Riemann–Hilbert approach for a coherently-coupled nonlinear Schrödinger system associated with a $4 \times 4$ matrix spectral problem

Hai-Qiang Zhang$^{a, *}$, Zhi-jie Pei$^{a}$, Wen-Xiu Ma$^{b, c, d, e}$

$^a$ College of Science, P. O. Box 253, University of Shanghai for Science and Technology, Shanghai 200093, China
$^b$ Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
$^c$ Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China
$^d$ College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China
$^e$ International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

A R T I C L E   I N F O

Article history:
Received 30 January 2019
Revised 1 April 2019
Accepted 10 April 2019
Available online 3 May 2019

Keywords:
Riemann–Hilbert problem
Coherently-coupled nonlinear Schrödinger system
Soliton

A B S T R A C T

In this paper, by applying the Riemann–Hilbert approach, we investigate an integrable coherently-coupled nonlinear Schrödinger (CCNLS) system associated with a $4 \times 4$ matrix spectral problem. Through spectral analysis, we formulate a $4 \times 4$ matrix Riemann–Hilbert problem on the real line. Furthermore, from a specific Riemann–Hilbert problem in which a jump matrix is taken to be the identity matrix, we derive the N-soliton solution of the CCNLS system.

© 2019 Elsevier Ltd. All rights reserved.

1. Introduction

One of the hallmarks of integrable nonlinear evolution equations (NLEEs) is that they can be written as the compatibility condition of linear eigenvalue equations which are usually referred to as a Lax pair and comprised of the spatial part and the temporal part. The Lax pair plays an important role in the study of integrable properties of NLEEs like the N-soliton solution. It has been well known that many NLEEs with Lax pairs can be solved by means of the inverse scattering transform (IST) method, such as the Korteweg-de Vries equation [1], the nonlinear Schrödinger (NLS) equation [2,3] and the coupled NLS equations [4,5]. For the IST method, each soliton is associated with a discrete eigenvalue for the scattering problem. Under the reflectionless coefficients, the N-soliton solutions of integrable NLEEs can be derived by solving the Gel’fand–Levitan–Marchenko (GLM) integral equations. Later on, Ref. [6] developed the Riemann–Hilbert formulation which simplifies the reconstruction of the potentials, instead of using the GLM integral equations. In general, by the analysis for analytical properties of the eigenfunction, the inverse problem can be formulated in terms of a Riemann–Hilbert problem. Furthermore, the N-soliton solution of a given NLEE is usually obtained from the asymptotic form of a rational matrix function which has $N$ distinct simple poles. In recent years, the solutions to many integrable equations can be formulated as a solution to an appropriate Riemann–Hilbert problem [7–12]. Moreover, the Riemann–Hilbert method has been generalized to solve initial-boundary value problems of integrable equations on the half-line [13,14].

The coupled NLS equations (Manakov system)

\begin{align}
& i u_{1,t} + u_{1,x} + 2(|u_1|^2 + |u_2|^2)u_1 = 0, \\
& i u_{2,t} + u_{2,x} + 2(|u_2|^2 + |u_1|^2)u_2 = 0,
\end{align}

are a physically and mathematically significant nonlinear model [15]. In optical fibers, the Manakov system (1) can be used to describe the propagation of two optical fields in the Kerr or Kerr-like media [16]. The initial value problems of system (1) both with vanishing and nonvanishing boundary conditions can be solved by the IST method [5,15,17]. The N-soliton solution to the Manakov system has been obtained by the Riemann–Hilbert problem approach [18]. The bright multi-soliton solution have been obtained by the Hirota method [19]. It has been shown that the collisions with complete or partial switching of energy between bright solitons can take place in Manakov system [20,21].
In this paper, we consider the following coherently-coupled NLS system

\begin{align}
&i u_{1,t} + u_{1,xx} + 2 \left( |u_1|^2 + 2 |u_2|^2 \right) u_1 - 2u_1^* u_2 = 0, \\
&i u_{2,t} + u_{2,xx} + 2 \left( |u_1|^2 + 2 |u_2|^2 \right) u_2 - 2u_2^* u_1 = 0,
\end{align}

