Big)_{\bar{l}\times l}, \qquad W(x,t) = I + E(x,t)E^T(x,t),$$

where I is an $\bar{l}\times\bar{l}$ unit matrix, $k_j,\bar{k}_j,c_j,\bar{c}_j$ are functions of t, and

$$\Lambda(x,t) = (c_1\mathrm{e}^{\mathrm{i}k_1x},c_2\mathrm{e}^{\mathrm{i}k_2x},\dots,c_l\mathrm{e}^{\mathrm{i}k_lx})^T, \qquad \bar{\Lambda}(x,t) = (\bar{c}_1\mathrm{e}^{-\mathrm{i}\bar{k}_1x},\bar{c}_2\mathrm{e}^{-\mathrm{i}\bar{k}_2x},\dots,\bar{c}_{\bar{l}}\mathrm{e}^{-\mathrm{i}\bar{k}_{\bar{l}}x})^T.$$

(i) When $l = \bar{l} = 1$, we have $\Lambda(x,t) = c_1\mathrm{e}^{\mathrm{i}k_1x}$ and $\bar{\Lambda}(x,t) = \bar{c}_1\mathrm{e}^{-\mathrm{i}\bar{k}_1x}$, and thus, we obtain the one-soliton solution of the spectral problem (6),

$$q(x,t) = \frac{2\bar{c}_1{}^2}{W(x,t)}\mathrm{e}^{-2\mathrm{i}\bar{k}_1x}, \qquad r(x,t) = \frac{2c_1{}^2}{W(x,t)}\mathrm{e}^{2\mathrm{i}k_1x}, \tag{37}$$

where

$$W(x,t) = 1 + \frac{c_1{}^2\bar{c}_1{}^2}{(k_1-\bar{k}_1)^2}\mathrm{e}^{2\mathrm{i}(k_1-\bar{k}_1)x}.$$

(ii) When $l = \bar{l} = 2$, we have $\Lambda(x,t) = (c_1\mathrm{e}^{\mathrm{i}k_1x},c_2\mathrm{e}^{\mathrm{i}k_2x})$ and $\bar{\Lambda}(x,t) = (\bar{c}_1\mathrm{e}^{-\mathrm{i}\bar{k}_1x},\bar{c}_2\mathrm{e}^{-\mathrm{i}\bar{k}_2x})$, and thus, we obtain the two-soliton solution of the spectral problem (6),

$$q(x,t) = \frac{2\triangle_1}{\det(W(x,t))}, \qquad r(x,t) = \frac{2\triangle_2}{\det(W(x,t))}, \tag{38}$$

where

$$\triangle_1 = \bar{c}_1{}^2\mathrm{e}^{-2\mathrm{i}\bar{k}_1x} + \bar{c}_2{}^2\mathrm{e}^{-2\mathrm{i}\bar{k}_2x} + \left[\frac{c_1\bar{c}_2\bar{c}_1(\bar{k}_2-\bar{k}_1)}{(k_1-\bar{k}_1)(k_1-\bar{k}_2)}\right]^2\mathrm{e}^{2\mathrm{i}(k_1-\bar{k}_1-\bar{k}_2)x}$$

$$+\left[\frac{\bar{c}_1 c_2 \bar{c}_2(\bar{k}_2-\bar{k}_1)}{(k_2-\bar{k}_1)(k_2-\bar{k}_2)}\right]^2 \mathrm{e}^{2\mathrm{i}(k_2-\bar{k}_1-\bar{k}_2)x},$$

$$\triangle_2 = {c_1}^2\mathrm{e}^{2\mathrm{i}k_1x} + {c_2}^2\mathrm{e}^{2\mathrm{i}k_2x} + \left[\frac{c_1c_2\bar{c}_1(k_2-k_1)}{(k_1-\bar{k}_1)(k_2-\bar{k}_1)}\right]^2 \mathrm{e}^{2\mathrm{i}(k_1+k_2-\bar{k}_1)x}$$

$$+\left[\frac{c_1c_2\bar{c}_2(k_2-k_1)}{(k_1-\bar{k}_2)(k_2-\bar{k}_2)}\right]^2 \mathrm{e}^{2\mathrm{i}(k_1+k_2-\bar{k}_2)x},$$

$$\det(W(x,t)) = 1 + \left(\frac{c_1\bar{c}_1}{k_1-\bar{k}_1}\right)^2 \mathrm{e}^{2\mathrm{i}(k_1-\bar{k}_1)x} + \left(\frac{c_1\bar{c}_2}{k_1-\bar{k}_2}\right)^2 \mathrm{e}^{2\mathrm{i}(k_1-\bar{k}_2)x}$$

$$+\left(\frac{c_2\bar{c}_1}{k_2-\bar{k}_1}\right)^2 \mathrm{e}^{2\mathrm{i}(k_2-\bar{k}_1)x} + \left(\frac{c_2\bar{c}_2}{k_2-\bar{k}_2}\right)^2 \mathrm{e}^{2\mathrm{i}(k_2-\bar{k}_2)x}$$

$$+\left[\frac{c_1c_2\bar{c}_1\bar{c}_2(k_2-k_1)(\bar{k}_2-\bar{k}_1)}{(k_1-\bar{k}_1)(k_1-\bar{k}_2)(k_2-\bar{k}_1)(k_2-\bar{k}_2)}\right]^2 \mathrm{e}^{2\mathrm{i}(k_1+k_2-\bar{k}_1-\bar{k}_2)x}.$$

C. Time evolution of the scattering data

From the above discussion, in order to obtain the solutions $\{q(x,t), r(x,t)\}$ of the nonlinear evolution equations of the nonisospectral AKNS hierarchy, we need to solve the inverse scattering problem. We are just in hand of a final step that is to determine the scattering data $k_j(t), \bar{k}_j(t)$ and $c_j(t), \bar{c}_j(t)$, and then we can obtain the potentials $q(x,t)$ and $r(x,t)$ from (36). In what follows, we compute the time evolution of the scattering data (35), which is determined by the linear evolution equation (8).

Recall that we have assumed that $k_j = k_j(t)(j = 1,2,\ldots,l)$ are the single roots of $a(k)$, and $\bar{k}_j = \bar{k}_j(t)(j = 1,2,\ldots,\bar{l})$ are the single roots of $\bar{a}(k)$. In addition, the spectral parameter $k(t)$ satisfies (7), and thus from (7), we directly obtain the time evolution of the scattering data $k_j(t)$ and $\bar{k}_j(t)$,

$$k_{j,t} = \sum_{j=0}^{n} f_j {k_j}^{n-j}, \qquad \bar{k}_{j,t} = \sum_{j=0}^{n} f_j {\bar{k}_j}^{n-j}. \tag{39}$$

The computation of the norming constants is a little complicated, but the procedure is standard. We need the following lemma, which can be found in Ref. 2.

Lemma 2. Assume that $\phi(x,k)$ is a solution of (6), and the matrices U and V satisfy the zero curvature condition $U_t - V_x + [U,V] = 0$. Then

$$P(x,k) = \phi_t(x,k) - V\phi(x,k)$$

is a solution of (6) as well.

Moreover, recall that the integration constant A_0 satisfies (10), such that

$$A_0(k_j) = \sum_{j=0}^{n} g_j {k_j}^{n-j}, \qquad A_0(\bar{k}_j) = \sum_{j=0}^{n} g_j {\bar{k}_j}^{n-j}. \tag{40}$$

Then following the idea in Ref. 11, we have the following theorem.

