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Kink solutions of two generalized fifth-order nonlinear evolution equations

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In this paper, two generalized fifth-order nonlinear evolution equations are introduced and investigated: One is (1+1)-dimensional, the other is (2+1)-dimensional. The Hereman–Nuseir method is used to derive the multiple kink solutions and singular kink solutions, and the conditions for the cases of complete integrability of these two equations. Meanwhile, it is found that these equations have completely different dispersion relations and physical structures. The corresponding graphs with specific parameters are given to show the effectiveness and validness of the obtained results.

Keywords: Fifth-order nonlinear evolution equations; kink solutions; singular kink solutions; Hereman–Nuseri method.

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1. Introduction

With the speedy development of nonlinear science and theoretical physics and computer technology, research object gradually went from linear models to nonlinear models, which have been applied many areas, for example, fluid, plasma, nonlinear optics, atmospheric science, marine science and so on. Researchers put a lot of research efforts into finding multiple soliton solutions of nonlinear systems.[1–5]

For the nonlinear science, the research on soliton solutions with super stability plays an important role and has attracted many researchers.[6–56] Up to now, a great deal of efficient approaches have been established: the Hirota bilinear method,[22,23] the homogeneous balance method,[24] the inverse scattering method,[25] the Bäcklund transformation method,[26–30] the Darboux transformation method[31,32] the exponential function method,[33] the Riemann–Hilbert problem method,[38–42] the Lie symmetry method[43–48] and so on. Amongst these approaches, the Hirota bilinear method possesses powerful features due to its simplicity and directness.[49–51] Hereman and his coworkers developed the so-called simplified Hirotas method (Hereman–Nuseri method)[52,53] which is very heuristic and of significance in handling nonlinear systems with constant coefficients. These two methods are effective for the determination of multiple soliton solutions of a great many nonlinear evolution equations. Furthermore, the Hereman–Nuseri method is independent of the construction of the bilinear forms; it supposes that the multi-soliton solutions can be expressed as polynomials of exponential functions. For more details of the Hereman–Nuseri method, see Refs. 52 and 53.

Using the Hereman–Nuseri method, Wazwaz introduced and investigated a fifth-order nonlinear integrable equation[54]

$$u_{ttt} - u_{txxxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x = 0. \tag{1}$$

Furthermore, Wazwaz extended the above equation

$$u_{ttt} - u_{txxxx} - u_{tyyyy} - \alpha(u_x u_{xt})_x = 0, \tag{2}$$

$$u_{ttt} - u_{txxxx} - u_{tyyyy} - u_{tzzzz} - \beta(u_x u_{xt})_x = 0, \tag{3}$$

whose soliton solutions of these two equations were studied.[55]

In this work, based on the research on Eqs. (1)–(3)[54,55] we introduce two generalized fifth-order nonlinear evolution equations which read

$$u_{ttt} - u_{txxxx} - \lambda u_{txx} - \alpha(u_x u_t)_{xx} - \beta(u_x u_{xt})_x = 0, \tag{4}$$

$$u_{ttt} - u_{tyyyy} - u_{txx} - \mu_1 u_{txxxx} - \mu_2 u_{tyy} - \alpha(u_y u_t)_{yy} - \beta(u_y u_{yt})_y = 0. \tag{5}$$

which will be studied, where $\lambda, \mu_1, \mu_2, \alpha, \beta$ are parameters. The conditions for the parameters that guarantee these generalized forms integrable will be developed. Moreover, it will be illustrated that multiple soliton solutions can be found for suitable parameters.

The structure of this paper is as follows. In Sec. 2, based on the Hereman–Nuseri method, the kink and singular kink solutions of Eq. (4) are obtained. In Sec. 3, we find the kink and singular kink solutions of Eq. (5). Particularly, the multiple soliton solutions are formally derived for the cases of integrability with specific values of parameters. Meanwhile, some corresponding graphs are given to illustrate the obtained results. Section 4 contains a short summary and some discussions.

2. A Generalized (1+1)-Dimensional Fifth-Order Nonlinear Evolution Equation

In this section, we will consider a generalized (1+1)-dimensional fifth-order nonlinear evolution equations, which reads

$$u_{ttt} - u_{txxxx} - \lambda u_{txx} - \alpha(u_x u_t)_{xx} - \beta(u_x u_{xt})_x = 0, \tag{6}$$

where $\lambda = 0, 1$ and α, β are arbitrary constants. When λ is equal to 0, then Eq. (6) is changed into (1).[54] In this research, we mainly handle the case of $\lambda = 1$, i.e.

$$u_{ttt} - u_{txxxx} - u_{txx} - \alpha(u_x u_t)_{xx} - \beta(u_x u_{xt})_x = 0. \tag{7}$$

Plugging $u = e^{\theta_i}$, $\theta_i = k_i x - c_i t$ into the linear terms of (7) yields the dispersion relation given by

$$c = \pm k\sqrt{1 + k^2}. \tag{8}$$

In turn, this leads to the following phase variable:

$$\theta = kx \pm k\sqrt{1 + k^2}t. \tag{9}$$

To determine the single soliton solutions, according to the Hereman–Nuseri method[52,53] we might assume

$$u = R(\ln(f))_x, \tag{10}$$

with the auxiliary function $f(x, t)$.