which can be used to describe the simultaneous propagation of polarized optical waves in an isotropic medium [22,23]. Compared with system (1), the additional last terms represent the coherent coupling that governs the energy exchange between two axes of the fiber [22,23]. For system (2), many exact solutions have been obtained such as the bright multi-soliton solutions [24–28] and rogue wave solutions [29] by the Hirota bilinear method and the Darboux transformation method. In Ref. [25], it has been shown that system (2) admits both degenerate and non-degenerate solitons in which the latter can take single-hump, double-hump and flat-top profiles. Moreover, in Refs. [25,26], the collision mechanisms of bright solitons in system (2) have been revealed, namely, the collisions among degenerate solitons alone or among non-degenerate solitons alone are elastic; the collision of a degenerate soliton with a non-degenerate soliton can undergo nontrivial behavior. Furthermore, the integrability and bright soliton solutions for N-coupled version of system (2) have also been studied extensively in Refs. [30–32].

Recently, Ref. [33] has studied a similar coupled NLS system by the Riemann–Hilbert method. Keeping in mind the features of system (2), we have found that the coupled system (2) is different from that in Ref. [33]. The main reason for this is that the potential matrices in linear spectral problem of two system have different forms (This issues can be seen in Ref. [24]). To the best of our knowledge, the Riemann–Hilbert problem of system (2) associated with a 4 × 4 matrix spectral problem has not been investigated.

In this paper, we will apply the Riemann–Hilbert approach to investigate the coherently-coupled NLS system (2). In Section 2, firstly we study the analyticity of the scattering eigenfunctions for the 4 × 4 Lax pair; Secondly, the inverse problem is formulated as a matrix Riemann–Hilbert problem associated with analytic eigenfunctions. By considering the asymptotic behavior of the eigenfunction for large values of the scattering parameters, we present the reconstruction expressions of the potentials; Finally, we discuss the involution properties and the time evolution of the scattering data. In Section 3, we present the N-soliton solution of the coherently-coupled NLS system (2) from a specific Riemann–Hilbert problem with vanishing scattering coefficients. In Section 4, we give our conclusions.

2. Riemann–Hilbert problem

In this section, we will consider the scattering and inverse scattering transforms, and formulate a Riemann–Hilbert problem on the real line for the coherently-coupled NLS system (2).

The Lax pair associated with Eqs. (2a) and (2b) can be written as the 4 × 4 Ablowitz–Kaup–Newell–Segur form

\begin{align}
\Lambda &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
Q &= \begin{pmatrix} 0 & 0 & u_1 & u_2 \\ 0 & 0 & -u_2 & -u_1 \\ -u_1^* & u_2^* & 0 & 0 \\ -u_2^* & -u_1^* & 0 & 0 \end{pmatrix},
\end{align}

with \( \zeta \) a spectral parameter, \( \mathbf{Y}(x, t, \zeta) \) is a matrix function, and the asterisk denotes the complex conjugate. One can check that the compatibility condition \( \mathbf{Y}_{xt} = \mathbf{Y}_{tx} \) is equivalent to Eqs. (2a) and (2b).

2.1. Spectral analysis of the Lax pair

In our analysis, we always assume that potential functions \( u_1(x, t) \) and \( u_2(x, t) \) decay to zero sufficiently fast as \( x \to \pm \infty \). Hence, we see from Eqs. (3a) and (3b) that \( Y \propto e^{-i \mathbf{L} x - 2i \mathbf{Q} t} \). By introducing the variable transformation

\( Y = J(x, t) e^{-i \mathbf{L} x - 2i \mathbf{Q} t}, \)

we find that the original forms of Lax pair (3a) and (3b) become

\begin{align}
J_t &= -i[\Lambda, J] + QJ, \\
J_x &= -2i\zeta^2 [\Lambda, J] + QJ + \bar{Q}J,
\end{align}

where \( [\Lambda, J] = AJ - JA \) is the matrix commutator, and \( \bar{Q} = \zeta Q + i \Lambda (Q_x - Q^2) \). Notice that the traces of both matrices \( J \) and \( \bar{Q} \) are equal to zero, i.e., \( \text{tr}(Q) = \text{tr}(\bar{Q}) = 0 \) ("\( \text{tr}^* \) denotes the trace of a matrix), and the potential matrix \( Q \) is anti-Hermitian, i.e., \( Q^T = -Q \).

