.

Binary Darboux transformation for the coupled Sasa-Satsuma equations

Hai-Qiang Zhang, Yue Wang, and Wen-Xiu Ma

Citation: Chaos **27**, 073102 (2017); doi: 10.1063/1.4986807
View online: http://dx.doi.org/10.1063/1.4986807
View Table of Contents: http://aip.scitation.org/toc/cha/27/7
Published by the American Institute of Physics

CHAOS **27**, 073102 (2017)

Binary Darboux transformation for the coupled Sasa-Satsuma equations

Hai-Qiang Zhang,[1,a)] Yue Wang,[1] and Wen-Xiu Ma[2,3,4]
[1]*College of Science, University of Shanghai for Science and Technology, P. O. Box 253, Shanghai 200093, China*
[2]*Department of Mathematics and Statistics, University of South Florida, Tampa, Florida 33620-5700, USA*
[3]*College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China*
[4]*Department of Mathematical Sciences, International Institute for Symmetry Analysis and Mathematical Modelling, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa*

(Received 6 June 2017; accepted 21 June 2017; published online 10 July 2017)

The binary Darboux transformation method is applied to the coupled Sasa-Satsuma equations, which can be used to describe the propagation dynamics of femtosecond vector solitons in the birefringent fibers with third-order dispersion, self-steepening, and stimulated Raman scattering higher-order effects. An N-fold iterative formula of the resulting binary Darboux transformation is presented in terms of the quasideterminants. Via the simplest case of this formula, a few of illustrative explicit solutions to the coupled Sasa-Satsuma equations are generated from vanishing and non-vanishing backgrounds, which include the breathers, single- and double-hump bright vector solitons, and anti-dark vector solitons. *Published by AIP Publishing*. [http://dx.doi.org/10.1063/1.4986807]

The binary Darboux transformation method is a very effective tool in soliton theory to generate wide classes of exact solutions of integrable nonlinear equations. The coupled Sasa-Satsuma equations are one of the integrable higher-order nonlinear Schrödinger systems. The binary Darboux transformation for the coupled Sasa-Satsuma equations is constructed by use of an appropriate dimensional reduction. Different types of exact analytical solutions in terms of the quasideterminants can be derived from vanishing and nonvanishing backgrounds.

I. INTRODUCTION

The Sasa-Satsuma equation

$$i\,q_T + \frac{1}{2}q_{XX} + q|q|^2 + i\,\varepsilon\Big[q_{XXX} + 6|q|^2 q_X + 3q\big(|q|^2\big)_X\Big] = 0, \quad (1)$$

where $q(T, X)$ is a complex-valued function, and the subscripts T and X denote the partial derivatives, is originally presented as a model for the femtosecond pulse propagation in a monomode fiber.[1,2] The last three terms on the left-hand side of Eq. (1) represent the third order dispersion, self-steepening, and stimulated Raman scattering effects, respectively. Equation (1) is a completely integrable higher-order nonlinear Schrödinger model in the sense of being solvable by the inverse scattering transform.[3,4] Its many integrable properties and exact solutions have been studied extensively by various methods.[5–9]

Equation (1) can be easily extended to a coupled case, the so-called coupled Sasa-Satsuma equations[10–12]

$$iq_{j,T} + \frac{1}{2}q_{j,XX} + q_j\sum_{k=1}^{2}|q_k|^2 + i\varepsilon\left[q_{j,XXX} + 6\,q_{j,X}\sum_{k=1}^{2}|q_k|^2 + 3\,q_j\left(\sum_{k=1}^{2}|q_k|^2\right)_X\right] = 0, \quad j = 1, 2, \quad (2)$$

which can be used to describe the propagation dynamics of femtosecond vector solitons in the birefringent or two-mode fibers.[10,11] Many integrable properties of Eqs. (2) have been studied, which include the Painlevé property,[11] infinitely many conserved laws,[12] the Hirota bilinear representation,[13] and the Bäcklund transformation.[10] Recently, the single- and double-hump soliton solutions have been obtained from the zero seed solution by the basic Darboux transformation.[14]

The binary Darboux transformation method is a very effective tool in soliton theory to construct wide classes of exact solutions of integrable nonlinear equations.[15–17] The general idea of this method is to keep both the spectral problem and the corresponding adjoint spectral problem associated with nonlinear equations invariant with respect to the action of the binary Darboux transformation. Moreover, the iterative formula of the binary Darboux transformation is expressible in terms of the quasideterminants. In this paper, by an appropriate reduction from $(2+1)$ to $(1+1)$ dimensions, we will give a systematic method to construct the binary Darboux transformation for Eqs. (2). We will present the N-fold iterative formula of the resulting binary Darboux transformation and construct a few of illustrative explicit solutions from vanishing and non-vanishing backgrounds.

The outline of this work is the following. In Sec. II, we will recall the Lax pair of Eqs. (2) and its important symmetry property. In Sec. III, via the theory of the quasideterminants,[18,19] we will consider the dimensional reductions of the binary Darboux transformation from $(2+1)$ to $(1+1)$ dimensions, and present an N-fold iterative transformation. In Sec. IV, we will give illustrative applications of the resulting binary Darboux transformation from vanishing and

[a)]Author to whom correspondence should be addressed: hqzhang@usst.edu.cn

non-vanishing backgrounds. Taking the once-iterated transformation as an example, we will derive the breathers, single- and double-hump bright vector solitons, and anti-dark vector soliton solutions. Finally, Sec. V gives some concluding remarks.