In order to get the single soliton solutions, the auxiliary function f is given by

$$f = 1 + C_1 e^{\theta_1}, \tag{11}$$

and $C_1 = 1, -1$ and θ_1 is given by (9). For $C_1 = 1$, substituting (10) and (11) into Eq. (7) and solving for R, we can find

$$R = \frac{12}{2\alpha + \beta}. \tag{12}$$

Furthermore, this in turn yields the following single soliton solution:

$$u = \frac{12}{2\alpha + \beta}\frac{f_x}{f} = \frac{12}{2\alpha + \beta}\frac{k_1 e^{k_1 x \pm k_1\sqrt{1+k_1{}^2}t}}{1 + e^{k_1 x \pm k_1\sqrt{1+k_1{}^2}t}}. \tag{13}$$

In order to find two-soliton solutions, the auxiliary function f can be taken as follows:

$$f = 1 + C_1 e^{\theta_1} + C_2 e^{\theta_2} + C_1 C_2 a_{12} e^{\theta_1 + \theta_2}. \tag{14}$$

Similarly, for $C_1 = C_2 = 1$, we can obtain the phase shift a_{12} by substituting (10) and (14) into Eq. (7). In particular, we herein take special values $k_1 = 3, k_2 = 4$ since the expression of a_{12} is too big. From the computations, it is found that there are two-soliton solutions for two cases $\alpha = 0, \beta \neq 0$ or $\alpha = \beta \neq 0$. For other cases, such as $\alpha \neq 0$, $\beta = 0$ or $\alpha \neq \beta$, $\alpha, \beta \neq 0$, there is no suitable solution of a_{12}, therefore, there are not any two-soliton solutions. Now, if we take $k_3 = 5$, then the solutions of $a_{ij}, 1 \leq i < j \leq 3$ are given by

(i) $\alpha = 0, \beta \neq 0$:

$$\begin{aligned} a_{12} &= \frac{-72\sqrt{17}\sqrt{10} + 1135}{7399}, \quad a_{13} = \frac{-45\sqrt{5}\sqrt{13} + 619}{3184}, \\ a_{23} &= \frac{-40\sqrt{26}\sqrt{17} + 949}{6669}, \end{aligned} \tag{15}$$

(ii) $\alpha = \beta \neq 0$:

$$\begin{aligned} a_{12} &= \frac{-12\sqrt{17}\sqrt{10} + 181}{637}, \quad a_{13} = \frac{-3\sqrt{5}\sqrt{13} + 37}{112}, \\ a_{23} &= \frac{-20\sqrt{26}\sqrt{17} + 461}{1701}. \end{aligned} \tag{16}$$

The two-soliton solution is presented as

$$u = \frac{12}{2\alpha + \beta} \frac{3e^{3x \pm 3\sqrt{10}t} + 4e^{4x \pm 4\sqrt{17}t} + 7a_{12}e^{7x \pm (3\sqrt{10} + 4\sqrt{17})t}}{1 + e^{3x \pm 3\sqrt{10}t} + e^{4x \pm 4\sqrt{17}t} + a_{12}e^{7x \pm (3\sqrt{10} + 4\sqrt{17})t}}. \tag{17}$$

When the auxiliary function f is set

$$\begin{aligned} f &= 1 + C_1 e^{\theta_1} + C_2 e^{\theta_2} + C_3 e^{\theta_3} + C_1 C_2 a_{12} e^{\theta_1 + \theta_2} + C_1 C_3 a_{13} e^{\theta_1 + \theta_3} \\ &\quad + C_2 C_3 a_{23} e^{\theta_2 + \theta_3} + C_1 C_2 C_3 b_{123} e^{\theta_1 + \theta_2 + \theta_3}, \end{aligned} \tag{18}$$

with $C_i = 1$ $(1 \leq i \leq 3)$, we can obtain b_{123} through substituting Eqs. (10) and (18) into Eq. (7). For the two above-mentioned cases, it can be readily found

$$b_{123} = a_{12} a_{13} a_{23}. \tag{19}$$

Similarly, we get three-soliton solutions as follows:

$$u = \frac{12}{2\alpha + \beta} \cdot \frac{p}{q}, \tag{20}$$

with

$$\begin{aligned} p &= 1 + 3e^{\theta_1} + 4e^{\theta_2} + 5e^{\theta_3} + 7a_{12}e^{\theta_1 + \theta_2} + 8a_{13}e^{\theta_1 + \theta_3} + 9a_{23}e^{\theta_2 + \theta_3} \\ &\quad + 12a_{12}a_{13}a_{23}e^{\theta_1 + \theta_2 + \theta_3}, \\ q &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1 + \theta_2} + a_{13}e^{\theta_1 + \theta_3} + a_{23}e^{\theta_2 + \theta_3} \\ &\quad + a_{12}a_{13}a_{23}e^{\theta_1 + \theta_2 + \theta_3}. \end{aligned} \tag{21}$$