In what follows, we only consider the scattering Eq. (5a) and will treat time \( t \) as a dummy variable. Let us construct two Jost matrix solutions \( J_{\pm} \) for Eq. (5a)

\begin{align}
J_{\pm} &= \begin{pmatrix} \psi_{\pm} & \chi_{\pm} \end{pmatrix}, \quad x \to \pm \infty,
\end{align}

where \( \psi_{\pm} \) is the \( j \)-th column of \( J_{\pm} \), if \( \int_{\pm \infty} \mathbf{Y}(x, t, \zeta) \) converges. Thus, by Eq. (5a), we can turn the scattering Eq. (5a) into Volterra integral equations

\begin{align}
J_{\pm}(x, \zeta) &= I + \int_{\pm \infty} \mathbf{Y}(x, t, \zeta) Q(x, t) \mathbf{Y}(x, t, \zeta) e^{i \mathbf{L}(t-x)} dt,
\end{align}

It can be noted that the existence and uniqueness of the Jost solutions \( J_{\pm} \) for integral Eqs. (8a) and (8b) can be proved according to the standard procedures [4]. Thus, \( J_{\pm} \) must allow analytic continuations off the real axis \( \zeta \in \mathbb{R} \) as long as the integrals on their right hand sides converge. In view of the structure of the potential matrix \( Q \), we find that \( J_{\pm} \) is \( \mathbb{C}^4 \)-valued, and the matrices \( J_{\pm} \) can be analytically continued to the upper half-plane \( \mathbb{C}^+ = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \). In a similar way, \( J_{\pm} \) is \( \mathbb{C}^4 \)-valued, and the matrices \( J_{\pm} \) are analytically continued to the upper half-plane \( \mathbb{C}^- = \{ z \in \mathbb{C} | \text{Im}(z) < 0 \} \).

Utilizing Abel's identity and the boundary condition (7), from Eq. (5a) we see that

\[ \det J_{\pm}(x, \zeta) = 1. \]

Since \( J_{\pm}(x, \zeta) \mathbf{E}_1 \) (\( \mathbf{E}_1 = e^{-i \mathbf{L} x} \)) are both solutions of Eq. (3a), they must be linearly related by a matrix \( S(\zeta) \)

\[ J_{\pm} E_1 = J_{\pm} E_1 S(\zeta), \quad \zeta \in \mathbb{R}, \]

where \( S(\zeta) \) is called the scattering matrix. It is obvious to verify that \( \text{det} S(\zeta) = 1 \) from Eqs. (9) to (10). Furthermore, from Eq. (10) we can derive the integral expression of the scattering matrix

\[ S(\zeta) = I + \int_{\pm \infty} e^{i \mathbf{L}(t-x)} \mathbf{Q}(x, t) e^{-i \mathbf{L}(t-x)} dx. \]

According to the above analytical properties of \( J_{\pm} \), we immediately see that the scattering matrix elements \( s_{11}, s_{12}, s_{21}, \) and \( s_{22} \) can be analytically extended to the upper half-plane \( \mathbb{C}^+ \), whereas \( s_{33}, s_{34}, s_{43}, \) and \( s_{44} \) can be analytically extended to the lower half-plane \( \mathbb{C}^- \). Other elements do not allow analytical extensions to \( \mathbb{C}^\pm \).
In order to present the formulation of a Riemann–Hilbert problem, we introduce the following matrix as a collection of columns in Jost solution $J_\pm(x, \zeta)$

$$\Phi^+(\zeta) = (J_1, \ldots, J_4)$$.  