II. LAX PAIR

Through the variable transformations[10]

$$q_j(X,T) = u_j(x,t)\exp\left\{\frac{i}{6\varepsilon}\left(X - \frac{T}{18\varepsilon}\right)\right\}, \quad j=1,2,$$
$$x = X - \frac{T}{12\varepsilon}, \; t = \varepsilon T. \tag{3}$$

Equations (2) are transformed into the following coupled complex modified Korteweg-de Vries equations

$$u_{j,t} + u_{j,xxx} + 6\,u_{j,x}\sum_{k=1}^{2}|u_k|^2 + 3\,u_j\left(\sum_{k=1}^{2}|u_k|^2\right)_x = 0,$$
$$j = 1,2, \tag{4}$$

which can be commonly regarded as the coupled Sasa-Satsuma equations. The Lax pair for Eqs. (4) can be presented as follows:[10]

$$L = \partial_x + J\lambda + R, \tag{5a}$$

$$M = \partial_t + 4J\lambda^3 + 4R\lambda^2 - 2Q\lambda + W, \tag{5b}$$

with

$$J = i\begin{pmatrix} 1 & 0 \\ 0^T & -I \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -\boldsymbol{u} \\ \boldsymbol{u}^\dagger & O \end{pmatrix},$$
$$Q = i\begin{pmatrix} \boldsymbol{u}\boldsymbol{u}^\dagger & \boldsymbol{u}_x \\ \boldsymbol{u}_x^\dagger & -\boldsymbol{u}^\dagger\boldsymbol{u} \end{pmatrix},$$
$$W = \begin{pmatrix} \boldsymbol{u}_x\boldsymbol{u}^\dagger - \boldsymbol{u}\boldsymbol{u}_x^\dagger & \boldsymbol{u}_{xx} + 2\boldsymbol{u}\boldsymbol{u}^\dagger\boldsymbol{u} \\ -\boldsymbol{u}_{xx}^\dagger - 2\boldsymbol{u}^\dagger\boldsymbol{u}\boldsymbol{u}^\dagger & \boldsymbol{u}_x^\dagger\boldsymbol{u} - \boldsymbol{u}^\dagger\boldsymbol{u}_x \end{pmatrix},$$
$$\boldsymbol{u} = (u_1, u_2, u_1^*, u_2^*),$$

where I and O denote the 4×4 identity matrix and zero matrix, T stands for the transpose of a vector, and the asterisk and the dagger denote the complex conjugate and the Hermitian conjugate, respectively. It is straightforward to verify that the compatibility condition of $[L, M] = 0$ is equivalent to Eqs. (4), indeed.

Since the potential matrix R is skew-Hermitian, the eigenvalue and the eigenfunction of the Lax pair (5) have the following important symmetry property.

Proposition 1.1 *If $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)^T$ is an eigenfunction of the Lax pair (5a)* and *(5b) with an eigenvalue λ, then $\psi = (\phi_1^*, \phi_4^*, \phi_5^*, \phi_2^*, \phi_3^*)^T$ is also an eigenfunction of the Lax pair (5a)* and *(5b) with an eigenvalue $-\lambda^*$.*

The proof is a straightforward computation. Due to the above-mentioned property, we can get the matrix solution of the Lax pair (5a) and (5b)

$$\theta_x + J\theta\Lambda + R\theta = 0, \tag{6a}$$

$$\theta_t + 4J\theta\Lambda^3 + 4R\theta\Lambda^2 - 2Q\theta\Lambda + W\theta = 0, \tag{6b}$$

with

$$\theta = \begin{pmatrix} \phi_1 & \phi_1^* \\ \phi_2 & \phi_4^* \\ \phi_3 & \phi_5^* \\ \phi_4 & \phi_2^* \\ \phi_5 & \phi_3^* \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^* \end{pmatrix}.$$

III. BINARY DARBOUX TRANSFORMATION

In this section, we first review the binary Darboux transformation in $(2+1)$ dimensions.[20] Then, by a separation of variable technique, we consider the dimensional reductions of the binary Darboux transformation from $(2+1)$ to $(1+1)$ dimensions.

Let us consider the linear differential operators

$$L = \partial_x + \sum_{j=0}^{N} a_j\partial_y^j, \quad M = \partial_t + \sum_{j=0}^{N} b_j\partial_y^j, \tag{7}$$

where a_j and b_j are $m\times m$ matrices.

A basic Darboux transformation for the matrix differential operators in Eqs. (7) is stated in the following proposition.

Proposition 3.1 *Let θ be a non-singular $m\times m$ matrix solution of the linear system $L(\phi) = M(\phi) = 0$. Then, the Darboux transformation*

$$\phi \to \tilde\phi = G_\theta(\phi) = \theta\partial_y\theta^{-1}\phi, \tag{8}$$

keeps the linear system $L(\phi) = M(\phi) = 0$ invariant, namely, $\tilde\phi$ also satisfies the same linear system $L(\tilde\phi) = M(\tilde\phi) = 0$ with different coefficients.

The corresponding binary Darboux transformation is given in the following proposition.

Proposition 3.2 *Let θ and ρ be $m\times k$ matrix solutions of the linear system $L(\phi) = M(\phi) = 0$ and its adjoint system $L^\dagger(\psi) = M^\dagger(\psi) = 0$, respectively. Then, the binary Darboux transformation*

$$\phi \to \tilde\phi = B_{\theta,\rho}(\phi) = \phi - \theta\,\omega(\theta,\rho)^{-1}\omega(\phi,\rho), \tag{9a}$$

$$\psi \to \tilde\psi = B_{\theta,\rho}^{-\dagger}(\psi) = \psi - \rho\,\omega(\theta,\rho)^{-\dagger}\omega(\theta,\psi), \tag{9b}$$

where $\omega(\theta,\rho)_y = \rho^\dagger\theta$, keeps the linear system $L(\phi) = M(\phi) = 0$ and its adjoint system $L^\dagger(\psi) = M^\dagger(\psi) = 0$ invariant, respectively.

We iterate the binary Darboux transformations (9a) and (9b) N times, and further present an N-fold iteration expression.