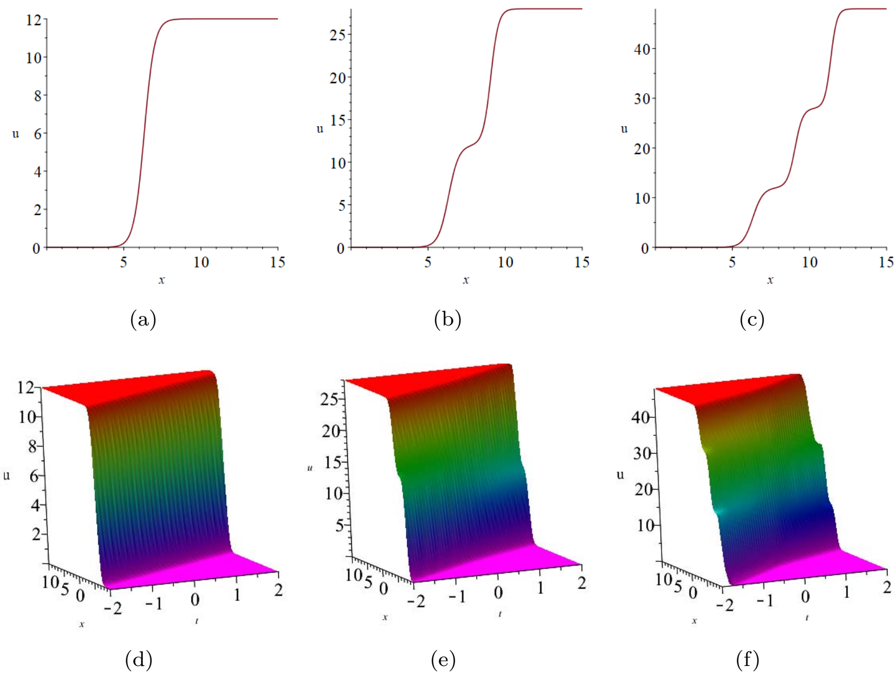


Fig. 1. (Color online) The kink solutions (13), (17), (21) of Eq. (7) when $\alpha = \beta = 1, k_1 = 3, k_2 = 4, k_3 = 5$. (a)–(c) 2D plots; (d)–(f) 3D plots.

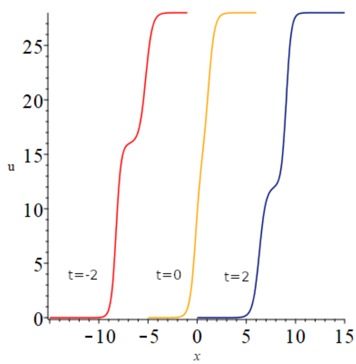
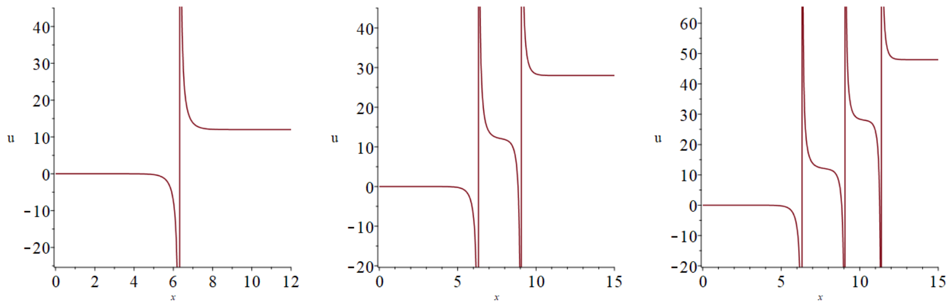


Fig. 2. (Color online) The two-kink solutions (17) of Eq. (7) at different time $t = -2, 0, 2$ with specific parameters $\alpha = \beta = 1, k_1 = 3, k_2 = 4, k_3 = 5$.

It is noted that (7) is transformed into the corresponding equations in Ref. 56 while $\alpha = \beta = 4$ or $\alpha = 0, \beta = 4$, and the results obtained in this work are consistent with those in Ref. 56. For these above-mentioned cases, this proves that this generalized nonlinear fifth-order equation is completely integrable and N soliton solutions can be obtained for the positive integer N.

Some graphs with specific parameters are given in Fig. 1 to illustrate the aforementioned results. From Fig. 1, the kink solution travels from left to right.



(a) (b) (c)

Fig. 3. (Color online) The singular kink solutions u of Eq. (7) for $t = 2$ when $\alpha = \beta = 1, k_1 = 3, k_2 = 4, k_3 = 5$. (a) Singular single kink solutions; (b) singular two-kink solutions; (c) singular three-kink solutions.

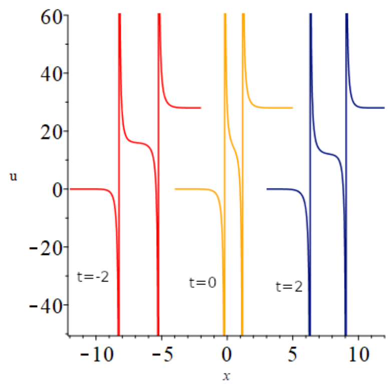


Fig. 4. (Color online) The singular two-kink solutions u of Eq. (7) at different time $t = -2, 0, 2$ with specific parameters $\alpha = \beta = 1, k_1 = 3, k_2 = 4, k_3 = 5$.

Figure 2 depicts the elastic collisions among two-soliton solutions at different time $t = -2, 0, 2$. It can be seen from 2 that the traveling wave of Eq. (7) can still keep its shape and speed unchanged after interacting with other traveling wave.

