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Localized solutions of (5+1)-dimensional evolution equations

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Abstract In this research, the general linear evolution equations (EEs) in (5+1) dimensions are studied. All the mixed second-order derivatives are included in this aforementioned model. Using the Hirota bilinear operator and symbolic computation, the localized solutions–the abundant lump solutions are constructed. Particularly, it is found that only four groups of linear (5+1)-dimensional EEs are found that they have abundant lump solutions, and no interactions between the lump and other solutions are found via the positive definite quadratic functions. Finally, four examples corresponding to the above-mentioned cases are given to validate the obtained results, and the corresponding graphs are presented to show the dynamic behaviors of the abundant lump solutions of these given examples.

Keywords Evolution equations · Localized solutions · Lump solutions · Symbolic computations

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1 Introduction

Evolution equations (EEs) are a special class of key partial differential equations containing temporal variable [1–17]. It has been applied to describe states or processes changing with time in many areas, such as physics and plasma. Moreover, a lot of complex phenomena have to be desired by spatiotemporal systems consisting of evolution equations. Based on the above-mentioned facts, the research on the evolution equations is attracting more attentions. Particularly, as an important part, the solitary wave theory refers to some kind of wave possessing the wave-particle duality and existing universally in nature. Up to now, the localized solitary wave theory has got extensive applications in nonlinear fields, for example, fluid mechanics, optical fiber communication, life science, and so on.

A lot of researchers have devoted to the study of exact solutions of EEs. With the help of symbolic computation and software Maple, many useful and interesting localized solutions are studied. For instance, rational solutions, periodic solutions and double period solutions [18] as well as kink and anti-kink wave solutions. With the rapid development of the solitary wave theory, it is found that many kinds of interesting exact solutions cannot be derived via the traditional methods. Then, some effective and approaches are presented, among which are the Darboux transformation method [19–21], the Bäcklund transformation method [22–25], the inverse scattering transformation approach [26,27], the Hirota bilinear method and generalized bilinear method [28–30]. Among all the aforementioned approaches, people found that the Hirota bilinear method is a direct and powerful tool for studying solitary wave solutions, because this method only makes use of derivatives, and does not need to deal with the spectral problems of the studied equations.

In the past decades, some kinds of localized solutions and interaction solutions, such as the lump solution, have been systematically studied by many researchers [31–38]. It is noted that lump solution is a particular rational functional solution that attenuates and remains present in all spatial directions, which is mainly obtained by using the Hirota bilinear operator method. The research on lump solutions has attracted much more attention and has become a hot spot. In 2015, an approach using the Hirota bilinear operators to construct the lump solution, rationally localized in all spatial directions, of the (2+1)-dimensional Kadomtsev–Petviashvili equation was proposed. This shows us the form and existence of the lump solutions of the EEs [39]. From then on, with the help of Maple, the lump solutions including multiple lump solutions of a lot of EEs were derived. In 2017, a positive definite quadratic method related to bilinear equations was used in the literature, which gives the lump solutions of (2+1)-dimensional Sawada–Kotera equation [40]. In 2018, lump and interaction solutions of a class of (3+1)-dimensional linear EEs were constructed [41].

This research will focus on constructing the lump solutions of the general linear (5+1)-dimensional EEs and the dynamic analysis of the corresponding lump solutions. Using this method, we not only construct the lump solutions but also found that there is no interaction among lump solutions and other solutions, which is completely different from the (3+1)-dimensional case. As follows is the structure of this work: In Sect. 2, the main results are derived. Some specific cases are given in Sect. 3. Moreover, some graphs are plotted to illustrate the obtained results. Finally, some results and remarks are given in Sect. 4.

2 Localized solutions

In this part, we will do some research on the localized solutions of the general linear (5+1)-dimensional EEs, which consist of all the second-order derivatives, given by

$$\begin{aligned}&\alpha_1 u_{x_1x_2}+\alpha_2 u_{x_1x_3}+\alpha_3 u_{x_1x_4}+\alpha_4 u_{x_1x_5}\\&\quad+\alpha_5 u_{x_1t}+\alpha_6 u_{x_2x_3}+\alpha_7 u_{x_2x_4}+\alpha_8 u_{x_2x_5}\\&\quad+\alpha_9 u_{x_2t}+\alpha_{10} u_{x_3x_4}+\alpha_{11} u_{x_3x_5}\\&\quad+\alpha_{12} u_{x_3t}+\alpha_{13} u_{x_4x_5}+\alpha_{14} u_{x_4t}+\alpha_{15} u_{x_5t}=0,\end{aligned}\tag{1}$$

where $u=u(x_1,x_2,x_3,x_4,x_5,t)$, is a function of spatial variables $x_i\,(i=1,2,3,4,5)$ and temporal variable t. Additionally, $\alpha_i\,(1\le i\le 15)$ are constants.

As above-mentioned, we are searching localized solutions of (1) in the following form

$$u=v(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6)\tag{2}$$

with functions ξ_i, $1\le i\le 6$, determined by

$$\begin{aligned}\xi_i&=a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+a_{i4}x_4\\&\quad+a_{i5}x_5+a_{i6}t+a_{i7},\end{aligned}\tag{3}$$

where a_{ij}, $1\le i\le 6$, $1\le j\le 7$ are real constants to be determined later.