Theorem 1. The time dependence of the discrete scattering data $c_j(t), \bar{c}_m(t)$, $j = 1,2,\ldots,l, m = 1,2,\ldots,\bar{l}$, for the spectral problem (6) is given by

$$c_{j,t} = (\alpha(k_j) - A_0(k_j))c_j, \qquad \bar{c}_{m,t} = (\bar{\alpha}(\bar{k}_m) + A_0(\bar{k}_m))\bar{c}_m, \tag{41}$$

where $A_0(k_j)$ and $A_0(\bar{k}_m)$ are defined in (40), and $\alpha(k_j), \bar{\alpha}(\bar{k}_m)$ are defined by

$$\alpha(k_j) = \frac{1}{2}\sum_{p=1}^{n}\sum_{l=0}^{p-1} {k_j}^{n-p+l} f_{p-1-l}, \qquad \bar{\alpha}(\bar{k}_m) = \frac{1}{2}\sum_{p=1}^{n}\sum_{l=0}^{p-1} \bar{k}_m^{\,n-p+l} f_{p-1-l}. \tag{42}$$

Proof. First, let us prove the first equation of (41). Taking $k = k_j$, consider $\psi(x,k_j)$ as the normalization eigenfunction $c_j\psi(x,k_j)$. From Lemma 2, we know that $P(x,k_j) = \psi_t(x,k_j) - V\psi(x,k_j)$ is a solution of (6), too, where V is defined in (8). Such that $P(x,k_j)$ can be represented linearly by $\psi(x,k_j)$ and $\chi(x,k_j)$, i.e., there exist two constants α and β such that

$$\psi_t(x,k_j) - V\psi(x,k_j) = \alpha\psi(x,k_j) + \beta\chi(x,k_j), \tag{43}$$

where $\chi(x,k_j)$ also satisfies (6) and is independent of $\psi(x,k_j)$. Due to the asymptotic condition (29), when $x \to +\infty$, we obtain $\beta = 0$, and thus (43) is reduced to

$$\psi_t(x,k_j) - V\psi(x,k_j) = \alpha\psi(x,k_j). \tag{44}$$

On the other hand, since $q(x,t)$ and $r(x,t)$ decay rapidly at infinity, from (16), we obtain $\lim_{|x|\to\infty} B(x,t) = 0$, $\lim_{|x|\to\infty} C(x,t) = 0$, and then from the first equation of (9), we get $\lim_{|x|\to\infty} A(x,t) = \bar{A}(k) = -\mathrm{i}k_t x + A_0$. Summarizing these results, we have

$$\lim_{|x|\to\infty} V(x,t) = \begin{pmatrix} \bar{A}(k) & 0 \\ 0 & -\bar{A}(k) \end{pmatrix}. \tag{45}$$

Moreover, from (29), we obtain

$$\lim_{x\to+\infty} \psi(x,k_j) = c_j e^{\mathrm{i}k_j x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{46}$$

Taking $x \to +\infty$ and substituting (45) and (46) into (44), we have

$$(c_{j,t} + \mathrm{i}k_{j,t}c_j x)e^{\mathrm{i}k_j x}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_j\bar{A}(k_j)e^{\mathrm{i}k_j x}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha c_j e^{\mathrm{i}k_j x}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and noting that $\bar{A}(k_j) = -\mathrm{i}k_{j,t}x + A_0(k_j)$, we thus have

$$c_{j,t} = (\alpha - A_0(k_j))c_j.$$

In what follows, we compute the constant α. Left-multiplying (44) by $(\psi_2(x,k_j),\psi_1(x,k_j))$ yields

$$\frac{\mathrm{d}}{\mathrm{dt}}(\psi_1(x,k_j)\psi_2(x,k_j)) - (C\psi_1(x,k_j)^2 + B\psi_2(x,k_j)^2) = 2\alpha\psi_1(x,k_j)\psi_2(x,k_j). \tag{47}$$

Integrating (47) with respect to x from $-\infty$ to $+\infty$, and using (34), we obtain

$$\alpha = -\int_{-\infty}^{+\infty} (C\psi_1(x,k_j)^2 + B\psi_2(x,k_j)^2)dx. \tag{48}$$

In the following, we suppress the variables x and k_j in $\psi(x,k_j)$ for convenience, and define an inner product

$$\langle\theta,\xi\rangle = \int_{-\infty}^{+\infty} \theta\xi dx = \int_{-\infty}^{+\infty} B{\psi_2}^2 + C{\psi_1}^2 dx, \quad \theta = ({\psi_2}^2,{\psi_1}^2), \quad \xi = (B,C)^T,$$

and then (48) is rewritten as

$$\alpha = -\langle\theta,\xi\rangle.$$

By using (16), we obtain

$$\alpha = -\sum_{p=1}^{n}\sum_{l=0}^{p-1}\langle\theta, L^l\mathcal{F}_{p-1-l}\rangle k_j^{n-p} = -\sum_{p=1}^{n}\sum_{l=0}^{p-1}\langle((L^*)^l\theta^T)^T,\mathcal{F}_{p-1-l}\rangle k_j^{n-p}, \tag{49}$$

where L^* is the adjoint operator of L, which satisfies

$$\langle\theta, L\mathcal{F}_p\rangle = \langle(L^*\theta^T)^T,\mathcal{F}_p\rangle. \tag{50}$$

The left hand side of (50) is computed as follows:

$$\langle\theta, L\mathcal{F}_p\rangle = \langle\theta, \frac{\mathrm{i}}{2}\begin{pmatrix} f_p(qx)_x + \mathrm{i}g_p q_x \\ -f_p(rx)_x - \mathrm{i}g_p r_x \end{pmatrix}\rangle$$

$$
\begin{aligned}
&= \frac{\mathrm{i}}{2} f_p \int_{-\infty}^{\infty} ((qx)_x \psi_2{}^2 - (rx)_x \psi_1{}^2) dx - \frac{1}{2} g_p \int_{-\infty}^{\infty} (q_x \psi_2^2 - r_x \psi_1^2) dx \\
&= \mathrm{i} f_p \int_{-\infty}^{\infty} x (r\psi_1 \psi_{1x} - q \psi_2 \psi_{2x}) dx - g_p \int_{-\infty}^{\infty} (r \psi_1 \psi_{1x} - q \psi_2 \psi_{2x}) dx \\
&= f_p k_j \int_{-\infty}^{\infty} x (q \psi_2{}^2 + r \psi_1{}^2) dx + \mathrm{i} g_p k_j \int_{-\infty}^{\infty} (\psi_1 \psi_{2x} + \psi_2 \psi_{1x}) dx \\
&= f_p k_j \int_{-\infty}^{\infty} x (q \psi_2{}^2 + r \psi_1{}^2) dx, \qquad (51)
\end{aligned}
$$

where we have used the fact,

$$
r\psi_1\psi_{1x} - q\psi_2\psi_{2x} = -\mathrm{i}k_j \left(q\psi_2{}^2 + r\psi_1{}^2\right) = -\mathrm{i}k_j \left(\psi_1\psi_{2x} + \psi_2\psi_{1x}\right), \qquad (52)
$$

which is computed from (6). On the other hand, a simple computation shows that

$$
\langle \theta, \mathcal{F}_p \rangle = f_p \int_{-\infty}^{\infty} x(q\psi_2{}^2 + r\psi_1{}^2)dx + \mathrm{i}g_p \int_{-\infty}^{\infty} (q\psi_2{}^2 + r\psi_1{}^2)dx = f_p \int_{-\infty}^{\infty} x(q\psi_2{}^2 + r\psi_1{}^2)dx. \qquad (53)
$$