which is analytic in $\zeta \in \mathbb{C}^+$, where

$$H_1 = \text{diag}(1, 1, 0, 0), \quad H_2 = \text{diag}(0, 0, 1, 1).$$

Furthermore, by considering the large-$\zeta$ asymptotic behavior of $\Phi^+(\zeta)$, we find that

$$\Phi^+(\zeta) \to I, \quad \zeta \in \mathbb{C}^+ \to \infty.$$  

Similarly, the Jost matrix solution

$$(J, J_-, J_1, J_\pm) = J_\pm(x, \zeta)H_1 + J_\zeta(x, \zeta)H_2,$$

is analytic in $\zeta \in \mathbb{C}^-$, and its large-$\zeta$ asymptotic is

$$(J, J_-, J_1, J_\pm) \to I, \quad \zeta \in \mathbb{C}^- \to \infty.$$  

In what follows, we construct the analytical counterpart of $\Phi^+(\zeta)$ in $\mathbb{C}^-$ to formulate a Riemann–Hilbert problem. We consider the adjoint equation of scattering Eq. (5a)

$$K_x = -i\zeta[\Lambda, K] - KQ,$$

where $K(x, t)$ is the Hermitian of $f(x, t)$. It is easy to see that $J^\pm(x, \zeta)$ satisfy adjoint Eq. (17), and have the boundary condition $J^\pm(x, \zeta) \to I$ as $x \to \pm \infty$. We express the inverse Jost matrices $J^\pm_-(x, \zeta)$ as a collection of rows

$$J^\pm_-(x, \zeta) = \begin{pmatrix} [-1]_j^\pm \nonumber \\ [-1]_j^\pm \\ [-1]_j^\pm \\ [-1]_j^\pm \nonumber \end{pmatrix},$$

where $[[J^\pm_\pm]]_j (j = 1, 2, 3, 4)$ denotes the $j$-th row of $J^\pm_-(x, \zeta)$.

By similar spectral analysis used above, we can show that four rows $[[-1]_1^\pm, [-1]_2^\pm, [-1]_3^\pm, [-1]_4^\pm]$ can be analytically extended to the lower half-plane $\mathbb{C}^-$, whereas $[[J^\pm_\pm]]_1, [[J^\pm_\pm]]_4$ allow analytic extensions to the upper half-plane $\mathbb{C}^+$. By collecting the rows in $J^\pm_-(x, \zeta)$, we can show that the matrix solution

$$\Phi^-\zeta = \begin{pmatrix} [-1]_j^\pm \nonumber \\ [-1]_j^\pm \\ [-1]_j^\pm \\ [-1]_j^\pm \nonumber \end{pmatrix} = H_1 J^\pm_1 + H_2 J^\pm_2,$$

is analytic for $\zeta \in \mathbb{C}^-$, and the other matrix solution

$$\begin{pmatrix} [-1]_j^\pm \\ [-1]_j^\pm \\ [-1]_j^\pm \\ [-1]_j^\pm \nonumber \end{pmatrix} = H_1 J^\pm_1 + H_2 J^\pm_2,$$

is analytic for $\zeta \in \mathbb{C}^+$. In the same way, by considering the large-$\zeta$ asymptotic behavior, we find that

$$\Phi^-\zeta \to I, \quad \zeta \in \mathbb{C}^- \to \infty.$$  

and

$$E_{i}^{-1}J^{-1}_i = R(\zeta)E_{i}^{-1}J^{-1}_i, \quad \zeta \in \mathbb{R},$$

where $R(\zeta) \equiv (r_{1i})_{4 \times 4} = S^{-1}(\zeta)$. Similarly, according to the above analytical properties of $J_\pm(x, \zeta)$, we show that the scattering matrix elements $r_{12}, r_{21}, r_{23} \text{ and } r_{24}$ can be analytically extended to the lower half-plane $\mathbb{C}^-$, whereas $r_{33}, r_{34}, r_{43}$ and $r_{44}$ can be analytically extended to the upper half-plane $\mathbb{C}^+$. Other elements do not allow analytical extensions to $\mathbb{C}^+$. Hence, we have constructed two matrix functions $\Phi^\pm(\zeta)$ and $\Phi^\pm(\zeta)$ which are analytic in $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. On the real line, they are related by