Substituting (2) into (1), we turned (1) into

$$\sum_{i=1}^{6}\sum_{j=i}^{6} w_{ij}v_{\xi_i\xi_j}=0\tag{4}$$

where $w_{ij}\,(1\le i\le j\le 6)$ are quadratic functions of a_{ij}, $1\le i,j\le 6$. Since (2) is the exact solutions of(1), (4) should always hold no matter what variables are selected. Setting $w_{ij}=0$, $1\le i\le j\le 6$, we obtain a system of algebraic equations as follows

$$\begin{cases}\alpha_1(a_{i1}a_{i2})+\alpha_2(a_{i1}a_{i3})+\alpha_3(a_{i1}a_{i4})\\\quad+\alpha_4(a_{i1}a_{i5})+\alpha_5(a_{i1}a_{i6})+\alpha_6(a_{i2}a_{i3})\\\quad+\alpha_7(a_{i2}a_{i4})+\alpha_8(a_{i2}a_{i5})+\alpha_9(a_{i2}a_{i6})\\\quad+\alpha_{10}(a_{i3}a_{i4})+\alpha_{11}(a_{i3}a_{i5})\\\quad+\alpha_{12}(a_{i3}a_{i6})+\alpha_{13}(a_{i4}a_{i5})+\alpha_{14}(a_{i4}a_{i6})\\\quad+\alpha_{15}(a_{i5}a_{i6})=0,\qquad 1\le i=j\le 6\\\alpha_1(a_{i1}a_{j2}+a_{j1}a_{i2})+\alpha_2(a_{i1}a_{j3}+a_{j1}a_{i3})\\\quad+\alpha_3(a_{i1}a_{j4}+a_{j1}a_{i4})+\alpha_4(a_{i1}a_{j5}+a_{j1}a_{i5})\\\quad+\alpha_5(a_{i1}a_{j6}+a_{j1}a_{i6})+\alpha_6(a_{i2}a_{j3}+a_{j2}a_{i3})\\\quad+\alpha_7(a_{i2}a_{j4}+a_{j2}a_{i4})+\alpha_8(a_{i2}a_{j5}+a_{j2}a_{i5})\\\quad+\alpha_9(a_{i2}a_{j6}+a_{j2}a_{i6})+\alpha_{10}(a_{i3}a_{j4}+a_{j3}a_{i4})\\\quad+\alpha_{11}(a_{i3}a_{j5}+a_{j3}a_{i5})+\alpha_{12}(a_{i3}a_{j6}+a_{j3}a_{i6})\\\quad+\alpha_{13}(a_{i4}a_{j5}+a_{j4}a_{i5})+\alpha_{14}(a_{i4}a_{j6}+a_{j4}a_{i6})\\\quad+\alpha_{15}(a_{i5}a_{j6}+a_{j5}a_{i6})=0,\qquad 1\le i<j\le 6\end{cases}\tag{5}$$

Through Maple symbolic computations, four categories of solutions of this system ((5)) are determined as follows

$$\{\alpha_1=\alpha_2=\alpha_3=\alpha_6=\alpha_7=\alpha_{10}=\alpha_{15}=0, \alpha_5 = \frac{\alpha_4\alpha_9}{\alpha_8}, \alpha_{11}=\frac{\alpha_8\alpha_{12}}{\alpha_9}, \alpha_{14}=\frac{\alpha_9\alpha_{13}}{\alpha_8}, a_{61}=a_{62}=a_{63}=a_{64}=a_{65}=a_{66}=0, a_{32} = a_{42}=a_{52}=a_{33}=a_{53}=a_{54}=0, a_{i5}=-\frac{\alpha_9 a_{i6}}{\alpha_8} \quad (1\le i\le 5)\} \tag{6}$$

$$\{\alpha_1=\alpha_2=\alpha_6=\alpha_{13}=\alpha_{14}=\alpha_{15}=0, \alpha_3 = \frac{\alpha_4\alpha_7}{\alpha_8}, \alpha_5=\frac{\alpha_4\alpha_9}{\alpha_8}, \alpha_{10} = \frac{\alpha_7\alpha_{12}}{\alpha_9}, \alpha_{11}=\frac{\alpha_8\alpha_{12}}{\alpha_9}, a_{61}=a_{62}=a_{63}=a_{64}=a_{65}=a_{66}=0, a_{32} = a_{42}=a_{52}=a_{53}=a_{54}=0, a_{i5}=-\frac{\alpha_7 a_{i4}+\alpha_9 a_{i6}}{\alpha_8} \quad (1\le i\le 5)\} \tag{7}$$

$$\{\alpha_2=\alpha_7=\alpha_8=\alpha_9=\alpha_{13}=\alpha_{14}=\alpha_{15}=0, \alpha_6 = \frac{\alpha_1\alpha_{12}}{\alpha_5}, \alpha_{10} = \frac{\alpha_3\alpha_{12}}{\alpha_5}, \alpha_{11} = \frac{\alpha_4\alpha_{12}}{\alpha_5}, a_{61}=a_{62}=a_{63}=a_{64}=a_{65}=a_{66}=0, a_{32} = a_{42}=a_{52}=a_{53}=a_{54}=0, a_{i5}=-\frac{\alpha_1 a_{i2}+\alpha_3 a_{i4}+\alpha_5 a_{i6}}{\alpha_4} \quad (1\le i\le 5)\} \tag{8}$$

$$\{\alpha_6=\alpha_7=\alpha_8=\alpha_9=\alpha_{10}=\alpha_{11}=\alpha_{12} =\alpha_{13}=\alpha_{14}=\alpha_{15}=0, a_{61}=a_{62}=a_{63}=a_{64}=a_{65}=a_{66}=0, a_{22} =a_{32}=a_{42}=a_{52}=a_{53}=a_{54}=0, a_{i5}=-\frac{\alpha_1 a_{i2}+\alpha_2 a_{i3}+\alpha_3 a_{i4}+\alpha_5 a_{i6}}{\alpha_4} \quad (1 \le i\le 5)\}. \tag{9}$$

For each set of the above solutions above, the constant parameters are arbitrary based on the fact that all expressions in the set are well defined. Then, we apply the positive definite quadratic function approach to construct the localized solutions, particularly the abundant lump solutions, of (5+1)-dimensional EEs. Since the localized solutions are required to be positive definite, then the constraint parameters in the above four sets have to satisfy the following conditions: the first condition is $det(a_{ij})_{5\times 5}\neq 0$, and the second is $det(a_{ij})_{6\times 6}=0$, which implies that we get some class of nonsingular rational solutions.