Comparing (51) and (53), we obtain

$$
\langle \theta, L\mathcal{F}_p \rangle = \langle k_j \theta, \mathcal{F}_p \rangle,
$$

and then from (50), we have

$$
L^* \theta^T = k_j \theta^T. \qquad (54)
$$

By using (34) and (53), we get

$$
\langle \theta, \mathcal{F}_p \rangle = f_p \int_{-\infty}^{\infty} x(q\psi_2{}^2 + r\psi_1{}^2)dx = f_p \int_{-\infty}^{\infty} x(\psi_1\psi_{2x} + \psi_2\psi_{1x})dx = -\frac{1}{2} f_p. \qquad (55)
$$

Then from (49), (54), and (55), we obtain

$$
\alpha = -\sum_{p=1}^{n} \sum_{l=0}^{p-1} \langle k_j^l \theta, \mathcal{F}_{p-1-l} \rangle k_j^{n-p} = \frac{1}{2} \sum_{p=1}^{n} \sum_{l=0}^{p-1} k_j^{n-p+l} f_{p-1-l}.
$$

Similarly, to prove the second equation of (41), taking $k = \bar{k}_m$, we consider $\bar{\psi}(x, \bar{k}_m)$ as the normalization eigenfunction $\bar{c}_m \bar{\psi}(x, \bar{k}_m)$ of (6). From Lemma 2, $\bar{P}(x, \bar{k}_m) = \bar{\psi}_t(x, \bar{k}_m) - V\bar{\psi}(x, \bar{k}_m)$ is a solution of (6), and it can be represented linearly by $\bar{\psi}(x, \bar{k}_m)$ and $\bar{\chi}(x, \bar{k}_m)$, i.e., there exist two constants $\bar{\alpha}$ and $\bar{\beta}$ such that

$$
\bar{\psi}_t(x, \bar{k}_m) - V\bar{\psi}(x, \bar{k}_m) = \bar{\alpha}\bar{\psi}(x, \bar{k}_m) + \bar{\beta}\bar{\chi}(x, \bar{k}_m), \qquad (56)
$$

where $\bar{\chi}(x, \bar{k}_m)$ also satisfies (6) and is independent of $\bar{\psi}(x, \bar{k}_m)$. When $x \to +\infty$, from the asymptotic condition (29), we obtain $\bar{\beta} = 0$ such that (56) is reduced to

$$
\bar{\psi}_t(x, \bar{k}_m) - V\bar{\psi}(x, \bar{k}_m) = \bar{\alpha}\bar{\psi}(x, \bar{k}_m). \qquad (57)
$$

On the other hand, from (29), we obtain

$$
\lim_{x \to +\infty} \bar{\psi}(x, \bar{k}_m) = \bar{c}_m \mathrm{e}^{-\mathrm{i}\bar{k}_m x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (58)
$$

Taking $x \to +\infty$ and substituting (45) and (58) into (57), we derive

$$
(\bar{c}_{m,t} - \mathrm{i}\bar{k}_{m,t}\bar{c}_m x)\mathrm{e}^{-\mathrm{i}\bar{k}_m x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \bar{c}_m \bar{A}(\bar{k}_m)\mathrm{e}^{-\mathrm{i}\bar{k}_m x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \bar{\alpha}\bar{c}_m \mathrm{e}^{-\mathrm{i}\bar{k}_m x} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

where $\bar{A}(\bar{k}_m) = -\mathrm{i}\bar{k}_{m,t} x + A_0(\bar{k}_m)$. Finally, we obtain

$$
\bar{c}_{m,t} = (\bar{\alpha} + A_0(\bar{k}_m))\bar{c}_m.
$$

The constant $\bar{\alpha}$ is computed analogously to α. □

Furthermore, by using Lemma 2, after a standard argument, we can prove the time evolution of the continuous scattering data.

Theorem 2. The time dependence of the continuous scattering data $R(k) = b(k)/a(k), \bar{R}(k) = \bar{b}(k)/\bar{a}(k)$ for the spectral problem (6) is given by

$$a_t = 0, \qquad b_t = -2A_0(k)b, \qquad \bar{a}_t = 0, \qquad \bar{b}_t = 2A_0(k)\bar{b}, \tag{59}$$

where $A_0(k)$ is defined in (10). Moreover, the time dependence of the reflection coefficients reads

$$R(k,t) = R(k,0)\mathrm{e}^{-2\int_0^t A_0(k)\mathrm{d}t}, \qquad \bar{R}(k,t) = \bar{R}(k,0)\mathrm{e}^{2\int_0^t A_0(k)\mathrm{d}t}, \tag{60}$$

where $R(k,0) = b(k,0)/a(k,0), \bar{R}(k,0) = \bar{b}(k,0)/\bar{a}(k,0)$.

IV. LOCAL AND NONLOCAL REDUCTIONS

In this section, we consider the reductions of the inverse scattering transformation of the AKNS hierarchy (15) under the conditions (20) and (21), which induce local and nonlocal integrable equations, respectively. The symmetry reduction $r(x,t) = -q^*(\epsilon x,t), \epsilon = \pm 1$ results in important symmetry relations of the eigenfunctions and thus imposes different symmetries in the scattering data, which leads to different solutions. We discuss it in detail as follows.

A. Symmetry of the eigenfunctions

Let $\chi(x,k) = (\chi_1(x,k), \chi_2(x,k))^T$ be a solution of system (6) with symmetry reduction $r(x,t) = -q^*(\epsilon x,t), \epsilon = \pm 1$. Then $\left(\chi_2^*(\epsilon x, \epsilon k^*), -\epsilon\chi_1^*(\epsilon x, \epsilon k^*)\right)$ also satisfies the scattering problem (6) such that we obtain important symmetry relations of the eigenfunctions,

$$\textbf{(i)} \quad \bar{\psi}(x,k) = \begin{pmatrix} \psi_2^*(\epsilon x, \epsilon k^*) \\ -\epsilon\psi_1^*(\epsilon x, \epsilon k^*) \end{pmatrix}, \qquad \bar{\phi}(x,k) = \begin{pmatrix} -\epsilon\phi_2^*(\epsilon x, \epsilon k^*) \\ \phi_1^*(\epsilon x, \epsilon k^*) \end{pmatrix} \tag{61}$$

or

$$\textbf{(ii)} \quad \psi(x,k) = \begin{pmatrix} \phi_2^*(\epsilon x, \epsilon k^*) \\ -\epsilon\phi_1^*(\epsilon x, \epsilon k^*) \end{pmatrix}, \qquad \bar{\psi}(x,k) = \begin{pmatrix} -\epsilon\bar{\phi}_2^*(\epsilon x, \epsilon k^*) \\ \bar{\phi}_1^*(\epsilon x, \epsilon k^*) \end{pmatrix}, \tag{62}$$

where the eigenfunctions $\phi(x,k)$ and $\bar{\phi}(x,k)$ satisfy the boundary conditions (28), and $\psi(x,k)$ and $\bar{\psi}(x,k)$ satisfy the boundary conditions (29). We mention that here we have two kinds of choices of the symmetry relations of the eigenfunctions. In the case of $\epsilon = -1$, the first choice is a new try and the second choice is similar to that in Ref. 19. In the case of $\epsilon = 1$, the first choice is the classical one but the second choice does not work.