Theorem 3.1 *Suppose that $\theta_1, \theta_2, \ldots, \theta_N$ are a set of N linearly independent solutions of the linear system $L(\phi) = M(\phi) = 0$, and $\rho_1, \rho_2, \ldots, \rho_N$ are a set of N linearly independent solutions of its adjoint system $L^\dagger(\psi) = M^\dagger(\psi) = 0$. Then, the N-fold iteration of the binary Darboux transformation is in terms of the quasideterminants*

$$\phi \to \phi[N] = \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Theta & \boxed{\phi} \end{vmatrix} = \phi - \Theta\,\Omega(\Theta, P)^{-1}\,\Omega(\phi, P), \tag{10a}$$

$$\psi \to \psi[N] = \begin{vmatrix} \Omega(\Theta, P)^{\dagger} & \Omega(\Theta, \psi)^{\dagger} \\ P & \boxed{\psi} \end{vmatrix} = \psi - P\,\Omega(\Theta, P)^{-\dagger}\,\Omega(\Theta, \psi)^{\dagger}, \tag{10b}$$

where $\Theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $P = (\rho_1, \rho_2, \ldots, \rho_N)$, $\Omega(\Theta, P) = \left(\omega(\theta_i, \rho_j)\right)_{i,j=1,2,\ldots,N}$ *is an* $N \times N$ *matrix,* $\Omega(\phi, P) = \left(\omega(\phi, \rho_j)\right)_{j=1,2,\ldots,N}$ *is an N-column vector, and* $\Omega(\Theta, \psi) = (\omega(\psi, \theta_i))_{i=1,2,\ldots,N}$ *is an N-row vector.*

In the following, we consider the dimensional reduction of the binary Darboux transformation from $(2+1)$ to $(1+1)$ dimensions by a separation of variable technique[20]

$$\phi(x,y,t) = \phi^r(x,t)e^{\lambda y}, \quad \psi(x,y,t) = \psi^r(x,t)e^{\mu y}, \tag{11}$$

$$\theta(x,y,t) = \theta^r(x,t)e^{\Lambda y}, \quad \rho(x,y,t) = \rho^r(x,t)e^{\Pi y}, \tag{12}$$

where λ and μ are two constants, and Λ and Π are two $k \times k$ matrices. Then, the matrix differential operators (7) become

$$L^r = \partial_x + \sum_{j=0}^{N} a_j \lambda^j, \quad M^r = \partial_t + \sum_{j=0}^{N} b_j \lambda^j. \tag{13}$$

If we assume that

$$\omega(\theta, \rho) = e^{\Pi^{\dagger}}\omega^r(\theta^r, \rho^r)e^{\Lambda y},$$
$$\omega(\phi, \rho) = e^{(\Pi^{\dagger} + \lambda I)y}\omega^r(\theta^r, \rho^r)e^{\Lambda y}, \tag{14}$$

according to Eqs. (11) and (12), we have

$$\Pi^{\dagger}\omega^r(\theta^r, \rho^r) + \omega^r(\theta^r, \rho^r)\Lambda = \rho^{r\dagger}\theta^r,$$
$$(\Pi^{\dagger} + \lambda I)\omega^r(\phi^r, \rho^r) = \rho^{r\dagger}\phi^r. \tag{15}$$

From now on, for notational simplicity, we omit the superscript r denoting reduced objects and only discuss the binary Darboux transformation in the reduced case. If we choose $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\Pi = diag(\xi_1, \xi_2, \ldots, \xi_N)$, then, from Eqs. (15), we can have

$$\omega(\theta, \rho) = \frac{(\rho^{\dagger}\theta)_{ij}}{\lambda_j + \xi_i^*}, \quad \omega(\phi, \rho) = \frac{(\rho^{\dagger}\phi)_{ij}}{\lambda + \xi_i^*}, \quad i, j = 1, 2, \ldots, N. \tag{16}$$

It is straightforward to check that the Lax operators L and M in Eqs. (5a) and (5b) are both anti-Hermitian, i.e., $L^{\dagger} = -L$ and $M^{\dagger} = -M$. Due to this property, we choose $\rho = \theta$ and $\Pi = -\Lambda$ to keep constraints among the potentials in the matrix R. Thus, the rest work is only to look for a relationship between new potentials and the original potentials. Furthermore, because the form of the operator L is invariant under the binary Darboux transformation

$$L \to \tilde{L} = B_{\theta,\rho} L B_{\theta,\rho}^{-1}, \tag{17}$$

where $\tilde{L}$ has the same form as L except that R is replaced by $\tilde{R}$, it follows from Eq. (17) that

$$\tilde{R} = R + \left[J, \theta\,\omega(\theta, \theta)^{-1}\theta^{\dagger}\right]. \tag{18}$$

By introducing the matrix

$$U = \frac{1}{2i}\begin{pmatrix} 0 & \boldsymbol{u} \\ \boldsymbol{u}^{\dagger} & O \end{pmatrix}, \tag{19}$$

the potential matrix R can be rewritten as $R = [U, J]$. Therefore, from Eq. (18), we can get the relation

$$\tilde{U} = U + \begin{vmatrix} \omega(\theta, \theta) & \theta^{\dagger} \\ \theta & \boxed{0} \end{vmatrix} = U - \theta\,\omega(\theta, \theta)^{-1}\theta^{\dagger}, \tag{20}$$

where $\omega(\theta, \theta)$ satisfies

$$\omega(\theta, \theta)\Lambda - \Lambda^{\dagger}\omega(\theta, \theta) = \theta^{\dagger}\theta,$$
$$\omega(\theta, \theta) = \frac{(\theta^{\dagger}\theta)_{ij}}{\lambda_j - \lambda_i^*} \quad (i, j = 1, 2). \tag{21}$$

Via the above explicit expression of $\omega(\theta, \theta)$ in Eq. (21), it is easy to verify that the reductions and constraints among the original potentials in the matrix R are consistent.

By iterating successively, the N-fold iterative potential transformation can be given in the following theorem.