It is interesting that we found that the parameters ξ_6 in solutions (6)-(9) are all constants. Therefore, during the rest of the calculations, we set $g(\xi_6)=\xi_6$. The corresponding cases with specific parameters are given in the following section to illustrate the above-discussed method.

3 Specific cases

Case 1: Considering the solution (6), we choose $\alpha_4 = \alpha_5=\alpha_8=\alpha_9=\alpha_{11}=\alpha_{12}=\alpha_{13}=\alpha_{14}=1$; then (1) is turned into the following form

$$u_{x_1x_5}+u_{x_1t}+u_{x_2x_5}+u_{x_2t} +u_{x_3x_5}+u_{x_3t}+u_{x_4x_5}+u_{x_4t}=0, \tag{10}$$

which admits the following kind of solution

$$u=2(\ln f)_{xx}, f=\xi_1^2+\xi_2^2+\xi_3^2 +\xi_4^2+\xi_5^2+g(\xi_6), \tag{11}$$

with arbitrary function g and wave variables ξ_i, $1\le i\le 6$, defined by the following constraint conditions

$$\begin{cases} \xi_1=a_{11}x_1+a_{12}x_2+a_{13}x_3+a_{14}x_4-a_{16}x_5+a_{16}t+a_{17}, \\ \xi_2=a_{21}x_1+a_{22}x_2+a_{23}x_3+a_{24}x_4-a_{26}x_5+a_{26}t+a_{27}, \\ \xi_3=a_{31}x_1+a_{34}x_4-a_{36}x_5+a_{36}t+a_{37}, \\ \xi_4=a_{41}x_1+a_{43}x_3+a_{44}x_4-a_{46}x_5+a_{46}t+a_{47}, \\ \xi_5=a_{51}x_1-a_{56}x_5+a_{56}t+a_{57}, \\ \xi_6=a_{67}, \end{cases} \tag{12}$$

Substituting (12) into (11) yields

$$u=2(\frac{f_{xx}f-f_x^2}{f^2}) =4(\frac{a_{11}^2+a_{22}^2+a_{33}^2+a_{44}^2+a_{55}^2}{f} -2\frac{(a_{11}\xi_1+a_{22}\xi_2+a_{33}\xi_3+a_{44}\xi_4+a_{55}\xi_5)^2}{f^2}). \tag{13}$$

Moreover, if parameters are assigned special values as in the following table

	i=1	i=2	i=3	i=4	i=5	i=6
a_{i1}	1	1	1	1	1	0
a_{i2}	1	2	0	0	0	0
a_{i3}	1	3	0	1	0	0
a_{i4}	1	4	1	0	0	0
a_{i5}	−1	−5	−1	−1	−1	0
a_{i6}	1	5	1	1	1	0
a_{i7}	1	6	1	1	1	2

then, we have

$$\begin{cases} \xi_1 = x_1 + x_2 + x_3 + x_4 - x_5 + t + 1, \\ \xi_2 = x_1 + 2x_2 + 3x_3 + 4x_4 - 5x_5 + 5t + 6, \\ \xi_3 = x_1 + x_4 - x_5 + t + 1, \\ \xi_4 = x_1 + x_3 - x_5 + t + 1, \\ \xi_5 = x_1 - x_5 + t + 1, \\ \xi_6 = 2, \end{cases} \tag{14}$$

Finally, one specific solution to (10) is

$$\begin{cases} u = 4(\frac{6f - 2(2x_1+5x_2+7x_3+9x_4-10x_5+10t+12)^2}{f^2}), \\ f = (x_1 + x_2 + x_3 + x_4 - x_5 + t + 1)^2 \\ \quad +(x_1 + 2x_2 + 3x_3 + 4x_4 - 5x_5 + 5t + 6)^2 \\ \quad +(x_1 + x_4 - x_5 + t + 1)^2 \\ \quad +(x_1 + x_3 - x_5 + t + 1)^2 \\ \quad +(x_1 - x_5 + t + 1)^2 + 2, \end{cases} \tag{15}$$

To illustrate the obtained results, the following graphs, three-dimensional plots, contour plots and density plots of those solutions with specific parameters are presented in Fig. 1.

Case 2: Now, we choose parameters $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 1$ in the solution (7), and the equation (1) is transformed into a linear (5+1)-dimensional PDEs that reads

$$u_{x_1x_4} + u_{x_1x_5} + u_{x_1t} + u_{x_2x_4} + u_{x_2x_5} \\ +u_{x_2t} + u_{x_3x_4} + u_{x_3x_5} + u_{x_3t} = 0, \tag{16}$$

which has a kind of solution as

$$u = 2(\ln f)_{xx},\ f = \xi_1^2 + \xi_2^2 \\ +\xi_3^2 + \xi_4^2 + \xi_5^2 + g(\xi_6), \tag{17}$$

where the function g is arbitrary and the spatial variables ξ_i, $1 \le i \le 6$, are defined by

$$\begin{cases} \xi_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ \quad -(a_{14} + a_{16})x_5 + a_{16}t + a_{17}, \\ \xi_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ \quad -(a_{24} + a_{26})x_5 + a_{26}t + a_{27}, \\ \xi_3 = a_{31}x_1 + a_{33}x_3 + a_{34}x_4 - (a_{34} \\ \quad +a_{36})x_5 + a_{36}t + a_{37}, \\ \xi_4 = a_{41}x_1 + a_{43}x_3 + a_{44}x_4 - (a_{44} \\ \quad +a_{46})x_5 + a_{46}t + a_{47}, \\ \xi_5 = a_{51}x_1 - a_{56}x_5 + a_{56}t + a_{57}, \\ \xi_6 = a_{67}, \end{cases} \tag{18}$$