Furthermore, proceeding as before for $C_1 = C_2 = C_3 = -1$, the singular multiple kink solutions can be found. The corresponding graphs for $t = 2$ of these singular multiple kink solutions are shown by Fig. 3. Figure 4 depicts a singular two-kink solution at different time $t = -2, 0, 2$.

3. A Generalized (2+1)-Dimensional Fifth-Order Nonlinear Evolution Equation

We introduce a generalized (2+1)-dimensional fifth-order nonlinear evolution equations of the form

$$u_{ttt} - u_{tyyyy} - u_{txx} - \mu_1 u_{txxxx} - \mu_2 u_{tyy} - \alpha(u_y u_t)_{yy} - \beta(u_y u_{yt})_y = 0, \quad (22)$$

where $\mu_1, \mu_2 = 0, 01$. By choosing different values of μ_1 and μ_2, Eq. (22) can be turned into the four classes

$$u_{ttt} - u_{tyyyy} - u_{txx} - \alpha(u_y u_t)_{yy} - \beta(u_y u_{yt})_y = 0, \tag{23}$$

$$u_{ttt} - u_{tyyyy} - u_{txx} - u_{txxxx} - \alpha(u_y u_t)_{yy} - \beta(u_y u_{yt})_y = 0, \tag{24}$$

$$u_{ttt} - u_{tyyyy} - u_{txx} - u_{tyy} - \alpha(u_y u_t)_{yy} - \beta(u_y u_{yt})_y = 0, \tag{25}$$

$$u_{ttt} - u_{tyyyy} - u_{txx} - u_{txxxx} - u_{tyy} - \alpha(u_y u_t)_{yy} - \beta(u_y u_{yt})_y = 0. \tag{26}$$

While taking $\alpha = 0$ and $\beta = 4$, (23) is reduced to the corresponding equation in Ref. 56. The obtained results of Case 1 are consistent with those in the literature.[56] To the best of our knowledge, Eqs. (24)–(26) have not been reported so far. It seems that these equations take the similar forms, but they are totally different such as the dispersion relations and phase shifts.

Case 1. Substituting $u = e^{\theta}$, $\theta = kx + ry - ct$ into the linear term of (23) gives the dispersion relation that reads

$$c = \pm\sqrt{r^4 + k^2}, \tag{27}$$

which implies the phase variable is

$$\theta = kx + ry \pm \sqrt{r^4 + k^2}t. \tag{28}$$

In order to get single kink solutions of (23), we set

$$u = R(\ln(f))_y, \tag{29}$$

where the auxiliary function $f(x, y, t)$ is determined by

$$f = 1 + C_1 e^{\theta_1}, \tag{30}$$

with $C_1 = 1$. Substituting (29) into (23) and solving for R give

$$R = \frac{12}{2\alpha + \beta}. \tag{31}$$

This in turn results in the single kink solution

$$u = \frac{12}{2\alpha + \beta}\frac{f_y}{f} = \frac{12}{2\alpha + \beta}\frac{r_1 e^{k_1 x + r_1 y \pm \sqrt{{r_1}^4 + {k_1}^2}t}}{1 + e^{k_1 x + r_1 y \pm \sqrt{{r_1}^4 + {k_1}^2}t}}. \tag{32}$$

For the two-kink solutions, we can assume the auxiliary function

$$f = 1 + C_1 e^{\theta_1} + C_2 e^{\theta_2} + C_1 C_2 a_{12} e^{\theta_1 + \theta_2}, \tag{33}$$

with $C_1 = C_2 = 1$, then substituting (29) and (33) into Eq. (23), we notice that the value of a_{12} is too long-winded to be listed. For simplicity, we take some special values of parameters $r_1 = 1, r_2 = 2, k_1 = 3, k_2 = 4$, then for the case $\alpha = 0, \beta \neq 0$, the two-kink solutions can be found. Similarly, when taking special values $r_3 = 3, k_3 = 5$, we can get a_{13} from k_1, r_1, k_3, r_3 and a_{23} from k_2, r_2, k_3, r_3. After the tedious and complicated computations, the solutions of $a_{ij}, 1 \leq i < j \leq 3$ are given by

$\alpha = 0, \beta \neq 0$:

$$a_{12} = \frac{-12\sqrt{5}+35}{101}, \quad a_{13} = \frac{-27\sqrt{5}\sqrt{53}+959}{2336}, \quad a_{23} = \frac{-108\sqrt{53}+995}{4829}. \tag{34}$$

Then, the two-kink solutions are given by

$$u = \frac{12}{\beta}\frac{e^{3x+y\pm\sqrt{10}t} + 2e^{4x+2y\pm4\sqrt{2}t} + 3\left(\frac{-12\sqrt{5}+35}{101}\right)e^{7x+3y\pm(\sqrt{10}+4\sqrt{2})t}}{1 + e^{3x+y\pm\sqrt{10}t} + e^{4x+2y\pm4\sqrt{2}t} + \left(\frac{-12\sqrt{5}+35}{101}\right)e^{7x+3y\pm(\sqrt{10}+4\sqrt{2})t}}. \tag{35}$$

Furthermore, if setting the auxiliary function

$$\begin{aligned} f = {} & 1 + C_1e^{\theta_1} + C_2e^{\theta_2} + C_3e^{\theta_3} + C_1C_2a_{12}e^{\theta_1+\theta_2} + C_1C_3a_{13}e^{\theta_1+\theta_3} \\ & + C_2C_3a_{23}e^{\theta_2+\theta_3} + C_1C_2C_3b_{123}e^{\theta_1+\theta_2+\theta_3}, \end{aligned} \tag{36}$$

where $C_i = 1, 1 \leq i \leq 3$. Substituting (29) and (36) into (23) leads to

$$b_{123} \neq a_{12}a_{13}a_{23}. \tag{37}$$

Therefore, this shows that Eq. (23) does not have three kink solutions in the sense of Hirota integrability.