Theorem 3.2 *Suppose that* $\theta_1, \theta_2, \ldots, \theta_N$ *are N linearly independent matrix solutions of linear system* $L(\phi) = M(\phi) = 0$ *corresponding to* λ_1, $\lambda_2 \cdots$, λ_N, *respectively. Then, the N-fold iterative potential transformation is expressed as*

$$U[N] = U + \begin{vmatrix} \Omega(\Theta, \Theta) & \Theta^{\dagger} \\ \Theta & \boxed{0} \end{vmatrix} = U - \Theta\,\Omega(\Theta, \Theta)^{-1}\Theta^{\dagger}, \tag{22}$$

with

$$\Omega(\Theta, \Theta) = \begin{pmatrix} \omega(\theta_1, \theta_1) & \omega(\theta_2, \theta_1) & \cdots & \omega(\theta_N, \theta_1) \\ \omega(\theta_1, \theta_2) & \omega(\theta_2, \theta_2) & \cdots & \omega(\theta_N, \theta_2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(\theta_1, \theta_N) & \omega(\theta_2, \theta_N) & \cdots & \omega(\theta_N, \theta_N) \end{pmatrix},$$

$$\Theta = (\theta_1, \theta_2, \ldots, \theta_N), \quad \theta_k = \begin{pmatrix} \phi_{5k-4} & \phi_{5k-4}^* \\ \phi_{5k-3} & \phi_{5k-1}^* \\ \phi_{5k-2} & \phi_{5k}^* \\ \phi_{5k-1} & \phi_{5k-3}^* \\ \phi_{5k} & \phi_{5k-2}^* \end{pmatrix},$$
$$(k = 1, 2, \ldots, N),$$

where $\omega(\theta_i, \theta_j)$ *satisfies*

$$\omega(\theta_i, \theta_j)\Lambda_i - \Lambda_j^{\dagger}\omega(\theta_i, \theta_j) = \theta_j^{\dagger}\theta_i, \quad \Lambda_i = diag(\lambda_i, -\lambda_i^*),$$
$$(i, j = 1, 2, \ldots, N). \tag{23}$$

From the above Eq. (23), we get

$$\omega(\theta_i,\theta_j)=\begin{pmatrix} F_{ij} & -G_{ij}^* \\ G_{ij} & -F_{ij}^* \end{pmatrix}, \tag{24}$$

with

$$F_{ij}=\frac{1}{\lambda_i-\lambda_j^*}\left(\phi_{5i}\phi_{5j}^*+\phi_{5i-1}\phi_{5j-1}^*+\phi_{5i-2}\phi_{5j-2}^*+\phi_{5i-3}\phi_{5j-3}^*+\phi_{5i-4}\phi_{5j-4}^*\right),$$
$$G_{ij}=\frac{1}{\lambda_i+\lambda_j}\left(\phi_{5i}\phi_{5j-2}+\phi_{5i-1}\phi_{5j-3}^*+\phi_{5i-2}\phi_{5j}+\phi_{5i-3}\phi_{5j-1}+\phi_{5i-4}\phi_{5j-4}\right).$$

By use of the above expression (22), we can obtain the concrete N-fold iterative potential transformations

$$u_1[N]=u_1+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_2^\dagger \\ \Phi_1 & \boxed{0}\end{vmatrix}=u_1+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_1^\dagger \\ \Phi_4 & \boxed{0}\end{vmatrix}, \tag{25}$$

$$u_2[N]=u_2+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_3^\dagger \\ \Phi_1 & \boxed{0}\end{vmatrix}=u_2+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_1^\dagger \\ \Phi_5 & \boxed{0}\end{vmatrix}, \tag{26}$$

$$u_1^*[N]=u_1^*+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_1^\dagger \\ \Phi_2 & \boxed{0}\end{vmatrix}=u_1^*+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_4^\dagger \\ \Phi_1 & \boxed{0}\end{vmatrix}, \tag{27}$$

$$u_2^*[N]=u_2^*+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_1^\dagger \\ \Phi_3 & \boxed{0}\end{vmatrix}=u_2^*+2i\begin{vmatrix}\Omega(\Theta,\Theta) & \Phi_5^\dagger \\ \Phi_1 & \boxed{0}\end{vmatrix}, \tag{28}$$

where $\Phi_j\,(j=1,2,\ldots,5)$ is the jth row vector of the matrix Θ. With the aid of the properties of quasideterminants, one can directly check that the above expressions are consistent.

IV. SOLUTIONS OF THE COUPLED SASA-SATSUMA EQUATIONS

In this section, applying the potential transformations (25) and (26), we will construct different types of exact analytical solutions of Eqs. (4) from vanishing and non-vanishing backgrounds.

Let us consider the once-iterated potential transformations (25) and (26)

$$u_1[1]=u_1+2i\begin{vmatrix}F_{11} & -G_{11}^* & \phi_2^* \\ G_{11} & -F_{11}^* & \phi_4 \\ \phi_1 & \phi_1^* & \boxed{0}\end{vmatrix}, \tag{29a}$$

$$u_2[1]=u_2+2i\begin{vmatrix}F_{11} & -G_{11}^* & \phi_3^* \\ G_{11} & -F_{11}^* & \phi_5 \\ \phi_1 & \phi_1^* & \boxed{0}\end{vmatrix}, \tag{29b}$$

where

$$F_{11}=\frac{1}{\lambda_1-\lambda_1^*}\sum_{j=1}^{5}|\phi_j|^2,\quad G_{11}=\frac{1}{2\lambda_1}\left(\phi_1^2+2\phi_2\phi_4+2\phi_3\phi_5\right).$$

A. Solutions with vanishing background

With $u_1=u_2=0$ as a seed solution of Eqs. (4), the linear system $L(\phi)=M(\phi)=0$ becomes

$$\phi_x+\lambda J\phi=0, \tag{30a}$$

$$\phi_t+4\lambda^3 J\phi=0. \tag{30b}$$

The fundamental solution of Eqs. (30a) and (30b) can be given as

$$\phi=\left(\alpha e^{-i\chi},\beta e^{i\chi},\gamma e^{i\chi},\delta e^{i\chi},\varrho e^{i\chi}\right)^T,\quad \chi=\lambda(x+\lambda^2 t), \tag{31}$$

where $\alpha,\beta,\gamma,\delta,$ and ϱ are all complex constants. By substituting the above solution (31) into Eqs. (29a) and (29b), we obtain

$$u[1]=\frac{2\eta}{|\alpha|^2}\frac{\alpha^*\left[(\tau_1+\tau_2)\cosh(2i\chi)+(\tau_2-\tau_1)\sinh(2i\chi)\right]+\alpha\left[(\tau_3+\tau_4)\cosh(2i\chi^*)+(\tau_3-\tau_4)\sinh(2i\chi^*)\right]}{\varpi\cosh\left[2i(\chi^*-\chi)-\frac{\varphi_1}{2}\right]+2\eta^2|\beta\delta+\gamma\varrho|\cosh\left[2i(\chi^*+\chi)-\frac{\varphi_2}{2}\right]+\Xi\left(\xi^2+\eta^2\right)}, \tag{32a}$$