Substituting (18) into (17), we might obtain

$$u = 2\left(\frac{f_{xx}f - f_x^2}{f^2}\right) \\ = 4\Big(\frac{a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{44}^2 + a_{55}^2}{f} \\ - 2\frac{(a_{11}\xi_1 + a_{22}\xi_2 + a_{33}\xi_3 + a_{44}\xi_4 + a_{55}\xi_5)^2}{f^2}\Big). \tag{19}$$

Particularly, parameters are set to special values as in the following table

	i=1	i=2	i=3	i=4	i=5	i=6
a_{i1}	1	1	1	1	1	0
a_{i2}	1	2	0	0	0	0
a_{i3}	1	3	1	2	0	0
a_{i4}	1	4	1	3	0	0
a_{i5}	−2	−9	−2	−7	−1	0
a_{i6}	1	5	1	4	1	0
a_{i7}	1	6	1	5	1	1

and in turn

$$\begin{cases} \xi_1 = x_1 + x_2 + x_3 + x_4 - 2x_5 + t + 1, \\ \xi_2 = x_1 + 2x_2 + 3x_3 + 4x_4 - 9x_5 + 5t + 6, \\ \xi_3 = x_1 + x_3 + x_4 - 2x_5 + t + 1, \\ \xi_4 = x_1 + 2x_3 + 3x_4 - 7x_5 + 4t + 5, \\ \xi_5 = x_1 - x_5 + t + 1, \\ \xi_6 = 1, \end{cases} \tag{20}$$

We can obtain specific solution to the equation (16)

$$\begin{cases} u = 4\left(\frac{16f - 2(6x_1+5x_2+14x_3+19x_4-42x_5+23t+28)^2}{f^2}\right), \\ f = (x_1 + x_2 + x_3 + x_4 - 2x_5 + t + 1)^2 \\ \quad +(x_1 + 2x_2 + 3x_3 + 4x_4 - 9x_5 + 5t + 6)^2 \\ \quad +(x_1 + x_3 + x_4 - 2x_5 + t + 1)^2 \\ \quad +(x_1 + 2x_3 + 3x_4 - 7x_5 + 4t + 5)^2 \\ \quad +(x_1 - x_5 + t + 1)^2 + 1, \end{cases} \tag{21}$$