$$\Phi^-(\zeta)\Phi^+(\zeta) = G(\zeta), \quad \zeta \in \mathbb{R},$$

where the jump matrix $G(\zeta)$ takes the form

$$G(\zeta) = E \begin{pmatrix} 1 & 0 & r_{13} & r_{14} \\ 0 & 1 & r_{23} & r_{24} \\ s_{31} & s_{32} & 4 & r_{23} \\ s_{41} & s_{42} & 4 & r_{24} \end{pmatrix} E^{-1}.$$  

In fact, by use of $R(\zeta) = S^{-1}(\zeta)$ and $\det(S(\zeta)) = 1$. the jump matrix $G(\zeta)$ can be simplified as

$$G(\zeta) = E \begin{pmatrix} 1 & 0 & r_{13} & r_{14} \\ 0 & 1 & r_{23} & r_{24} \\ s_{31} & s_{32} & 1 & 0 \\ s_{41} & s_{42} & 0 & 1 \end{pmatrix} E^{-1}.$$  

Hence, Eq. (24) determines a matrix Riemann–Hilbert problem on the real $\zeta$-line for the coherently coupled system (2). The canonical normalization condition for this Riemann–Hilbert problem has been obtained from Eqs. (14) to (21)
From Eqs. (30) and (31), it follows that
\begin{equation}
(\Phi_{1}^{+})_{13} = (\Phi_{1}^{+})_{24} = -(\Phi_{1}^{+})_{31} = -(\Phi_{1}^{+})_{42},
\end{equation}
where the element in matrix $M$ is given by
\begin{equation}
M_{lm,ij} = \tilde{v}_{lm,ij}^j, \quad (1 \leq m, l \leq 2, 1 \leq i, j \leq N).
\end{equation}

In general, the Riemann–Hilbert problem (24) is not regular and does not have a unique solution because $\det\Phi^{+}(\zeta)$ and $\det\Phi^{-}(\zeta)$ may have zero roots at certain discrete locations $\zeta_k \in \mathbb{C}^+ \text{ and } \zeta_k \in \mathbb{C}^-$. 

2.2. Riemann–Hilbert problem

In this subsection, we solve the Riemann–Hilbert problem by considering the second order zeros of determinants of eigenfunction matrices. Let $\Phi^{+}(\zeta)$ and $\Phi^{-}(\zeta)$ each have but one second-order zero $\zeta_1$ and $\zeta_1$, respectively:
\begin{equation}
\det\Phi^{+}(\zeta) = (\zeta - \zeta_1)^2 \varphi(\zeta), \quad \det\Phi^{-}(\zeta) = (\zeta - \zeta_1)^2 \varphi(\zeta),
\end{equation}
where $\varphi(\zeta_1) \neq 0$ and $\varphi(\zeta_1) \neq 0$. Suppose the geometric multiplicities of $\zeta_1$ and $\zeta_1$ are the same and equal to its algebraic multiplicity. Thus, in the kernels of the matrices $\Phi^{+}(\zeta)$ and $\Phi^{-}(\zeta)$, vectors $v_{j,1}$ and $\bar{v}_{j,1}$ ($j = 1, 2$) satisfy
\begin{equation}
\Phi^{+}(\zeta)v_{j,1} = 0, \quad \bar{v}_{j,1}\Phi^{-}(\zeta) = 0, \quad (j = 1, 2).
\end{equation}