$$u[2]=\frac{2\eta}{|\alpha|^2}\frac{\alpha^*\left[(\varsigma_1+\varsigma_2)\cosh(2i\chi)+(\varsigma_2-\varsigma_1)\sinh(2i\chi)\right]+\alpha\left[(\varsigma_3+\varsigma_4)\cosh(2i\chi^*)+(\varsigma_3-\varsigma_4)\sinh(2i\chi^*)\right]}{\varpi\cosh\left[2i(\chi^*-\chi)-\frac{\varphi_1}{2}\right]+2\eta^2|\beta\delta+\gamma\varrho|\cosh\left[2i(\chi^*+\chi)-\frac{\varphi_2}{2}\right]+\Xi\left(\xi^2+\eta^2\right)}, \tag{32b}$$

with

$$\tau_1=\alpha^2\beta^*\xi(i\eta-\xi),\quad \tau_2=2i\beta^*\eta(\beta\delta+\gamma\varrho)(\xi-i\eta)-\delta\Xi\left(\xi^2+\eta^2\right),$$
$$\tau_3=\alpha^{2*}\delta\xi(-i\eta-\xi),\quad \tau_4=-2i\delta\eta(\beta^*\delta^*+\gamma^*\varrho^*)(\xi+i\eta)-\beta^*\Xi\left(\xi^2+\eta^2\right),$$
$$\varsigma_1=\alpha^2\gamma^*\xi(i\eta-\xi),\quad \varsigma_2=2i\gamma^*\eta(\beta\delta+\gamma\varrho)(\xi-i\eta)-\varrho\Xi\left(\xi^2+\eta^2\right),$$
$$\varsigma_3=\alpha^{2*}\varrho\xi(-i\eta-\xi),\quad \varsigma_4=-2i\varrho\eta(\beta^*\delta^*+\gamma^*\varrho^*)(\xi+i\eta)-\gamma^*\Xi\left(\xi^2+\eta^2\right),$$
$$\varpi=\sqrt{\left(\xi^2+\eta^2\right)\Xi^2+4\eta^2|\beta\delta+\gamma\varrho|^2},\quad \Xi=|\beta|^2+|\gamma|^2+|\delta|^2+|\varrho|^2,$$
$$\varphi_1=\ln\frac{\Xi^2\left(\xi^2+\eta^2\right)+4\eta^2|\beta\delta+\gamma\rho|^2}{|\alpha|^4\left(\xi^2+2\eta^2\right)},\quad \varphi_2=\ln\frac{\alpha^2(\beta^*\delta^*+\gamma^*\varrho^*)}{\alpha^{2*}(\beta\delta+\gamma\varrho)},$$

where ξ and $\eta\,(\neq 0)$ are the real and imaginary parts of λ.

The above presented solutions (32a) and (32b) represent the complicated breather transition dynamics in the space and time coordinates, which is characterized by seven involved parameters of α, β, γ, δ, ϱ, ξ, and η. Figures 1(a) and 1(b) depict the evolution plot of a vector breather in the two components u_1 and u_2. Note that when either $\beta=\gamma=0$, $\beta=\varrho=0$, $\delta=\gamma=0$, or $\delta=\varrho=0$ holds, the solutions (32a) and (32b) degenerate to the bright vector solitons including the single- and double-hump bright solitons. The values of parameters ξ and η determine the type of bright vector soliton. If $|\xi|\geq\sqrt{3}|\eta|$, the solutions (32a) and (32b) represent a single-hump bright vector soliton, but they are a double-hump typed vector soliton for the case $|\xi|<\sqrt{3}|\eta|$. Figures 2 and 3 show the propagation dynamics of a single-hump vector soliton and a double-hump vector soliton, respectively.

In high bit-rate optical communication systems, the double-hump or multi-hump soliton has been proposed as an appropriate information carrier.[21,22] The main advantage of the double-hump soltions lies in the fact that they are not affected with time position shifts arising from intra-channel interaction in high bit-rate systems.[21] Moreover, this property of double-hump soltions can be exploited profitably to develop error preventable line-coding schemes, in which binary data are assigned to the single and double-hump solitons.[23] We hope that the single- and double-hump vector solitons in Eqs. (4) will be valuable to the study of the future development of high bit-rate optical communication systems by use of vector solitons as carriers of information.

B. Solutions with a non-vanishing background

Starting from the seed solution $u_j=c_j\,(j=1,2)$ with c_j as complex constants, we solve the linear system $L(\phi)=M(\phi)=0$, and obtain the general solution $\phi=(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)^T$

$$\phi_1=\beta\mathrm{e}^{\vartheta}+\gamma\mathrm{e}^{-\vartheta}, \tag{33a}$$

$$\phi_2=\frac{\alpha}{2}c_2\mathrm{e}^{\zeta}+c_1^*\left(\beta\chi^+\mathrm{e}^{\vartheta}+\gamma\chi^-\mathrm{e}^{-\vartheta}\right), \tag{33b}$$

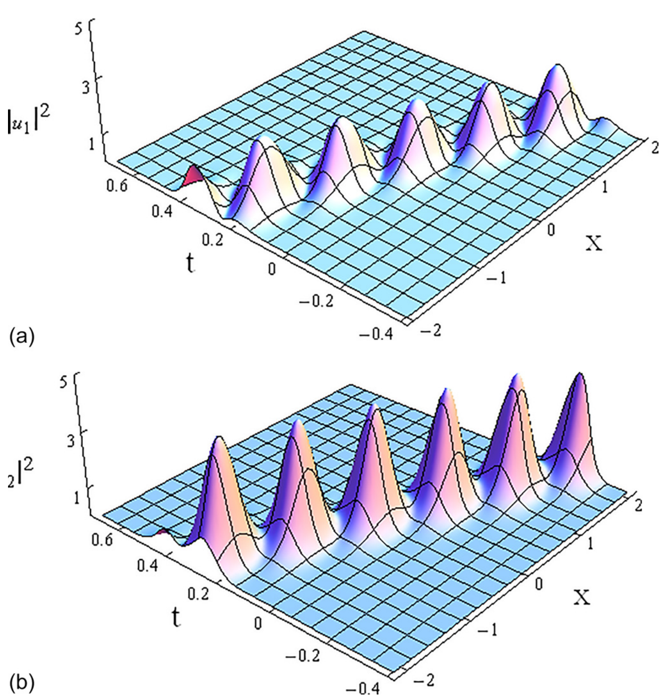

FIG. 1. The propagation dynamics of a vector breather via solutions (32a) and (32b). The parameters of relevant quantities are, respectively, $\alpha=\beta=\gamma=\delta=1, \varrho=-2,\ \xi=-1$, and $\eta=1$.