B. Symmetry of the scattering data

From (30), we know that the Wronskian representations (33) of the scattering data do not depend on x. Now from (33) and the symmetry relations of the eigenfunctions (61) and (62), we derive symmetry relations of the scattering data. We discuss the two cases (61) and (62) separately.

1. Symmetry of the scattering data under the first symmetry relations of the eigenfunctions

First, let us consider the symmetry relation (61). From the representation (33) and the condition (61), we obtain a symmetry of the scattering coefficients,

$$\bar{a}(k) = \epsilon a^*(\epsilon k^*), \qquad \bar{b}(k) = -b^*(\epsilon k^*). \tag{63}$$

In this case, if k_j is a zero of $a(k)$, ϵk_j^* is a zero of $\bar{a}(k)$ such that the roots of $a(k)$ and $\bar{a}(k)$ appear in pairs, $l = \bar{l}$, and

$$\bar{k}_j = \epsilon k_j^*. \tag{64}$$

Thus the spectral parameters k_j and $\bar{k}_j$ are both in the whole axis and satisfy (64). In order to obtain the symmetry reductions of the normalization constants c_j and $\bar{c}_j$, recalling (34), we need to compute $\dot{a}(k_j)$ and $\dot{\bar{a}}(\bar{k}_j)$. In fact, for a general N eigenvalues $k_j, \bar{k}_j, j = 1, 2, \ldots, N$, where we have set $l = \bar{l} = N$, from the trace formula[3]

$$a(k) = \prod_{j=1}^{N} \frac{k - k_j}{k - \bar{k}_j}, \qquad \bar{a}(k) = \prod_{j=1}^{N} \frac{k - \bar{k}_j}{k - k_j},$$

we get

$$\dot{a}(k_n) = \lim_{k \to k_n} \prod_{j=1}^{N} \frac{k - k_j}{k - \bar{k}_j} \sum_{l=1}^{N} \frac{k_l - \bar{k}_l}{(k - k_l)(k - \bar{k}_l)}, \tag{65}$$

$$\dot{\bar{a}}(\bar{k}_n) = \lim_{k \to \bar{k}_n} \prod_{j=1}^{N} \frac{k - \bar{k}_j}{k - k_j} \sum_{l=1}^{N} \frac{\bar{k}_l - k_l}{(k - k_l)(k - \bar{k}_l)}. \tag{66}$$

By using (64), we obtain

$$\dot{\bar{a}}(\bar{k}_j) = \epsilon \dot{a}^*(k_j). \tag{67}$$

On the other hand, the relations (63) and (64) lead to

$$\bar{b}_j = \bar{b}(\bar{k}_j) = -b^*(\epsilon \bar{k}_j^*) = -b^*(k_j) = -b_j^*. \tag{68}$$

Now substituting (67) and (68) into (34), we obtain the symmetry reductions of the normalization constants c_j and $\bar{c}_j$,

$$\bar{c}_j^{\,2} = -c_j^{*2},$$

i.e.,

$$\bar{c}_j = \pm \mathrm{i} c_j^*. \tag{69}$$

2. Symmetry of the scattering data under the second symmetry relations of the eigenfunctions

The symmetry relations of the eigenfunctions (62) lead to a symmetry of the scattering coefficients,

$$a(k) = a^*(\epsilon k^*), \qquad \bar{a}(k) = \bar{a}^*(\epsilon k^*), \qquad \bar{b}(k) = \epsilon b^*(\epsilon k^*). \tag{70}$$

In this case, if k_j is a zero of $a(k)$, ϵk_j^* is a zero of $a(k)$, too. Similarly, if $\bar{k}_j$ is a zero of $\bar{a}(k)$, so is $\epsilon \bar{k}_j^*$. Therefore the zeros $k_j, j = 1, \ldots, l$ of $a(k)$ and the zeros $\bar{k}_j, j = 1, \ldots, \bar{l}$ of $\bar{a}(k)$ should satisfy

$$k_j = \epsilon k_j^*, \qquad \bar{k}_j = \epsilon \bar{k}_j^*, \tag{71}$$

which implies that in the case $\epsilon = 1$, k_j and $\bar{k}_j$ are all real functions, and in the case $\epsilon = -1$, they are all imaginary functions. Ablowitz considered such a kind of IST transform for the nonlocal nonlinear Schrödinger equation ($\epsilon = -1$) through the left-right Riemann-Hilbert boundary value problem.[12] We are now in the step to compute the symmetry reductions of the normalization constants c_j and $\bar{c}_j$. The following analysis is similar to that in Ref. 12.

From the trace formula (65) and the first equation of (71), we obtain

$$\dot{a}(k_j) = \epsilon \dot{a}^*(k_j). \tag{72}$$

Similarly, from the trace formula (66) and the second equation of (71), we obtain

$$\dot{\bar{a}}(\bar{k}_j) = \epsilon \dot{\bar{a}}^*(\bar{k}_j). \tag{73}$$

In order to compute the symmetry reductions of b_j and $\bar{b}_j$, recall that k_j is the roots of $a(k)$ whereas $\bar{k}_j$ is the roots of $\bar{a}(k)$, in other words, $a(k_j) = 0$, $\bar{a}(\bar{k}_j) = 0$. Therefore, substituting $k = k_j$ into the first equation of (31), and substituting $k = \bar{k}_j$ into the second equation of (31), respectively, we obtain

$$\phi(x, k_j) = b_j \psi(x, k_j), \qquad \bar{\phi}(x, \bar{k}_j) = \bar{b}_j \bar{\psi}(x, \bar{k}_j). \tag{74}$$

On the other hand, we know from the symmetry relations of the eigenfunctions (62) that

$$\psi_1(x, k_j) = \phi_2^*(\epsilon x, \epsilon k_j^*), \qquad \psi_2(x, k_j) = -\epsilon \phi_1^*(\epsilon x, \epsilon k_j^*). \tag{75}$$

Thus

$$\phi_2(x, k_j) = b_j \psi_2(x, k_j) = -\epsilon b_j \phi_1^*(\epsilon x, \epsilon k_j^*). \tag{76}$$

From the first equation of (74), we know that

$$\phi_1(x, k_j) = b_j \psi_1(x, k_j),$$

which implies

$$\phi_1^*(\epsilon x, \epsilon k_j^*) = b_j^* \psi_1^*(\epsilon x, \epsilon k_j^*). \tag{77}$$

Then from the first equation of (75), we obtain

$$\psi_1^*(\epsilon x, \epsilon k_j^*) = \phi_2(x, k_j). \tag{78}$$

Substituting (78) into (77), and then substituting (77) into (76), we obtain

$$\phi_2(x, k_j) = b_j \psi_2(x, k_j) = -\epsilon b_j b_j^* \phi_2(x, k_j), \tag{79}$$

and thus b_j satisfies

$$-\epsilon |b_j|^2 = 1, \tag{80}$$

i.e.,

$$|b_j|^2 = -\epsilon. \tag{81}$$

(81) implies that $\epsilon = -1$, and the phase of b_j is arbitrary. Thus this kind of symmetry reduction (62) is only valid for the nonlocal case $\epsilon = -1$. A similar result holds for $\bar{b}_j$,

$$-\epsilon |\bar{b}_j|^2 = 1, \qquad |\bar{b}_j|^2 = -\epsilon.$$

In what follows, we only consider the case $\epsilon = -1$ and set

$$k_j = \mathrm{i}\eta_j, \qquad \bar{k}_j = -\mathrm{i}\bar{\eta}_j, \tag{82}$$

where $\eta_j, \bar{\eta}_j$ are all real functions of t, and we have suppressed the variable. Now from the expression (34), we obtain

$$|c_j|^2 = 1/|\dot{a}(k_j)|, \qquad |\bar{c}_j|^2 = 1/|\dot{\bar{a}}(\bar{k}_j)|, \tag{83}$$

where $\dot{a}(k_j)$ is expressed in (65), and $\dot{\bar{a}}(\bar{k}_1)$ is expressed in (66).