We introduce two new matrices to cancel the zeros $\zeta_1$ of $\Phi^{+}(\zeta)$ and $\zeta_1$ of $\Phi^{-}(\zeta)$
\begin{equation}
\Phi^{+}_M(\zeta) = \Phi^{+}(\zeta)\chi_1^{-1}(\zeta), \quad \Phi^{-}_M(\zeta) = \chi_1(\zeta)\Phi^{-}(\zeta),
\end{equation}
with
\begin{equation}
\chi_1(\zeta) = I - \frac{\zeta_1 - \zeta_1}{\zeta - \zeta_1} P_k, \quad \chi_1^{-1}(\zeta) = I + \frac{\zeta_1 - \zeta_1}{\zeta - \zeta_1} P_k,
\end{equation}
where $P_k$ is a projector matrix and expressible in the form
\begin{equation}
P_k = \sum_{j,k=1}^{2} v_{j,1} K_{jk} v_{k,1}^\dagger, \quad K_{jk} = v_{j,1}^\dagger v_{k,1}.
\end{equation}

It is easy to see that $\det\chi_1(\zeta) = (\zeta - \zeta_1)^2 (\zeta - \zeta_1)^2$. Then, $\Phi^{+}_M(\zeta)$ is the unique solution to the following regular Riemann–Hilbert problem
\begin{equation}
\Phi^{+}_M(\zeta)\Phi^{-}_M(\zeta) = \chi_1(\zeta)G(\zeta)\chi_1^{-1}(\zeta), \quad \zeta \in \mathbb{R},
\end{equation}
where $\Phi^{+}_M(\zeta)$ and $\Phi^{-}_M(\zeta)$ are nondegenerate and analytic in the domains $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively, and $\Phi^{+}_M(\zeta) \to I$ as $\zeta \to \infty$. By the matrix $\chi_1(\zeta)$, it has been shown that a Riemann–Hilbert problem with zeros is reduced to another without zeros.

In general case, we can consider the case of $N$ pairs of second-order zeros $\{\zeta_k\}_{k=1}^N$ and $\{\zeta_k\}_{k=1}^N$. The geometric multiplicities of $\zeta_k$ and $\zeta_k$ are the same and equal to 2, and the null vectors $[v_{1,k}, v_{2,k}]^\dagger_{k=1}$ and $[\bar{v}_{1,k}, \bar{v}_{2,k}]^\dagger_{k=1}$ from the respective kernels:
\begin{equation}
\Phi^{+}(\zeta_1) v_{j,k} = 0, \quad \bar{v}_{j,k}\Phi^{-}(\zeta_1) = 0, \quad (k = 1, 2, \ldots, N; \ j = 1, 2).
\end{equation}

By repeating the above process of Eq. (36), we have the following regular Riemann–Hilbert problem:
\begin{equation}
\Phi^{+}_M(\zeta)\Phi^{-}_M(\zeta) = \Gamma(\zeta)\Gamma^{-1}(\zeta), \quad \zeta \in \mathbb{R},
\end{equation}
with
\begin{equation}
\Gamma(\zeta) = I - \sum_{i,j=1}^{N} \sum_{m=1}^{2} \frac{2}{v_{lm,j}(\mathbf{M}^{-1})_{lm,ij}} \frac{\bar{v}_{lm,j}}{\zeta - \zeta_k},
\end{equation}
\begin{equation}
\Gamma^{-1}(\zeta) = I + \sum_{i,j=1}^{N} \sum_{m=1}^{2} \frac{1}{v_{lm,j}(\mathbf{M}^{-1})_{lm,ij}} \frac{\bar{v}_{lm,j}}{\zeta - \zeta_k}.
\end{equation}

2.3. Scattering data evolution

In this subsection, we investigate the time evolution of the scattering data (44). Using the vanishing conditions of the potentials and taking the $t$-derivative of Eq. (5b), we get
\begin{equation}
S_k = -2i\zeta_k^2 [\Lambda, S],
\end{equation}
from which the time evolution of the scattering data (44) can be written as
\begin{equation}
s_{3j}(t; \zeta) = s_{3j}(0; \zeta)e^{4i\zeta_k^2 t}, \quad s_{4j}(t; \zeta) = s_{4j}(0; \zeta)e^{4i\zeta_k^2 t},
\end{equation}
\begin{equation}
(j = 1, 2).
\end{equation}