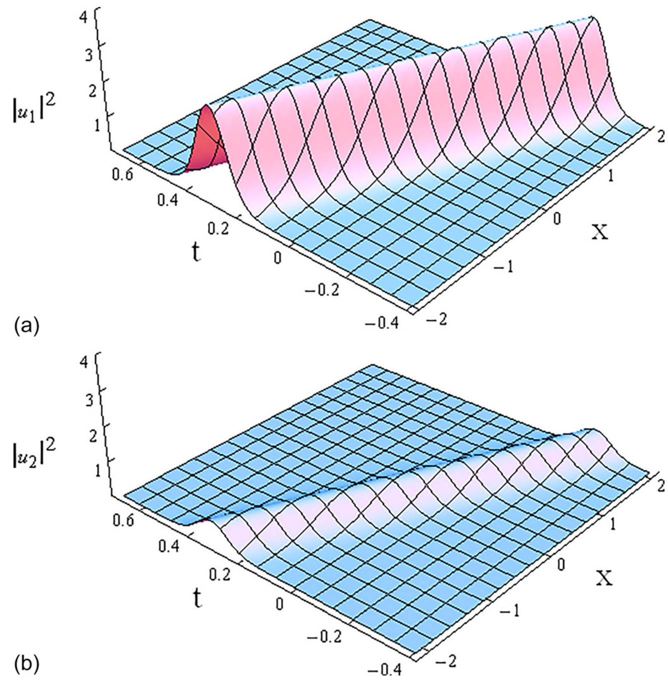

FIG. 2. The propagation dynamics of a single-hump bright vector soliton via solutions (32a) and (32b). The parameters of relevant quantities are, respectively, $\alpha=\gamma=1, \beta=\varrho=0, \delta=-2,\ \xi=1$, and $\eta=1$.

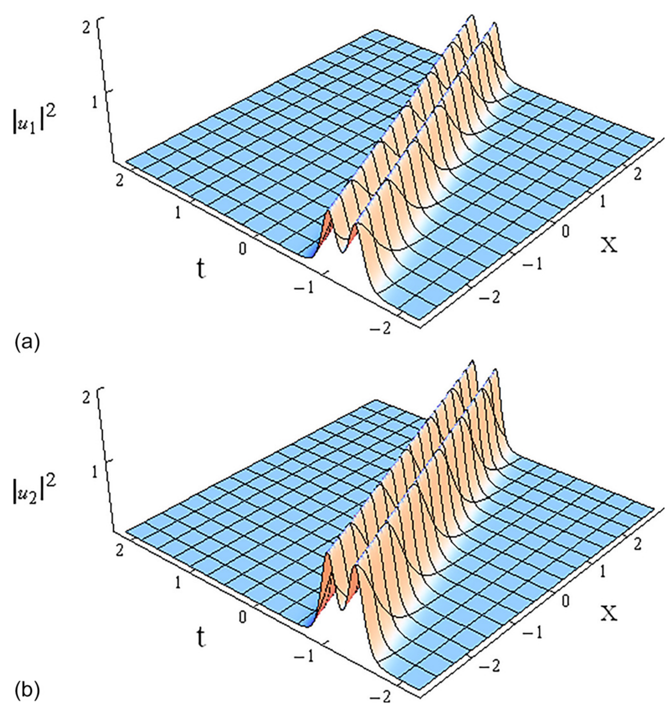

FIG. 3. The propagation dynamics of a double-hump bright vector soliton via solutions (32a) and (32b). The parameters of relevant quantities are, respectively, $\alpha=1, \beta=\gamma=0, \delta=5, \varrho=6,\ \xi=\frac{1}{8}$, and $\eta=1$.

$$\phi_3 = -\frac{\alpha}{2}c_1 e^{\zeta} + c_2^*\left(\beta\chi^+ e^{\vartheta} + \gamma\chi^- e^{-\vartheta}\right), \tag{33c}$$

$$\phi_4 = \frac{\alpha}{2}c_2^* e^{\zeta} + c_1\left(\beta\chi^+ e^{\vartheta} + \gamma\chi^- e^{-\vartheta}\right), \tag{33d}$$

$$\phi_5 = -\frac{\alpha}{2}c_1^* e^{\zeta} + c_2\left(\beta\chi^+ e^{\vartheta} + \gamma\chi^- e^{-\vartheta}\right), \tag{33e}$$

where α, β, and γ are all complex constants, and

$$\vartheta = i\sqrt{\lambda^2 + 2\left(|c_1|^2 + |c_2|^2\right)}\left[x + 4\left(\lambda^2 - |c_1|^2 - |c_2|^2\right)t\right],$$

$$\zeta = i\lambda(x + 4\lambda^2 t), \quad \chi^{\pm} = \frac{\lambda i \pm i\sqrt{\lambda^2 + 2\left(|c_1|^2 + |c_2|^2\right)}}{2\left(|c_1|^2 + |c_2|^2\right)}.$$

Inserting the above solution (33a)–(33e) into Eqs. (29a) and (29b), one can construct explicit solutions of Eqs. (4), including the periodic solutions, the resonant soliton solutions, and the anti-dark vector soliton solutions.