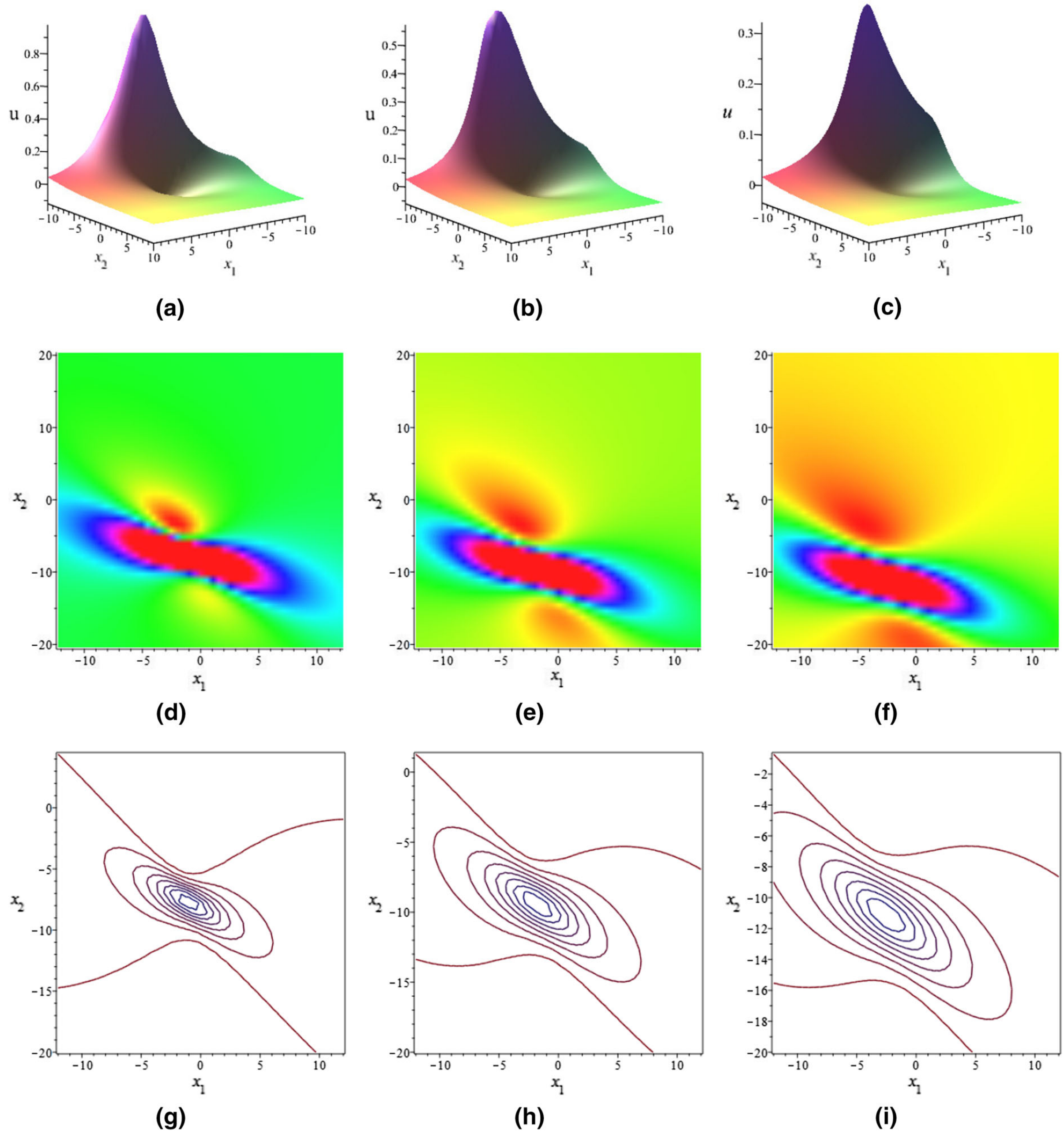

Fig. 1 Profiles of the solution u in (15) at different time $t = 0, 1, 2$ with specific parameters $x_3 = 3, x_4 = 2, x_5 = 1$. (1) **a**–**c** 3D plots; (2) **d**–**f** contour plots; (3) **g**–**i** density plots

Three-dimensional plots, contour plots and density plots of this solutions are presented in Fig. 2.

Case 3: Selecting the parameters as $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 1$ in the solution (8), the equation (1) becomes the following linear (5+1)-dimensional EEs

$$
\begin{aligned}
&u_{x_1x_2} + u_{x_1x_4} + u_{x_1x_5} + u_{x_1t} \\
&\quad + u_{x_2x_3} + u_{x_3x_4} + u_{x_3x_5} + u_{x_3t} = 0,
\end{aligned} \tag{22}
$$

which has a kind of solution

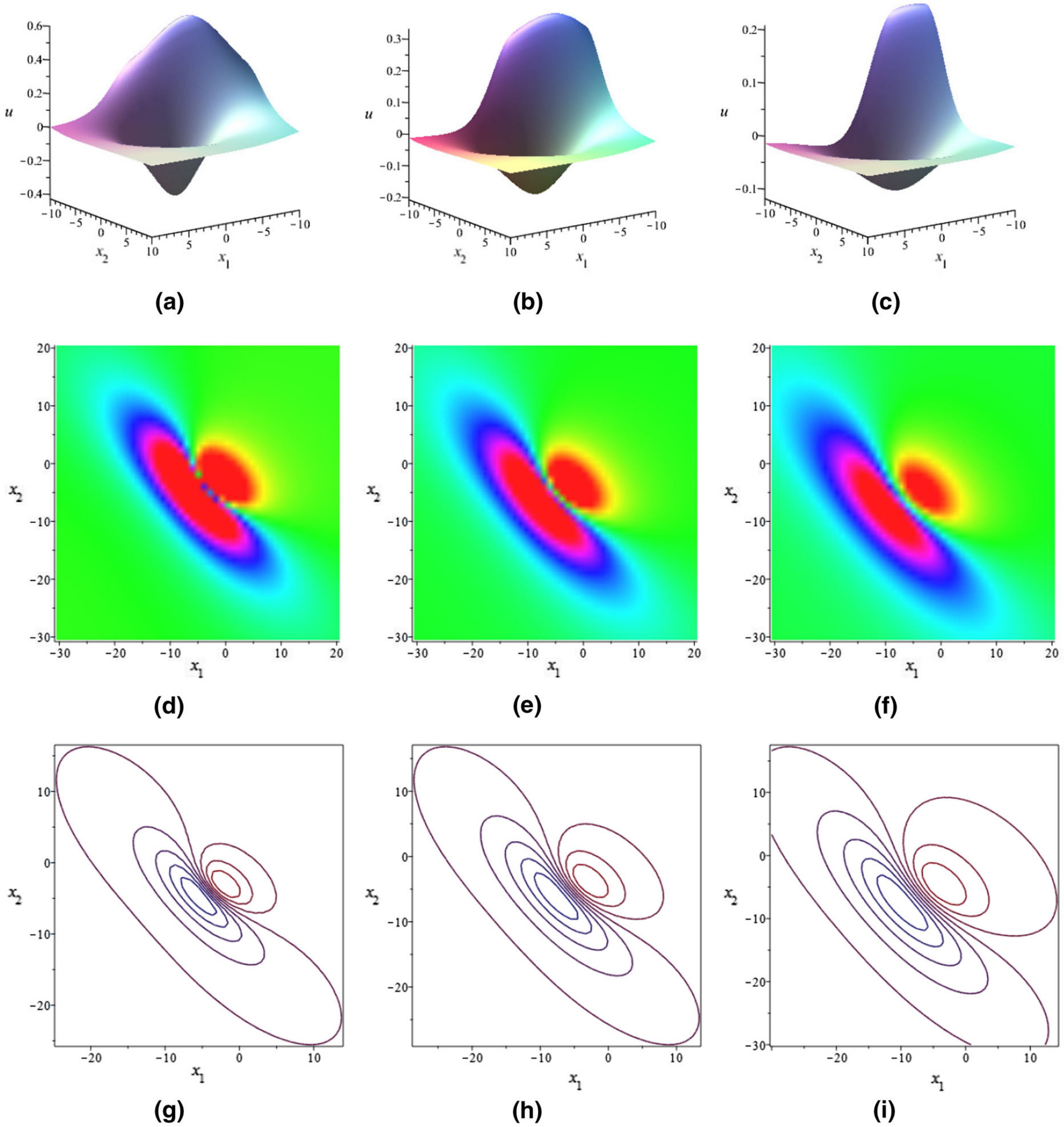

Fig. 2 Profiles of the solution u in (21) at different time $t = 0, 1, 2$ with specific parameters $x_3 = 3$, $x_4 = 2$, $x_5 = 1$. (1) **a**–**c** 3D plots; (2) **d**–**f** contour plots; (3) **g**–**i** density plots