Now we can derive the time evolution for vectors $v_{j,k}$ by taking the $t$-derivative of Eq. (40)
\begin{equation}
\frac{\partial v_{j,k}}{\partial t} + 2\zeta_k^2 \Lambda v_{j,k} = 0.
\end{equation}

Therefore, recalling the spatial dependence (50), we present the expressions of $v_{j,k}$ and $\bar{v}_{j,k}$
\begin{equation}
v_{j,k} = e^{-4i\zeta_k^2 \Lambda - 2i\zeta_k^2 \Lambda t} p_{j,k}, \quad \bar{v}_{j,k} = p_{j,k}^\dagger e^{4i\zeta_k^2 \Lambda - 2i\zeta_k^2 \Lambda t},
\end{equation}
\begin{equation}
(j = 1, 2; 1 \leq k \leq N).
\end{equation}
3. N-soliton solution

It is well known that the soliton solution corresponds to the vanishing of scattering data \([s_j], r_j\), \(j = 1, \ldots, 4\). For this case, the jump matrix \(G(\zeta)\) is a \(4 \times 4\) identity matrix, and corresponding solutions \(u_1(x, t)\) and \(u_2(x, t)\) are called the reflectionless potentials. According to the solution to the regular Riemann–Hilbert problem (47) and the asymptotic expression of \(\Gamma(\zeta)\) in Eq. (42) as \(\zeta \to \infty\), we can derive the matrix function \(\Phi_1^+\) in the expansion (28)

\[
\Phi_1^+(x, t) = -\sum_{i,j=1}^{N} \sum_{m,l=1}^{2} v_{m,i}(M^{-1})_{m,j} \tilde{b}_{i,j},
\]

where the matrix \(M\) has been given in Eq. (43).

Thus, from Eq. (30) the \(N\)-soliton solution of system (2) can be written explicitly as

\[
u_1 = 2\tilde{u}(\Phi_1^+)_{13} = -2i \left( \sum_{i,j=1}^{N} \sum_{m,l=1}^{2} v_{m,i}(M^{-1})_{m,j} \tilde{b}_{i,j} \right)_{13},
\]

and

\[
u_2 = 2\tilde{u}(\Phi_1^+)_{14} = -2i \left( \sum_{i,j=1}^{N} \sum_{m,l=1}^{2} v_{m,i}(M^{-1})_{m,j} \tilde{b}_{i,j} \right)_{14}.
\]

By letting \(p_{1,k} = (\alpha_k, \beta_k, \gamma_k, \delta_k)^T\) and \(p_{2,k} = (-\beta_k, \alpha_k, -\delta_k, \gamma_k)^T\) with \(\alpha_k, \beta_k, \gamma_k\) and \(\delta_k\) as complex constants, one can easily check the identities in Eqs. (32) and (33).

When \(N = 1\), we get the one-soliton solution from the above formulas as

\[
u_1 = \frac{2\xi_1^r \sqrt{\kappa_1 \kappa_2} e^{\rho_1 \cdot t} \cosh(\xi_1 + \eta)}{\sqrt{ab} \left[ 2 \cosh^2(\xi_1 + \eta) - 1 \right] + \frac{\rho}{2}},
\]

\[
u_2 = \frac{2\xi_1^r \sqrt{\mu_1 \mu_2} e^{\rho_1 \cdot t} \cosh(\xi_1^r + \nu)}{\sqrt{ab} \left[ 2 \cosh^2(\xi_1^r + \nu) - 1 \right] + \frac{\rho}{2}},
\]

with

\[
\xi_1 = \theta_1 + \theta_1^*, \quad \theta_1 = -i \xi_1 x - 2i \xi_1^2 t, \quad \epsilon^{2n} = \frac{\kappa_1}{\kappa_2}, \quad \epsilon^{2n} = \frac{\mu_1}{\mu_2},
\]

\[
\kappa_1 = (\alpha_1^2 + \beta_1^2)(\alpha_1^2 \gamma_1^2 + \beta_1^2 \delta_1^2), \quad \kappa_2 = (\gamma_1^2 + \delta_1^2)(\alpha_1 \gamma_1 + \beta_1 \delta_1),
\]