For example, by substituting the solutions (33a)–(33e) with $\lambda = i\sqrt{2\left(|c_1|^2 + |c_2|^2\right)}$ and $\beta = 0$ into Eqs. (29a) and (29b), we have

$$u[1] = c_1\left[1 - 2(\alpha^*\gamma + \alpha\gamma^*)\frac{(\alpha^*\gamma - \alpha\gamma^*) - \dfrac{2\sqrt{2}|\gamma|^2 c_2^* e^{\zeta}}{c_1\sqrt{\left(|c_1|^2 + |c_2|^2\right)}}}{(\alpha^{2*}\gamma^2 + \alpha^2\gamma^{2*}) + \dfrac{4|\gamma|^4 e^{2\zeta}}{|c_1|^2 + |c_2|^2}}\right], \tag{34a}$$

$$u[2] = c_2\left[1 - 2(\alpha^*\gamma + \alpha\gamma^*)\frac{(\alpha^*\gamma - \alpha\gamma^*) + \dfrac{2\sqrt{2}|\gamma|^2 c_1^* e^{\zeta}}{c_2\sqrt{\left(|c_1|^2 + |c_2|^2\right)}}}{(\alpha^{2*}\gamma^2 + \alpha^2\gamma^{2*}) + \dfrac{4|\gamma|^4 e^{2\zeta}}{|c_1|^2 + |c_2|^2}}\right], \tag{34b}$$

with $\zeta = \sqrt{2\left(|c_1|^2 + |c_2|^2\right)}\left[x - 8\left(|c_1|^2 + |c_2|^2\right)t\right]$.

It is found that the solutions (34a) and (34b) can exhibit anti-dark vector solitons, as seen in Fig. 4. Obviously, when $t \to \pm\infty$, the intensity of an anti-dark soliton is an infinitely extended constant, which is different from that of the bright soliton (the bright solitons have vanishing amplitude as $t \to \pm\infty$). Recently, anti-dark solitons have received considerable attention in the last few years.[24–28] As one specific type of soliton, anti-dark solitons are localized excitations on a continuous wave background. Experimentally, Ref. 24 has firstly observed anti-dark optical solitons in non-instantaneous nonlinear media. The instability of anti-dark optical solitons can be totally eliminated by properly engineering the incoherence of background beam.[24]

V. CONCLUDING REMARKS

In this paper, we have studied the coupled Sasa-Satsuma equations describing the propagation dynamics of femtosecond vector solitons in the birefringent fibers with third-order dispersion, self-steepening, and stimulated Raman scattering higher-order effects. Based on the linear spectral problem of this model, we have given a systematic approach for constructing the binary Darboux transformation. With the theory of quasideterminants, we have presented a determinant form of the N-fold iterative binary Darboux transformation. From the once-iterated transformation, we have derived the breathers, single- and double-hump bright vector solitons, and anti-dark vector solitons.

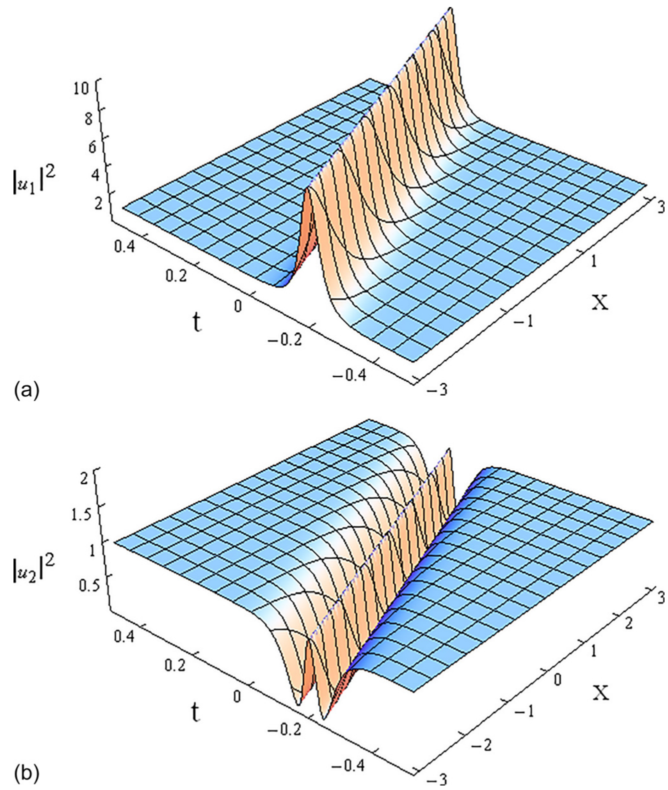

FIG. 4. The propagation dynamics of an anti-dark vector soliton via solutions (34a) and (34b). The parameters of relevant quantities are, respectively, $\alpha = \gamma = 1$ and $c_1 = c_2 = 1$.

In Refs. 10–14, some solutions of Eqs. (2) have been studied by various methods. Through comparing our obtained results with those published previously, we have the following remarks:

(1) In this paper, by a separation of variable technique, we have given a systematic approach for constructing the binary Darboux transformation of the coupled Sasa-Satsuma equations. The technique presented may also be applicable to other nonlinear integrable systems.
(2) By the N-fold iterative binary Darboux transformation, we have presented exact solutions of the coupled Sasa-Satsuma equations in terms of compact quasideterminant forms. The resulting solutions in this paper cover many previously published solutions. The obtained formulas allow one to construct various new solutions, including bright solitons, anti-dark solitons, breathers, rational solutions, and interaction solutions.
(3) Although our explicit solutions exhibited here are from the first iteration, it is feasible to construct more complicated Nth-order bright vector soliton, breather, and anti-dark soliton solutions. Based on those solutions, collision dynamics among solitons or breathers can be discussed, occurring between two components. Finally, it is also

interesting to explore rogue wave solutions of the coupled Sasa-Satsuma equations by the presented binary Darboux transformation.

ACKNOWLEDGMENTS

This work was supported by the Shanghai Leading Academic Discipline Project under Grant No. XTKX2012, by the Technology Research and Development Program of University of Shanghai for Science and Technology under Grant No. 2017KJFZ122 and by Hujiang Foundation of China under Grant No. B14005. The third author is supported in part by NSFC under the Grant Nos. 11371326, 11301331, and 11371086, NSF under the Grant No. DMS-1664561, the 111 project of China (B16002), and the Distinguished Professorships by Shanghai University of Electric Power and Shanghai Second Polytechnic University.