$$u = 2(\ln f)_{xx},\ f = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + g(\xi_6), \tag{23}$$

where the function g is arbitrary and the wave spatial variables ξ_i, $1 \le i \le 6$, are defined by

$$\begin{cases} \xi_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ \quad -(a_{12} + a_{14} + a_{16})x_5 + a_{16}t + a_{17}, \\ \xi_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ \quad -(a_{22} + a_{24} + a_{26})x_5 + a_{26}t + a_{27}, \\ \xi_3 = a_{31}x_1 + a_{33}x_3 + a_{34}x_4 \\ \quad -(a_{34} + a_{36})x_5 + a_{36}t + a_{37}, \\ \xi_4 = a_{41}x_1 + a_{43}x_3 + a_{44}x_4 \\ \quad -(a_{44} + a_{46})x_5 + a_{46}t + a_{47}, \\ \xi_5 = a_{51}x_1 - a_{56}x_5 + a_{56}t + a_{57}, \\ \xi_6 = a_{67}, \end{cases} \tag{24}$$

Substituting (24) into (23), we have

$$\begin{aligned} u &= 2\left(\frac{f_{xx}f - f_x^2}{f^2}\right) \\ &= 4\left(\frac{a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{44}^2 + a_{55}^2}{f} \right.\\ &\quad \left. - 2\frac{(a_{11}\xi_1 + a_{22}\xi_2 + a_{33}\xi_3 + a_{44}\xi_4 + a_{55}\xi_5)^2}{f^2}\right). \end{aligned} \tag{25}$$

Now, we assign specific values to the parameters as in the following table

	i=1	i=2	i=3	i=4	i=5	i=6
a_{i1}	1	1	1	1	1	0
a_{i2}	1	2	0	0	0	0
a_{i3}	1	3	1	1	0	0
a_{i4}	1	4	1	1	0	0
a_{i5}	−3	−8	−2	0	−1	0
a_{i6}	1	2	1	-1	1	0
a_{i7}	1	1	1	5	1	3

and then

$$\begin{cases} \xi_1 = x_1 + x_2 + x_3 + x_4 - 3x_5 + t + 1, \\ \xi_2 = x_1 + 2x_2 + 3x_3 + 4x_4 - 8x_5 + 2t + 1, \\ \xi_3 = x_1 + x_3 + x_4 - 2x_5 + t + 1, \\ \xi_4 = x_1 + x_3 + x_4 - t + 1, \\ \xi_5 = x_1 - x_5 + t + 1, \\ \xi_6 = 3, \end{cases} \tag{26}$$

We can obtain specific solution to the equation (22)

$$\begin{cases} u = 4\left(\frac{8f - 2(4x_1 + 5x_2 + 9x_3 + 11x_4 - 20x_5 + 4t + 4)^2}{f^2}\right), \\ f = (x_1 + x_2 + x_3 + x_4 - 3x_5 + t + 1)^2 \\ \quad +(x_1 + 2x_2 + 3x_3 + 4x_4 - 8x_5 + 2t + 1)^2 \\ \quad +(x_1 + x_3 + x_4 - 2x_5 + t + 1)^2 \\ \quad +(x_1 + x_3 + x_4 - t + 1)^2 \\ \quad +(x_1 - x_5 + t + 1)^2 + 3, \end{cases} \tag{27}$$

Three-dimensional plots, contour plots and density plots of that solution are presented in Fig. 3.

Case 4: We consider the parameters $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1$ in solution (9); then the equation (1) is turned into the following EEs

$$u_{x_1x_2} + u_{x_1x_3} + u_{x_1x_4} + u_{x_1x_5} + u_{x_1t} = 0, \tag{28}$$

which has a kind of solution

$$u = 2(\ln f)_{xx},\ f = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + g(\xi_6), \tag{29}$$

where the function g is arbitrary and the wave variables ξ_i, $1 \le i \le 6$, are defined by

$$\begin{cases} \xi_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ \quad -(a_{12} + a_{13} + a_{14} + a_{16})x_5 + a_{16}t + a_{17}, \\ \xi_2 = a_{22}x_2 + a_{23}x_3 + a_{24}x_4 - (a_{22} \\ \quad +a_{23} + a_{24} + a_{26})x_5 + a_{26}t + a_{27}, \\ \xi_3 = a_{31}x_1 + a_{33}x_3 + a_{34}x_4 - (a_{33} \\ \quad +a_{34} + a_{36})x_5 + a_{36}t + a_{37}, \\ \xi_4 = a_{41}x_1 + a_{43}x_3 + a_{44}x_4 - (a_{43} \\ \quad +a_{44} + a_{46})x_5 + a_{46}t + a_{47}, \\ \xi_5 = a_{51}x_1 - a_{56}x_5 + a_{56}t + a_{57}, \\ \xi_6 = a_{67}, \end{cases} \tag{30}$$