However, it should be noted when taking $k_i = r_i, 1 \leq i \leq 3$: while $\alpha = 0, \beta \neq 0$ and $\alpha = \beta \neq 0$ the two-soliton solutions exist, and while $\alpha = 0, \beta \neq 0$ the three-soliton solutions do exist since $b_{123} = a_{12}a_{13}a_{23}$. The solutions of the parameters are as follows with choosing $k_1 = r_1 = 1, k_2 = r_2 = 2, k_3 = r_3 = 3$.

(i) $\alpha = 0, \beta \neq 0$:

$$a_{12} = \frac{-20\sqrt{10}+125}{465}, \quad a_{13} = \frac{-9\sqrt{5}+85}{220}, \quad a_{23} = \frac{-36\sqrt{2}+71}{395}. \tag{38}$$

(ii) $\alpha = \beta \neq 0$:

$$a_{12} = \frac{-6\sqrt{10}+33}{81}, \quad a_{13} = \frac{-3\sqrt{5}+23}{44}, \quad a_{23} = \frac{-6\sqrt{2}+11}{35}. \tag{39}$$

Taking $\alpha = 0, \beta = 1, k_1 = 3, k_2 = 4, k_3 = 5$ with $y = 1$, the three-dimensional plots and two-dimensional plots at different time $t = -5, 0, 5$ of the single kink solutions and two-kink solutions are presented in Figs. 5 and 6. At the same time, the singular single kink solutions and singular two-kink solutions are presented as follows with choosing $C_i = -1$ $(i = 1, 2, 3)$

$$u = \frac{12}{2\alpha+\beta}\frac{-r_1e^{k_1x+r_1y\pm\sqrt{{r_1}^4+{k_1}^2}t}}{1 - e^{k_1x+r_1y\pm\sqrt{{r_1}^4+{k_1}^2}t}}, \tag{40}$$

$$u = -\frac{12}{\beta}\frac{e^{3x+y\pm\sqrt{10}t} + 2e^{4x+2y\pm4\sqrt{2}t} - 3\left(\frac{-12\sqrt{5}+35}{101}\right)e^{7x+3y\pm(\sqrt{10}+4\sqrt{2})t}}{1 - e^{3x+y\pm\sqrt{10}t} - e^{4x+2y\pm4\sqrt{2}t} + \left(\frac{-12\sqrt{5}+35}{101}\right)e^{7x+3y\pm(\sqrt{10}+4\sqrt{2})t}}. \tag{41}$$

The two-dimensional plots for $y = 1$ at different times $t = -5, 0, 5$ of those single singular kink solutions and singular two-kink solutions are presented in Fig. 7.

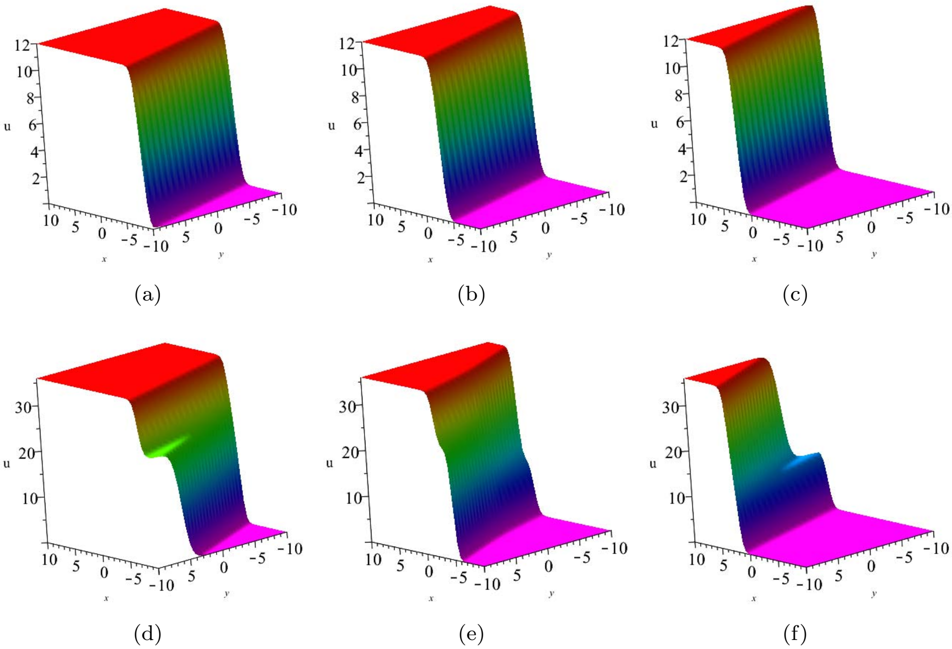


Fig. 5. (Color online) The 3D plots for kink solutions (32) and (35) of Eq. (23) at different time $t = -5, 0, 5$ when $\alpha = 0, \beta = 1, r_1 = 1, r_2 = 2, r_3 = 3, k_1 = 3, k_2 = 4, k_3 = 5$. (a)–(c) Single kink solutions; (d)–(f) two-kink solutions.