[1]Y. Kodama, "Optical solitons in a monomode fiber," J. Stat. Phys. **39**, 597 (1985).
[2]Y. Kodama and A. Hasegawa, "Nonlinear pulse propagation in a monomode dielectric guide," IEEE J. Quantum Electron. **23**, 510 (1987).
[3]N. Sasa and J. Satsuma, "New type of soliton solutions for a higher-order nonlinear Schrödinger equation," J. Phys. Soc. Jpn. **60**, 409 (1991).
[4]J. Yang and D. J. Kaup, "Squared eigenfunctions for the Sasa-Satsuma equation," J. Math. Phys. **50**, 023504 (2009).
[5]C. Gilson, J. Hietarinta, J. J. C. Nimmo, and Y. Ohta, "Sasa-Satsuma higher-order nonlinear Schrödinger equation and its bilinearization and multisoliton solutions," Phys. Rev. E **68**, 016614 (2003).
[6]O. C. Wright III, "Sasa-Satsuma equation, unstable plane waves and heteroclinic connections," Chaos, Solitons Fractals **33**, 374 (2007).
[7]Y. Liu, Y. T. Gao, T. Xu, X. Lü, Z. Y. Sun, X. H. Meng, X. Yu, and X. L. Gai, "Soliton solution, Bäklund transformation, and conservation laws for the Sasa-Satsuma equation in the optical fiber communications," Z. Naturforsch. A **65**, 291 (2010).
[8]T. Xu, D. H. Wang, M. Li, and H. Liang, "Soliton and breather solutions of the Sasa-Satsuma equation via the Darboux transformation," Phys. Scr. **89**, 075207 (2014).
[9]J. J. C. Nimmo and H. Yilmaz, "Binary Darboux transformation for the Sasa-Satsuma equation," J. Phys. A: Math. Theor. **48**, 425202 (2015).
[10]K. Nakkeeran, K. Porsezian, P. S. Sundaram, and A. Mahalingam, "Optical solitons in N-coupled higher order nonlinear Schrödinger equations," Phys. Rev. Lett. **80**, 1425 (1998).
[11]K. Porsezian, P. S. Sundaram, and A. Mahalingam, "Coupled higher-order nonlinear Schrödinger equations in nonlinear optics: Painlevé analysis and integrability," Phys. Rev. E **50**, 1543 (1994).
[12]S. Nandy, "Inverse scattering approach to coupled higher-order nonlinear Schrödinger equation and N-soliton," Nucl. Phys. B **679**, 647 (2004).
[13]M. N. Vinoj and V. C. Kuriakose, "Multisoliton solutions and integrability aspects of coupled higher-order nonlinear Schrödinger equations," Phys. Rev. E **62**, 8719 (2000).
[14]T. Xu and X. M. Xu, "Single- and double-hump femtosecond vector solitons in the coupled Sasa-Satsuma system," Phys. Rev. E **87**, 032913 (2013).
[15]V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer Press, Berlin, 1991).
[16]C. H. Gu, H. S. Hu, and Z. X. Zhou, *Darboux Transformation in Soliton Theory and Its Geometric Applications* (Shanghai Scientific and Technical Publishers, Shanghai, 2005).
[17]H. Q. Zhang, B. Tian, J. Li, T. Xu, and Y. X. Zhang, "Symbolic-computation study of integrable properties for the $(2+1)$-dimensional Gardner equation with the two-singular manifold method," IMA J. Appl. Math. **74**, 46 (2009).
[18]I. Gelfand, S. Gelfand, V. Retakh, and R. L. Wilson, "Quasideterminants," Adv. Math. **193**, 56 (2005).
[19]E. V. Doktorov and S. B. Leble, *A Dressing Method in Mathematical Physics* (Springer Press, Dordrecht, 2007).
[20]J. J. C. Nimmo, C. R. Gilson, and Y. Ohta, "Applications of Darboux transformations to the selfdual Yang-Mills equations," Theor. Math. Phys. **122**, 239 (2000).
[21]K. Porsezian and V. C. Kuriakose, *Optical Solitons: Theoretical and Experimental Challenges* (Springer Press, New York, 2003).
[22]M. H. Jakubowski, K. Steiglitz, and R. Squier, "State transformations of colliding optical solitons and possible application to computation in bulk media," Phys. Rev. E **58**, 6752 (1998).
[23]A. Maruta, T. Inoue, Y. Nonaka, and Y. Yoshika, "Bisoliton propagating in dispersion-managed system and its application to high-speed and long-haul optical transmission," IEEE J. Sel. Top. Quantum Electron. **8**, 640 (2002).
[24]T. H. Coskun, D. N. Christodoulides, Y. R. Kim, Z. Chen, M. Soljacic, and M. Segev, "Anti-dark spatial solitons with incoherent light," Quantum Electron. Laser Sci. Conf. Tech. Dig. **40**, 130 (2000).
[25]M. Crosta, A. Fratalocchi, and S. Trillo, "Bistability and instability of dark-antidark solitons in the cubic-quintic nonlinear Schrödinger equation," Phys. Rev. A **84**, 063809 (2011).
[26]R. Cowey, J. M. Christian, and G. S. McDonald, "Spatiotemporal dark and anti-dark solitons in cubic-quintic photonic systems," Salford Postgraduate Annual Research Conference, University of Salford (2013).
[27]H. Q. Zhang, B. G. Zhai, and X. L. Wang, "Dark and antidark soliton solutions in the modified nonlinear Schrödinger equation with distributed coefficients in inhomogeneous fibers," Phys. Scr. **85**, 015006 (2012).
[28]D. Y. Tang, J. Guo, Y. J. Xiang, G. D. Shao, Y. F. Song, L. M. Zhao, and D. Y. Shen, "Observation of anti-dark solitons in fiber lasers," Photonics Fiber Technol. **2016**, JM6A.17.