Especially, for the case $N = 1$, from (65) and (66), we derive

$$\dot{a}(k_1) = 1/(k_1 - \bar{k}_1), \quad \dot{\bar{a}}(\bar{k}_1) = 1/(\bar{k}_1 - k_1), \tag{84}$$

and substituting (82) into (84), we obtain

$$|\dot{a}(k_1)| = |\dot{\bar{a}}(\bar{k}_1)| = 1/(|\eta_1 + \bar{\eta}_1|). \tag{85}$$

Moreover, substituting (85) into (83), we obtain

$$|c_1|^2 = |\bar{c}_1|^2 = |\eta_1 + \bar{\eta}_1|. \tag{86}$$

V. APPLICATIONS TO THE GP_+ AND THE GP_-

In this section, we apply our results to the local GP_+ equation and the nonlocal GP_- equation and present some explicit solutions for Equations (4) and (5).

A. The scattering data of the $n = 2$ case

First, let us refine our discussion in the $n = 2$ case and compute the scattering data concretely. From the expansion conditions (7) and (10), we obtain

$$k_{j,t} = f_0 k_j^2 + f_1 k_j + f_2, \qquad \bar{k}_{j,t} = f_0 \bar{k}_j^2 + f_1 \bar{k}_j + f_2,$$
$$A_0(k_j) = g_0 k_j^2 + g_1 k_j + g_2, \qquad \bar{A}_0(\bar{k}_j) = g_0 \bar{k}_j^2 + g_1 \bar{k}_j + g_2.$$

In addition, it is easy to get from (42) that

$$\alpha(k_j) = \frac{1}{2} f_1 + k_j f_0, \qquad \bar{\alpha}(\bar{k}_j) = \frac{1}{2} f_1 + \bar{k}_j f_0.$$

Thus from Theorem 1, we obtain

$$c_{j,t} = (\frac{1}{2} f_1 + f_0 k_j - g_0 k_j^2 - g_1 k_j - g_2) c_j, \tag{87}$$
$$\bar{c}_{j,t} = (\frac{1}{2} f_1 + f_0 \bar{k}_j + g_0 \bar{k}_j^2 + g_1 \bar{k}_j + g_2) \bar{c}_j. \tag{88}$$

Equations (87), (88), and (89) determine the scattering data $k_j, \bar{k}_j$ and $c_j, \bar{c}_j$. Especially, for the GP equation, since $f_0 = 0$, we obtain

$$\begin{cases} k_j = (\int_0^t f_2 \mathrm{e}^{-\int f_1 dt} dt + \omega_{1j}) \mathrm{e}^{\int_0^t f_1 dt}, \\ \bar{k}_j = (\int_0^t f_2 \mathrm{e}^{-\int_0^t f_1 dt} dt + \omega_{3j}) \mathrm{e}^{\int_0^t f_1 dt}, \\ c_j = \omega_{2j} \mathrm{e}^{\int_0^t (\frac{1}{2} f_1 - g_0 k_j^2 - g_1 k_j - g_2) dt}, \\ \bar{c}_j = \omega_{4j} \mathrm{e}^{\int_0^t (\frac{1}{2} f_1 + g_0 \bar{k}_j^2 + g_1 \bar{k}_j + g_2) dt}, \end{cases} \tag{89}$$

where ω_{ij}, $i = 1,2,3,4, j = 1,\ldots,N$, are arbitrary complex numbers.

B. Solutions of the local GP_+ equation

Setting $\epsilon = +1$, substituting (22) and (23) into (89), and denoting

$$\Gamma = \int_0^t \gamma dt, \quad b = \left(\ln \frac{g}{f}\right)_t - 2\gamma, \quad M = \frac{g}{f} \mathrm{e}^{-2\Gamma}, \tag{90}$$
$$L = \int \frac{V_1}{M} dt, \tag{91}$$

we obtain

$$k_j(t) = M\big(-\frac{1}{2} L + \omega_{1j}\big), \qquad \bar{k}_j(t) = M\big(-\frac{1}{2} L + \omega_{3j}\big), \tag{92}$$
$$c_j(t) = \omega_{2j} \sqrt{\frac{g}{f}} \mathrm{e}^{\frac{1}{2} \int_0^t \vartheta_j(t) \mathrm{d}t}, \qquad \vartheta_j(t) = \mathrm{i} f M^2 (L - 2\omega_{1j})^2 - \mathrm{i} V_0 - 2\gamma, \tag{93}$$
$$\bar{c}_j(t) = \omega_{4j} \sqrt{\frac{g}{f}} \mathrm{e}^{\frac{1}{2} \int_0^t \bar{\vartheta}_j(t) \mathrm{d}t}, \qquad \bar{\vartheta}_j(t) = -\mathrm{i} f M^2 (L - 2\omega_{3j})^2 + \mathrm{i} V_0 - 2\gamma. \tag{94}$$

Hereafter, we suppress the variable t for convenience. Since $\epsilon = +1$, i.e., $r(x,t) = -q^*(x,t)$, the second symmetry constraint (62) of the eigenfunctions is not valid, and we use the first symmetry constraint of eigenfunctions (61) such that the eigenvalues satisfy (64), and the norming constants satisfy (69).