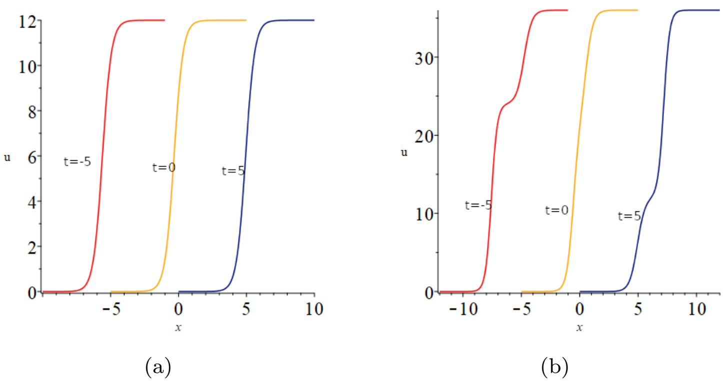


Fig. 6. (Color online) The 2D plots for kink solutions (32) and (35) of Eq. (23) at different time $t = -5, 0, 5$ when $\alpha = 0, \beta = 1, r_1 = 1, r_2 = 2, r_3 = 3, k_1 = 3, k_2 = 4, k_3 = 5, y = 1$. (a) Single kink solutions; (b) two-kink solutions.

Case 2. Consider Eq. (24). Similarly, the dispersion relationship and the phase variable can be obtained

$$c = \pm\sqrt{r^4 + k^2 + k^4}, \tag{42}$$

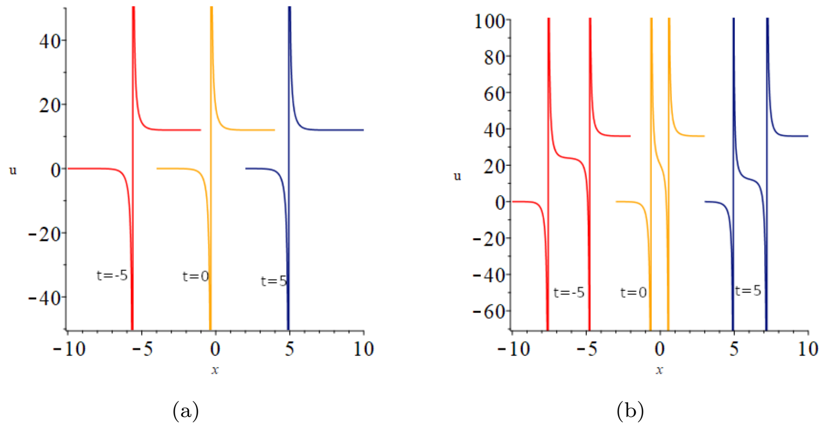


Fig. 7. (Color online) The 2D plots for singular kink solutions u of Eq. (23) at different time $t = -5, 0, 5$ when $\alpha = 0, \beta = 1, r_1 = 1, r_2 = 2, r_3 = 3, k_1 = 3, k_2 = 4, k_3 = 5, y = 1$. (a) Singular single kink solutions; (b) singular two-kink solutions.

$$\theta = kx + ry \pm \sqrt{r^4 + k^2 + k^4}t. \tag{43}$$

Proceeding as before, substituting (29) and the auxiliary function (30) into Eq. (24), we have

$$R = \frac{12({k_1}^4 + {r_1}^4)}{{r_1}^4(2\alpha + \beta)}. \tag{44}$$

This in turn gives the single kink solution

$$u = \frac{12({k_1}^4 + {r_1}^4)}{{r_1}^4(2\alpha + \beta)} \frac{r_1 e^{k_1 x + r_1 y \pm \sqrt{{r_1}^4 + {k_1}^2 + {k_1}^4}t}}{1 + e^{k_1 x + r_1 y \pm \sqrt{{r_1}^4 + {k_1}^2 + {k_1}^4}t}}. \tag{45}$$

In particular, if setting k_1 to r_1, then the parameter R is reduced to

$$R = \frac{24}{2\alpha + \beta}. \tag{46}$$

Furthermore, if setting the auxiliary function $f(x, y, t)$ to be (33), we find that Eq. (24) has no two-soliton solutions for any α and β when k_i is not equal to r_i, $1 \le i \le 3$. However, for $k_1 = r_1 = 1, k_2 = r_2 = 2, k_3 = r_3 = 3$, and $\alpha = 0, \beta \ne 0$ or $\alpha = \beta \ne 0$, the two-kink solutions can be found. The phase shifts $a_{ij}, 1 \le i < j \le 3$ can be listed as follows:

(i) $\alpha = 0, \beta \ne 0$:

$$a_{12} = \frac{-4\sqrt{3} + 15}{59}, \quad a_{13} = \frac{-9\sqrt{19}\sqrt{3} + 325}{856}, \quad a_{23} = \frac{-108\sqrt{19} + 671}{3875}, \tag{47}$$