Substituting (30) into (29), we could obtain

$$\begin{aligned} u &= 2\left(\frac{f_{xx}f - f_x^2}{f^2}\right) \\ &= 4\left(\frac{a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{44}^2 + a_{55}^2}{f} \right.\\ &\quad \left. - 2\frac{(a_{11}\xi_1 + a_{22}\xi_2 + a_{33}\xi_3 + a_{44}\xi_4 + a_{55}\xi_5)^2}{f^2}\right). \end{aligned} \tag{31}$$

Then, the parameters are assigned specific values as in the following table
and then

$$\begin{cases} \xi_1 = x_1 + x_2 + x_3 + x_4 - 4x_5 + t + 1, \\ \xi_2 = x_2 + x_3 + x_4 - 4x_5 + t + 1, \\ \xi_3 = x_1 + x_3 + x_4 - 3x_5 + t + 1, \\ \xi_4 = x_1 - x_3 - x_4 + x_5 + t + 1, \\ \xi_5 = x_1 - x_5 + t + 1, \\ \xi_6 = 1, \end{cases} \tag{32}$$

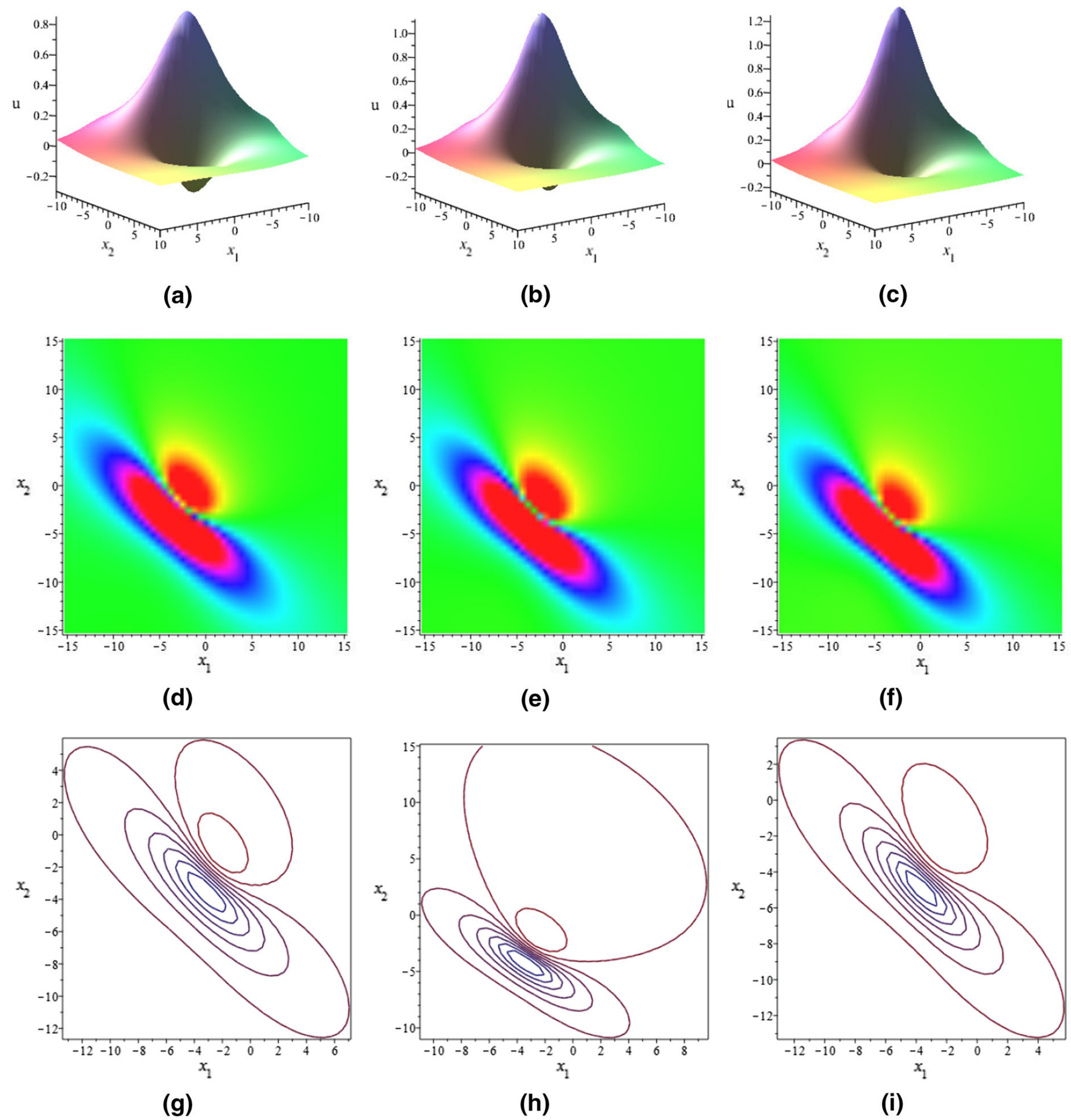

Fig. 3 Profiles of the solution u in (27) at different time $t = 0, 1, 2$ with specific parameters $x_3 = 3, x_4 = 2, x_5 = 1$. (1) **a–c** 3D plots; (2) **d–f** contour plots; (3) **g–i** density plots

We could get the specific solution to equation (28)

$$\begin{cases} u = 4\left(\frac{5f - 2(2x_2+4x_3+4x_4-11x_5+t+1)^2}{f^2}\right), \\ f = (x_1 + x_2 + x_3 + x_4 - 4x_5 + t + 1)^2 \\ \quad +(x_2 + x_3 + x_4 - 4x_5 + t + 1)^2 \\ \quad +(x_1 + x_3 + x_4 - 3x_5 + t + 1)^2 \\ \quad +(x_1 - x_3 - x_4 + x_5 + t + 1)^2 \\ \quad +(x_1 - x_5 + t + 1)^2 + 1, \end{cases} \tag{33}$$

Three-dimensional plots, contour plots and density plots of this solution are presented in Fig. 4.