1. 1-soliton solution

Set $l = \bar{l} = 1$, $\omega_{11} = \kappa_1 + \mathrm{i}\kappa_2$, where κ_1, κ_2 are arbitrary real constants. Substituting it into $k_1(t)$ in (92), we then see that $\bar{k}_1(t)$ is obtained from the reduction relation (64), i.e., $\bar{k}_1(t) = k_1^*(t)$. Similarly, set $\omega_{21} = c_1 + \mathrm{i}c_2$, where c_1, c_2 are arbitrary real constants, too. Substituting it into $c_1(t)$ in (93), we see that $\bar{c}_1(t)$ is obtained from the relation (69). Here the two choices of $\bar{c}_1(t)$ generate the same solution. Then substituting the scattering data $k_1(t), c_1(t), \bar{k}_1(t)$, and $\bar{c}_1(t)$ into (37), we obtain the solutions $q(x,t)$ and $r(x,t)$. Moreover, by using the transformation (26), we obtain the 1-soliton solution of the GP_+ equation (4),

$$Q_1(x,t) = -2\sqrt{2}\sqrt{\frac{f}{g}}\mathrm{e}^{\mathrm{i}(MLx-\frac{b}{4f}x^2)}\frac{(c_1-\mathrm{i}c_2)^2\mathrm{e}^{\int_0^t \mu_1+\mathrm{i}\nu_1 dt-2(\kappa_2+\mathrm{i}\kappa_1)Mx}}{1+\left(\frac{c_1{}^2+c_2{}^2}{2k_2M}\mathrm{e}^{\int_0^t \mu_1 dt-2\kappa_2 Mx}\right)^2}, \tag{95}$$

where Γ, b, M, L are defined in (90) and (91), and

$$\mu_1 = b + 4fM^2\kappa_2(L-2\kappa_1), \qquad \nu_1 = fM^2\left[4\kappa_2{}^2 - (L-2\kappa_1)^2\right] + V_0. \tag{96}$$

2. 2-soliton solution

Similarly, set $l = \bar{l} = 2$, $\omega_{11} = \kappa_1 + \mathrm{i}\kappa_2, \omega_{21} = c_1 + \mathrm{i}c_2$, and substitute them into $k_1(t)$ and $c_1(t)$. Let $\omega_{12} = \kappa_3 + \mathrm{i}\kappa_4$, $\omega_{22} = c_3 + \mathrm{i}c_4$ be another two arbitrary complex numbers and substitute them into $k_2(t)$ and $c_2(t)$. Then from (38) and the transformation (26), we obtain the 2-soliton solution of the GP_+ equation (4),

$$Q_2(x,t) = -2\sqrt{2}\sqrt{\frac{f}{g}}\mathrm{e}^{\mathrm{i}(MLx-\frac{b}{4f}x^2)}\frac{F(x,t)}{G(x,t)}, \tag{97}$$

where Γ, b, M, L are defined in (90) and (91), μ_1, ν_1 are defined in (96), and

$$\begin{aligned}
F(x,t) &= (c_1-\mathrm{i}c_2)^2\mathrm{e}^{\int \mu_1+\mathrm{i}\nu_1 dt-2M(\kappa_2+\mathrm{i}\kappa_1)x} + (c_3-\mathrm{i}c_4)^2\mathrm{e}^{\int \mu_2+\mathrm{i}\nu_2 dt-2M(\kappa_4+\mathrm{i}\kappa_3)x}\\
&\quad + \frac{\chi_4^2}{4M^2}\Big\{\left[(c_1{}^2+c_2{}^2)(c_3-\mathrm{i}c_4)/(\kappa_2\chi_1)\right]^2\mathrm{e}^{\int 2\mu_1+\mu_2+\mathrm{i}\nu_2 dt-2M(\kappa_4+\mathrm{i}\kappa_3+2\kappa_2)x}\\
&\quad + \left[(c_3{}^2+c_4{}^2)(c_1-\mathrm{i}c_2)/(\kappa_4\chi_2)\right]^2\mathrm{e}^{\int 2\mu_2+\mu_1+\mathrm{i}\nu_1 dt-2M(\kappa_2+\mathrm{i}\kappa_1+2\kappa_4)x}\Big\},\\
G(x,t) &= 1 + \frac{1}{4M^2}\Big\{\left[(c_1{}^2+c_2{}^2)/\kappa_2\right]^2\mathrm{e}^{\int 2\mu_1 dt-4\kappa_2 Mx} + \left[(c_3{}^2+c_4{}^2)/\kappa_4\right]^2\mathrm{e}^{\int 2\mu_2 dt-4\kappa_4 Mx}\\
&\quad - \left[2(c_1+\mathrm{i}c_2)(c_3-\mathrm{i}c_4)/\chi_1\right]^2\mathrm{e}^{\int \mu_2+\mathrm{i}\nu_2+\mu_1-\mathrm{i}\nu_1 dt+2\mathrm{i}\chi_1 Mx}\\
&\quad - \left[2(c_3+\mathrm{i}c_4)(c_1-\mathrm{i}c_2)/\chi_2\right]^2\mathrm{e}^{\int \mu_1+\mathrm{i}\nu_1+\mu_2-\mathrm{i}\nu_2 dt+2\mathrm{i}\chi_2 Mx}\\
&\quad + \left[(c_1{}^2+c_2{}^2)(c_3{}^2+c_4{}^2)\chi_3\chi_4/(\kappa_2\kappa_4\chi_1\chi_2 M)\right]^2\mathrm{e}^{\int 2\mu_1+2\mu_2 dt-4(\kappa_2+\kappa_4)Mx}\Big\},
\end{aligned}$$

with

$$\chi_1 = \kappa_1 + \mathrm{i}\kappa_2 - \kappa_3 + \mathrm{i}\kappa_4, \quad \chi_2 = \kappa_3 + \mathrm{i}\kappa_4 - \kappa_1 + \mathrm{i}\kappa_2, \tag{98}$$

$$\chi_3 = \kappa_1 + \mathrm{i}\kappa_2 - \kappa_3 - \mathrm{i}\kappa_4, \quad \chi_4 = \kappa_3 - \mathrm{i}\kappa_4 - \kappa_1 + \mathrm{i}\kappa_2, \tag{99}$$

and

$$\mu_2 = b + 4fM^2\kappa_4(L-2\kappa_3), \qquad \nu_2 = fM^2\left[4\kappa_4{}^2 - (L-2\kappa_3)^2\right] + V_0. \tag{100}$$

C. Solutions of the nonlocal GP_- equation

In this case, since $\epsilon = -1$, i.e., $r(x,t) = -q^*(-x,t)$, the two symmetry constraints of eigenfunctions (61) and (62) are both valid. The first symmetry constraint yields eigenvalues on the whole axis, and the second symmetry constraint yields eigenvalues on the imaginary axis. Thus they result in different kinds of solutions, and we discuss them separately.

First, let us compute the scattering data $k_j(t)$ and $c_j(t)$. Setting $\epsilon = -1$, substituting (22) and (23) into (89), and denoting

$$\tilde{L} = \int \frac{\tilde{V}_1}{M} dt, \tag{101}$$

we obtain

$$k_j(t) = M(-\frac{\mathrm{i}}{2}\tilde{L} + \omega_{1j}), \qquad \bar{k}_j(t) = M(-\frac{\mathrm{i}}{2}\tilde{L} + \omega_{3j}), \tag{102}$$

$$c_j(t) = \omega_{2j}\sqrt{\frac{g}{f}}\mathrm{e}^{\frac{1}{2}\int_0^t \varrho_j(t)\mathrm{d}t}, \qquad \varrho_j(t) = \mathrm{i} f M^2(\mathrm{i}\tilde{L} - 2\,\omega_{1j})^2 - \mathrm{i}V_0 - 2\gamma, \tag{103}$$

$$\bar{c}_j(t) = \omega_{4j}\sqrt{\frac{g}{f}}\mathrm{e}^{\frac{1}{2}\int_0^t \bar{\varrho}_j(t)\mathrm{d}t}, \qquad \bar{\varrho}_j(t) = -\mathrm{i} f M^2(\mathrm{i}\tilde{L} - 2\omega_{3j})^2 + \mathrm{i}V_0 - 2\gamma, \tag{104}$$

where Γ, b, M are defined in (90).