(ii) $\alpha = \beta \ne 0$:

$$a_{12} = \frac{-4\sqrt{3} + 13}{33}, \quad a_{13} = \frac{-\sqrt{19}\sqrt{3} + 29}{56}, \quad a_{23} = \frac{-4\sqrt{19} + 23}{75}. \tag{48}$$

Consequently, the two-kink solutions follow immediately

$$u = \frac{24}{\beta}\frac{e^{x+y\pm\sqrt{3}t} + 2e^{2x+2y\pm 6t} + 3\left(\frac{-4\sqrt{3}+15}{59}\right)e^{3x+3y\pm(\sqrt{3}+6)t}}{1 + e^{x+y\pm\sqrt{3}t} + e^{2x+2y\pm 6t} + \left(\frac{-4\sqrt{3}+15}{59}\right)e^{3x+3y\pm(\sqrt{3}+6)t}}, \tag{49}$$

$$u = \frac{8}{\alpha}\frac{e^{x+y\pm\sqrt{3}t} + 2e^{2x+2y\pm 6t} + 3\left(\frac{-4\sqrt{3}+13}{33}\right)e^{3x+3y\pm(\sqrt{3}+6)t}}{1 + e^{x+y\pm\sqrt{3}t} + e^{2x+2y\pm 6t} + \left(\frac{-4\sqrt{3}+13}{33}\right)e^{3x+3y\pm(\sqrt{3}+6)t}}. \tag{50}$$

Moreover, by setting the auxiliary function $f(x, y, t)$ to be (36) and proceeding as before, we find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{51}$$

It is observed that choosing $C_1 = C_2 = C_3 = -1$ and proceeding as before can yield single singular kink solutions, two singular kink solutions and three singular kink solutions.

Case 3. Consider Eq. (25). In the same way, the dispersion relationship and the phase variable are given by

$$c = \pm\sqrt{r^4 + r^2 + k^2}, \tag{52}$$

$$\theta = kx + ry \pm \sqrt{r^4 + r^2 + k^2}t. \tag{53}$$

Proceeding as before, substituting the transformation (29) and the auxiliary function (30) into Eq. (25), we find that

$$R = \frac{12}{2\alpha + \beta}. \tag{54}$$

So, the single kink solution follows immediately

$$u = \frac{12}{2\alpha + \beta}\frac{r_1 e^{k_1x + r_1y \pm\sqrt{{r_1}^4 + {r_1}^2 + {k_1}^2}t}}{1 + e^{k_1x + r_1y \pm\sqrt{{r_1}^4 + {r_1}^2 + {k_1}^2}t}}. \tag{55}$$

Setting the auxiliary function $f(x, y, t)$ to (33) and taking $r_1 = 1, r_2 = 2, r_3 = 3, k_1 = 3, k_2 = 4, k_3 = 5$, the two-kink solutions do exist for $\alpha = 0, \beta \neq 0$. The phase shifts $a_{ij}, 1 \leq i < j \leq 3$ are given by

$\alpha = 0, \beta \neq 0$:

$$a_{12} = \frac{-18\sqrt{11} + 77}{215}, \quad a_{13} = \frac{-27\sqrt{115}\sqrt{11} + 1366060}{3259840}, \quad a_{23} = \frac{-162\sqrt{115} + 2165}{9995}, \tag{56}$$

which in turn gives a two-kink solution

$$u = \frac{12}{\beta}\frac{e^{3x+y\pm\sqrt{11}t} + 2e^{4x+2y\pm 6t} + 3\left(\frac{-18\sqrt{11}+77}{215}\right)e^{7x+3y\pm(\sqrt{11}+6)t}}{1 + e^{3x+y\pm\sqrt{11}t} + e^{4x+2y\pm 6t} + \left(\frac{-18\sqrt{11}+77}{215}\right)e^{7x+3y\pm(\sqrt{11}+6)t}}. \tag{57}$$

However, for $k_i = r_i, 1 \leq i \leq 3$, the two-kink solutions are subsistent, the three kink solutions do not exist when $\alpha = 0, \beta \neq 0$ or $\alpha = \beta \neq 0$. The solutions of the parameters are given by

(i) $\alpha = 0, \beta \neq 0$:

$$a_{12} = \frac{-2\sqrt{2}+5}{17}, \quad a_{13} = \frac{-9\sqrt{33}+185}{464}, \quad a_{23} = \frac{-18\sqrt{66}+197}{1025}. \tag{58}$$

(ii) $\alpha = \beta \neq 0$:

$$a_{12} = \frac{-4\sqrt{2}+9}{21}, \quad a_{13} = \frac{-\sqrt{33}+17}{32}, \quad a_{23} = \frac{-4\sqrt{66}+41}{125}. \tag{59}$$

Setting the auxiliary function $f(x, y, t)$ to be (36) and proceeding as before, we can have

$$b_{123} \neq a_{12}a_{13}a_{23}. \tag{60}$$

Therefore, the three-kink solutions are not subsistent and (25) does not has multiple soliton solutions for the positive integer N. Additionally, when $C_1 = C_2 = C_3 = -1$, the single singular kink solutions and two singular kink solutions can be found by proceeding as before.