\[
\mu_1 = (\alpha_1^2 + \beta_1^2)(\alpha_1 \gamma_1 - \beta_1 \delta_1), \quad \mu_2 = (\gamma_1^2 + \delta_1^2)(\alpha_1 \delta_1 - \beta_1 \gamma_1),
\]

\[
a = |\alpha_1|^4 + |\beta_1|^4 + |\alpha_1| \beta_1^2 + |\beta_1| \alpha_1^2, \quad b = |\gamma_1|^4 + |\delta_1|^4 + |\gamma_1| \delta_1^2 + |\delta_1| \gamma_1^2,
\]

\[
\rho = (|\alpha_1|^2 + |\beta_1|^2)(|\gamma_1|^2 + |\delta_1|^2) + (|\alpha_1 \beta_1^* - \alpha_1^* \beta_1|)(|\gamma_1 \delta_1^* - \gamma_1^* \delta_1|).
\]

The intensity profiles of the presented one-soliton solution (57a) and (57b) have two kinds of shapes, i.e., the single- and double-hump solitons. Moreover, these solitons can vary their profile from a single hump to a double hump through a flat-top profile. The reason for this is that this one-soliton solution (57a) or

![Fig. 1. A non-degenerate soliton via solution (57): (a) a single-hump non-degenerate soliton in the \(u_1\) component; (b) a double-hump non-degenerate soliton in the \(u_2\) component.](image1)

![Fig. 2. Elastic collision of non-degenerate solitons.](image2)
(57b) corresponds the Riemann-Hilbert problem (39) with the second-order zero. By analyzing the form of solution (57), we find that the one-soliton solution (57) is the same as that (the non-degenerate case) obtained in Refs. [24,25,28]. When the parameters are chosen as \( \gamma_1 = \pm i \delta_1 \) or \( \beta_1 = \pm i \). solution (57) only has the single-hump soliton which is referred to as the degenerate case in Refs. [25,28].

For illustrative purpose, we plot the single- and double-hump solitons from solution (57a) and (57b), as shown in Fig. 1(a) and (b) for the parameters \( \alpha_1 = 0.5 \), \( \beta_1 = 1 - 2i \), \( \gamma_1 = 2i \), \( \delta_1 = 4 \) and \( \xi_1 = 0.1 - i \). For \( N = 2 \), two-soliton solution (56a) and (56b) can describe the collision dynamics of between two single-hump solitons, two double-hump solitons, or single- and double-hump solitons. In three kinds of collisions, under certain parametric conditions the interacting solitons can undergo shape-preserving or shape-changing behaviors between two components. Since the explicit expressions of the two-soliton solution is fairly complicated, we just show an illustrative example in Fig. 2 for the parameters \( \alpha_1 = 0.5 \), \( \beta_1 = 1 - 2i \), \( \gamma_1 = 2i \), \( \delta_1 = 4 \), \( \alpha_2 = 1 \), \( \beta_2 = 2 \), \( \gamma_2 = 1 + i \). \( \delta_2 = 1 \), \( \xi_1 = 0.1 - i \) and \( \xi_2 = -0.1 - i \).  

4. Conclusions

By using the Riemann–Hilbert approach, we have studied an integrable coherently-coupled nonlinear Schrödinger system associated with a 4 \times 4 matrix spectral problem. By analyzing the spectral problem and considering the second order zeros of determinants of eigenfunction matrices, we have presented the Riemann–Hilbert problem. Moreover, from the identity jump matrix corresponding to the reflectionless state, we have generated the \( N \)-soliton solution to the coherently-coupled nonlinear Schrödinger system.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work is supported by the Natural Science Foundation of Shanghai under Grant No. 18ZR1426600, Science and Technology Commission of Shanghai municipality, by the Technology Research and Development Program of University of Shanghai for Science and Technology under Grant No. 2017KJFZ122, The third author is supported in part by NSF under Grant DMS-1664561, the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17KJB110020), and the Distinguished Professorships by Shanghai University of Electric Power, China and North-West University, South Africa.

References