1. *Solutions with eigenvalues on the whole axes*

Under the first symmetry constraint (61), the eigenvalues satisfy (71), and the norming constants satisfy (69).

a. The case of $n = 1$ solution. Setting $l = \bar{l} = 1$, $\omega_{11} = \kappa_1 + \mathrm{i}\kappa_2$, and substituting it into $k_1(t)$ in (102), then we see that $\bar{k}_1(t)$ is obtained from the reduction relation (64), i.e., $\bar{k}_1(t) = -k_1^*(t)$. Similarly, setting $\omega_{21} = c_1 + \mathrm{i}c_2$, and substituting it into $c_1(t)$ in (103), we see that $\bar{c}_1(t)$ is obtained from the relation (69). Here the two choices of $\bar{c}_1(t)$ generate the same solution, too. Then substituting the scattering data $k_1(t), c_1(t), \bar{k}_1(t)$, and $\bar{c}_1(t)$ into (37), we obtain the solutions $q(x,t)$ and $r(x,t)$. Finally, from the transformation (26), we obtain the solution of the GP_- equation (5),

$$\tilde{Q}_1(x,t) = -2\sqrt{2}\sqrt{\frac{f}{g}}\mathrm{e}^{-M\tilde{L}x - \frac{\mathrm{i}b}{4f}x^2}\frac{(c_1 - \mathrm{i}c_2)^2\mathrm{e}^{\int \tilde{\mu}_1 + \mathrm{i}\tilde{\nu}_1 dt + 2(\kappa_2 + \mathrm{i}\kappa_1)Mx}}{1 - \left(\frac{c_1{}^2 + c_2{}^2}{2\kappa_1 M}\mathrm{e}^{\int \tilde{\mu}_1 dt + 2\mathrm{i}\kappa_1 Mx}\right)^2}, \tag{105}$$

where Γ, b, M are defined in (90), $\tilde{L}$ is defined in (101), and

$$\tilde{\mu}_1 = b + 4fM^2\kappa_1\left(\tilde{L} - 2\kappa_2\right), \qquad \tilde{\nu}_1 = fM^2\left((\tilde{L} - 2\kappa_2)^2 - 4\kappa_1{}^2\right) + V_0. \tag{106}$$

b. The case of $n = 2$ solution. Set $l = \bar{l} = 2$, $\omega_{11} = \kappa_1 + \mathrm{i}\kappa_2, \omega_{21} = c_1 + \mathrm{i}c_2$, and substitute them into $k_1(t)$ and $c_1(t)$, and let $\omega_{12} = \kappa_3 + \mathrm{i}\kappa_4, \omega_{22} = c_3 + \mathrm{i}c_4$, and substitute them into $k_2(t)$ and $c_2(t)$. Then from (38), we obtain the solution of the GP_- equation (5),

$$\tilde{Q}_2(x,t) = -2\sqrt{2}\sqrt{\frac{f}{g}}\mathrm{e}^{-M\tilde{L}x - \frac{\mathrm{i}b}{4f}x^2}\frac{\tilde{F}(x,t)}{\tilde{G}(x,t)}, \tag{107}$$

where Γ, b, M are defined in (90), $\tilde{L}$ is defined in (101), $\tilde{\mu}_1, \tilde{\nu}_1$ are defined in (106), and

$$\begin{aligned}
\tilde{F}(x,t) &= (c_1 - \mathrm{i}c_2)^2\mathrm{e}^{\int \tilde{\mu}_1 + \mathrm{i}\tilde{\nu}_1 dt + 2M(\kappa_2 + \mathrm{i}\kappa_1)x} + (c_3 - \mathrm{i}c_4)^2\mathrm{e}^{\int \tilde{\mu}_2 + \mathrm{i}\tilde{\nu}_2 dt + 2M(\kappa_4 + \mathrm{i}\kappa_3)x} \\
&\quad - \frac{\chi_4^2}{4M^2}\left\{\left[(c_1{}^2 + c_2{}^2)(c_3 - \mathrm{i}c_4)/(\kappa_1\tilde{\chi}_1)\right]^2\mathrm{e}^{\int 2\tilde{\mu}_1 + \tilde{\mu}_2 + \mathrm{i}\tilde{\nu}_2 dt + 2M(\kappa_4 + \mathrm{i}\kappa_3 + 2\mathrm{i}\kappa_1)x}\right. \\
&\quad \left. + \left[(c_3{}^2 + c_4{}^2)(c_1 - \mathrm{i}c_2)/(\kappa_3\tilde{\chi}_2)\right]^2\mathrm{e}^{\int 2\tilde{\mu}_2 + \tilde{\mu}_1 + \mathrm{i}\tilde{\nu}_1 dt + 2M(\kappa_2 + \mathrm{i}\kappa_1 + 2\mathrm{i}\kappa_3)x}\right\}, \\
\tilde{G}(x,t) &= 1 - \frac{1}{4M^2}\left\{\left[(c_1{}^2 + c_2{}^2)/\kappa_1\right]^2\mathrm{e}^{\int 2\tilde{\mu}_1 dt + 4\mathrm{i}\kappa_1 Mx} + \left[(c_3{}^2 + c_4{}^2)/\kappa_3\right]^2\mathrm{e}^{\int 2\tilde{\mu}_2 dt + 4\mathrm{i}\kappa_3 Mx}\right. \\
&\quad + \left[2(c_1 + \mathrm{i}c_2)(c_3 - \mathrm{i}c_4)/\tilde{\chi}_1\right]^2\mathrm{e}^{\int \tilde{\mu}_2 + \mathrm{i}\tilde{\nu}_2 + \tilde{\mu}_1 - \mathrm{i}\tilde{\nu}_1 dt + 2\mathrm{i}\tilde{\chi}_1 Mx} \\
&\quad + \left[2(c_3 + \mathrm{i}c_4)(c_1 - \mathrm{i}c_2)/\tilde{\chi}_2\right]^2\mathrm{e}^{\int \tilde{\mu}_1 + \mathrm{i}\tilde{\nu}_1 + \tilde{\mu}_2 - \mathrm{i}\tilde{\nu}_2 dt + 2\mathrm{i}\tilde{\chi}_2 Mx} \\
&\quad \left. - \left[(c_1{}^2 + c_2{}^2)(c_3^2 + c_4{}^2)\chi_3\chi_4/(\kappa_1\kappa_3\tilde{\chi}_1\tilde{\chi}_2 M)\right]^2\mathrm{e}^{\int 2\tilde{\mu}_1 + 2\tilde{\mu}_2 dt + 4\mathrm{i}(\kappa_1 + \kappa_3)Mx}\right\},
\end{aligned}$$

with

$$\tilde{\chi}_1 = \kappa_1 + \mathrm{i}\kappa_2 + \kappa_3 - \mathrm{i}\kappa_4, \qquad \tilde{\chi}_2 = \kappa_3 + \mathrm{i}\kappa_4 + \kappa_1 - \mathrm{i}\kappa_2, \tag{108}$$

χ_3, χ_4 are defined in (99), and

$$\tilde{\mu}_2 = b + 4fM^2\kappa_3\left(\tilde{L} - 2\kappa_4\right), \qquad \tilde{\nu}_2 = fM^2\left((\tilde{L} - 2\kappa_4)^2 - 4\kappa_3{}^2\right) + V_0. \tag{109}$$

c. A special case: The solution of the focusing nonlocal nonlinear Schrödinger equation. To compare our results with the one discussed in Ref. 12, here we specially give the expression of the solution for the focusing nonlocal nonlinear Schrödinger equation (1) (with sign "+"), which is a special case of the nonlocal GP_- equation (5) under the condition,

$$f(t) = -1, \quad g(t) = -2, \quad V_0(t) = 0, \quad \tilde{V}_1(t) = 0, \quad \gamma(t) = 0. \tag{110}$$