Case 4. Similarly, the dispersion relationship and the phase variable of Eq. (26) are given by

$$c = \pm\sqrt{r^4 + k^4 + r^2 + k^2}, \tag{61}$$

$$\theta = kx + ry \pm \sqrt{r^4 + k^4 + r^2 + k^2}t. \tag{62}$$

Plugging (29) and the auxiliary function (30) into (26) can give

$$R = \frac{12({k_1}^4 + {r_1}^4)}{{r_1}^4(2\alpha + \beta)}. \tag{63}$$

So, the single kink solution follows immediately

$$u = \frac{12({k_1}^4 + {r_1}^4)}{{r_1}^4(2\alpha + \beta)} \frac{r_1 e^{k_1x + r_1y \pm \sqrt{{r_1}^4 + {k_1}^4 + {r_1}^2 + {k_1}^2}t}}{1 + e^{k_1x + r_1y \pm \sqrt{{r_1}^4 + {k_1}^4 + {r_1}^2 + {k_1}^2}t}}. \tag{64}$$

Furthermore, equating k_1 and r_1 reduces R to

$$R = \frac{24}{2\alpha + \beta}. \tag{65}$$

Taking the auxiliary function $f(x, y, t)$ in (33), we can find that (26) has no two-soliton solutions for any α and β when k_i is not equal to r_i, $1 \leq i \leq 3$. But, if taking $k_1 = r_1 = 1, k_2 = r_2 = 2, k_3 = r_3 = 3$ yields two-kink solutions for $\alpha = 0, \beta \neq 0$ or $\alpha = \beta \neq 0$. Meanwhile, the phase shifts $a_{ij}, 1 \leq i < j \leq 3$ are given by

(i) $\alpha = 0, \beta \neq 0$:

$$a_{12} = \frac{-4\sqrt{10}+25}{93}, \quad a_{13} = \frac{-9\sqrt{5}+85}{220}, \quad a_{23} = \frac{-36\sqrt{2}+71}{395}. \tag{66}$$

(ii) $\alpha = \beta \neq 0$:

$$a_{12} = \frac{-2\sqrt{10}+11}{27}, \quad a_{13} = \frac{-3\sqrt{5}+23}{44}, \quad a_{23} = \frac{-6\sqrt{2}+11}{35}. \tag{67}$$

These give two-kink solutions while $r_1 = 1, r_2 = 2, k_1 = 1, k_2 = 2$

$$u = \frac{24}{\beta}\frac{e^{x+y\pm 2t} + 2e^{2x+2y\pm 2\sqrt{10}t} + 3\left(\frac{-4\sqrt{10}+25}{93}\right)e^{3x+3y\pm(2+2\sqrt{10})t}}{1 + e^{x+y\pm 2t} + e^{2x+2y\pm 2\sqrt{10}t} + \left(\frac{-4\sqrt{10}+25}{93}\right)e^{3x+3y\pm(2+2\sqrt{10})t}}, \tag{68}$$

$$u = \frac{8}{\alpha}\frac{e^{x+y\pm 2t} + 2e^{2x+2y\pm 2\sqrt{10}t} + 3\left(\frac{-2\sqrt{10}+11}{27}\right)e^{3x+3y\pm(2+2\sqrt{10})t}}{1 + e^{x+y\pm 2t} + e^{2x+2y\pm 2\sqrt{10}t} + \left(\frac{-2\sqrt{10}+11}{27}\right)e^{3x+3y\pm(2+2\sqrt{10})t}}. \tag{69}$$

By setting the auxiliary function $f(x, y, t)$ to be (36), and proceeding as before, we find

$$b_{123} = a_{12}a_{13}a_{23}. \tag{70}$$

It is noted that $C_1 = C_2 = C_3 = -1$ gives the singular single kink solutions, singular two-kink solutions and singular three kink solutions by proceeding as before.

4. Conclusion

In this research, we carry out the analysis of two generalized fifth-order nonlinear evolution equations with different dimensions by using the Hereman–Nuseir method developed by Hereman and Nuseir.[52,53] It is found that the parameters are very important for classifying the resulting equations into integrable equations or non-integrable equations. Using the Hereman–Nuseir method, we not only get the necessary conditions for the parameters α and β guaranteeing these generalized equations integrable in the sense of Hirota's integrability but also obtain the multiple soliton solutions, including kink solutions and singular kink solutions. For the (1+1)-dimensional equations, we find that it is integrable while $\alpha = 0, \beta \neq 0$ or $\alpha = \beta$, and obtained its soliton solutions. These solutions and their dynamic behaviors were described by corresponding graphs (seeing Figs. 1–4). For the (2+1)-dimensional equations, Cases 1 and 3 only have two-soliton solutions if $\alpha = 0, \beta \neq 0$ and $k_i \neq r_i, 1 \leq i \leq 3$, three soliton solutions do not exist, therefore these equations do not admit multiple solitons. Cases 2 and 4 are integrable when $\alpha = 0, \beta \neq 0$ or $\alpha = \beta$ with $k_i = r_i, 1 \leq i \leq 3$, and we can obtain their multiple solitons.

As a future work, we might study other types of solutions such as lumps, breathers and rogue waves of Eqs. (6) and (22).

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