4 Conclusions

It is well known that evolution equations (EEs) have been extensively used to describe a lot of complex

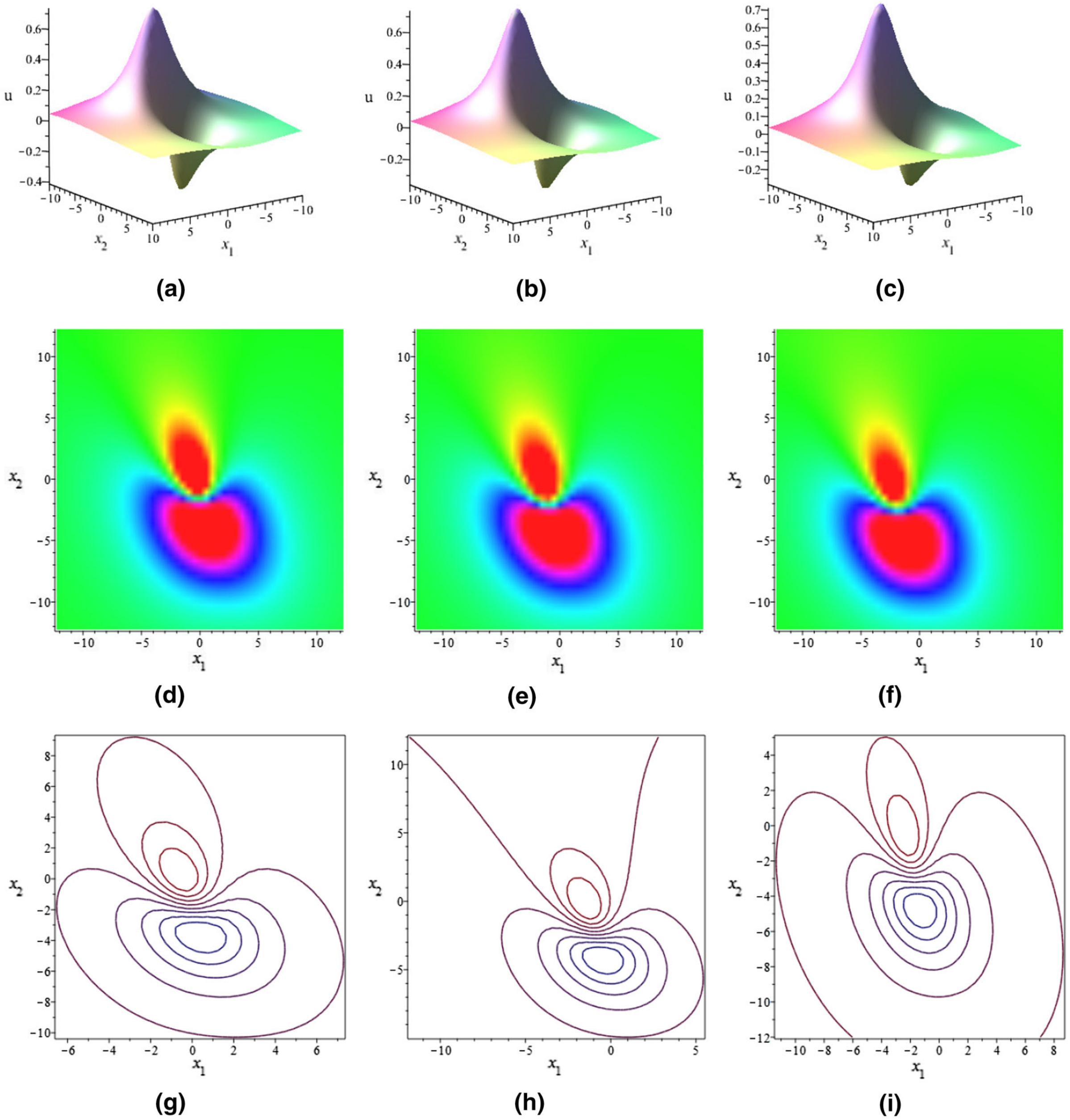

Fig. 4 Profiles of the solution u in (33) at different time $t = 0, 1, 2$ with specific parameters $x_3 = 3$, $x_4 = 2$, $x_5 = 1$. (1) **a**–**c** 3D plots; (2) **d**–**f** contour plots; (3) **g**–**i** density plots

spatiotemporal integrable systems. In this paper, the general (5+1)-dimensional linear EEs have been studied: the exact solutions are constructed successfully and analyzed. Four specific cases are given and the corresponding graphs are plotted, which illustrate the dynamic properties of the solutions with specific parameters. There is an interesting phenomenon that there are not any interaction solutions of the (5+1)-dimensional linear EEs by using the proposed method. Therefore, as a future work, we will do some research on the interaction solutions. Moreover, we will apply the proposed method or its modified versions to study complex nonlinear EEs.

	i=1	i=2	i=3	i=4	i=5	i=6
a_{i1}	1	0	1	1	1	0
a_{i2}	1	1	0	0	0	0
a_{i3}	1	1	1	-1	0	0
a_{i4}	1	1	1	-1	0	0
a_{i5}	-4	-4	-3	1	-1	0
a_{i6}	1	1	1	1	1	0
a_{i7}	1	1	1	1	1	1

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Declarations

Conflict of interest The author declares that there is no conflict of interest.

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