Then the scattering data are reduced to

$$k_1(t) = \kappa_1 + \mathrm{i}\kappa_2, \qquad \bar{k}_1(t) = -\kappa_1 + \mathrm{i}\kappa_2,$$
$$c_1(t) = (c_1 + \mathrm{i}c_2)\,\mathrm{e}^{-2\mathrm{i}(\kappa_1+\mathrm{i}\kappa_2)^2 t}, \qquad \bar{c}_1(t) = -\mathrm{i}\,(c_1 - \mathrm{i}c_2)\,\mathrm{e}^{2\mathrm{i}(\kappa_1-\mathrm{i}\kappa_2)^2 t},$$

where $\kappa_1, \kappa_2, c_1, c_2$ are arbitrary real constants. The solution (105) is reduced to

$$Q_1(x,t) = \frac{-8\kappa_1{}^2(c_1 - \mathrm{i}c_1)^2\,\mathrm{e}^{2\mathrm{i}(\kappa_1-\mathrm{i}\kappa_2)(2\kappa_1 t - 2\mathrm{i}\kappa_2 t + x)}}{(c_1{}^2 + c_2{}^2)^2\mathrm{e}^{-4\mathrm{i}\kappa_1(4\mathrm{i}\kappa_2 t - x)} - 4\kappa_1{}^2}. \tag{111}$$

In Subsection V C 2, we compare the solution (111) with the one obtained from the second symmetry constraint.

2. Solutions with eigenvalues on the imaginary axes

For the second symmetry constraint (62), the eigenvalues satisfy (82), which is only on the imaginary axes. While the norming constants satisfy (83). Setting $l = \bar{l} = 1$, $\omega_{11} = \mathrm{i}\kappa_1, \omega_{31} = -\mathrm{i}\kappa_2$, and substituting them into (102), we obtain

$$k_1(t) = \mathrm{i}M\left(-\frac{1}{2}\tilde{L} + \kappa_1\right), \qquad \bar{k}_1(t) = -\mathrm{i}M\left(\frac{1}{2}\tilde{L} + \kappa_2\right). \tag{112}$$

On the other hand, noticing the denotation $k_j(t)$ and $\bar{k}_j(t)$ in (82), we have

$$\eta_1(t) = M\left(-\frac{1}{2}\tilde{L} + \kappa_1\right), \qquad \bar{\eta}_1(t) = M\left(\frac{1}{2}\tilde{L} + \kappa_2\right). \tag{113}$$

Substituting (113) into (86), then

$$|c_1|^2 = |\bar{c}_1|^2 = M(\kappa_1 + \kappa_2). \tag{114}$$

Thus we assume $\omega_{21} = \sqrt{\kappa_1 + \kappa_2}\mathrm{e}^{\mathrm{i}\theta_1}, \omega_{41} = \sqrt{\kappa_1 + \kappa_2}\mathrm{e}^{\mathrm{i}\theta_2}$, and substituting them into (103) and (104), we obtain

$$\begin{cases} c_1(t) = \sqrt{\kappa_1 + \kappa_2}\sqrt{\dfrac{g}{f}}\mathrm{e}^{-\int_0^t \gamma dt}\mathrm{e}^{\mathrm{i}(\frac{1}{2}\int_0^t \varsigma(t)\mathrm{d}t + \theta_1)}, & \varsigma(t) = -fM^2(\tilde{L} - 2\kappa_1)^2 - V_0, \\ \bar{c}_1(t) = \sqrt{\kappa_1 + \kappa_2}\sqrt{\dfrac{g}{f}}\mathrm{e}^{-\int_0^t \gamma dt}\mathrm{e}^{\mathrm{i}(\frac{1}{2}\int_0^t \bar{\varsigma}(t)\mathrm{d}t + \theta_2)}, & \bar{\varsigma}(t) = fM^2(\tilde{L} + 2\kappa_2)^2 + V_0, \end{cases} \tag{115}$$

such that the condition (114) is satisfied. Now substituting (112), (115) into (37), together with the transformation (26), we obtain the solution of the GP_- equation (5).

Ablowitz and Musslimani[19] has considered the second symmetry constraint for the nonlocal nonlinear Schrödinger equation (1), which is a special case of (5). To compare with Ref. 19, here we just present a solution of the focusing nonlocal nonlinear Schrödinger equation (1). In this case, under the condition (110), the scattering data read

$$k_1(t) = \mathrm{i}\kappa_1, \qquad \bar{k}_1(t) = -\mathrm{i}\kappa_2,$$
$$c_1(t) = \sqrt{\kappa_1 + \kappa_2}\mathrm{e}^{\mathrm{i}(2\kappa_1{}^2 t + \theta_1)}, \qquad \bar{c}_1(t) = \sqrt{\kappa_1 + \kappa_2}\mathrm{e}^{-\mathrm{i}(2\kappa_2{}^2 t - \theta_2)}.$$

The solution is given by

$$Q_1(x,t) = 2\,\frac{e^{-2\kappa_2 x - 4i\kappa_2{}^2 t + 2i\theta_2}(\kappa_1 + \kappa_2)}{e^{4i(\kappa_1{}^2 - \kappa_2{}^2)t + 2i(\theta_1+\theta_2) - 2(\kappa_1+\kappa_2)x} - 1}. \tag{116}$$

Similar to the discussion in Ref. 19, if we set $\kappa_2 = \kappa_1$ and $\theta_2 = -\theta_1 + \frac{\pi}{2}$, then $\bar{k}_1(t) = k_1^*(t), \bar{c}_1(t) = ic_1^*(t)$, and the solution (116) is reduced to the solution of the classical local nonlinear Schrödinger equation. But for the nonlocal GP_- equation (5), from (112), we failed to obtain such a result.

Furthermore, since the solution (116) involves eigenvalues defined only on the imaginary axis, and the solution (111), eigenvalues defined on the whole axis, they are different. Moreover, they both have singularities and are not the standard soliton solutions.

VI. SUMMARY

In this paper, a general hierarchy of nonisospectral AKNS integrable equations was constructed in which the spectral parameter is determined by an ordinary differential equation with polynomial nonlinearity. Such an AKNS hierarchy contains some new nonlocal integrable equations such as the nonlocal KdV-like equation and nonlinear Schrödinger equation. Based on the spectral problem of the general AKNS hierarchy, we presented a unified inverse scattering transformation for the local and nonlocal nonautonomous Gross-Pitaevskii equations arising from soliton management in optical fibers and computed the inverse scattering solutions to the local GP_+ equation and the nonlocal GP_- equation. Unlike the local GP_+ equation, we found that the nonlocal GP_- equation possesses two different kinds of symmetry relations of the eigenfunctions and thus has two different kinds of inverse scattering solutions. Our study added a supplement to the known results on nonlocal equations, and we hope it will help understand the wave propagation governed by the nonlocal nonautonomous Gross-Pitaevskii equation with $\mathcal{PT}$-symmetry self-induced potential.

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