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Riemann–Hilbert approach for a coherently-coupled nonlinear Schrödinger system associated with a 4 × 4 matrix spectral problem

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ABSTRACT

In this paper, by applying the Riemann–Hilbert approach, we investigate an integrable coherently-coupled nonlinear Schrödinger (CCNLS) system associated with a 4 × 4 matrix spectral problem. Through spectral analysis, we formulate a 4 × 4 matrix Riemann–Hilbert problem on the real line. Furthermore, from a specific Riemann–Hilbert problem in which a jump matrix is taken to be the identity matrix, we derive the *N*-soliton solution of the CCNLS system.

1. Introduction

One of the hallmarks of integrable nonlinear evolution equations (NLEEs) is that they can be written as the compatibility condition of linear eigenvalue equations which are usually referred as a Lax pair and comprised of the spatial part and the temporal part. The Lax pair plays an important role in the study of integrable properties of NLEEs like the *N*-soliton solution. It has been well known that many NLEEs with Lax pairs can be solved by means of the inverse scattering transform (IST) method, such as the Korteweg-de Vries equation [1], the nonlinear Schrödinger (NLS) equation [2,3] and the coupled NLS equations [4,5]. For the IST method, each soliton is associated with a discrete eigenvalue for the scattering problem. Under the reflectionless coefficients, the *N*-soliton solutions of integrable NLEEs can be derived by solving the Gel'fand–Levitan–Marchenko (GLM) integral equations. Later on, Ref. [6] developed the Riemann–Hilbert formulation which simplifies the reconstruction of the potentials, instead of using the GLM integral equations. In general, by the analysis for analytical properties of the eigenfunction, the inverse problem can be formulated in terms of a Riemann–Hilbert problem. Furthermore, the *N*-soliton solution of a given NLEE is usually obtained from the asymptotic form of a rational matrix function which has *N* distinct simple poles. In recent years, the solutions to many integrable equations can be formulated as a solution to an appropriate Riemann–Hilbert problem [7–12]. Moreover, the Riemann–Hilbert method has been generalized to solve initial-boundary value problems of integrable equations on the half-line [13,14].

The coupled NLS equations (Manakov system)

$$i u_{1,t} + u_{1,xx} + 2\left(|u_1|^2 + |u_2|^2\right)u_1 = 0, \tag{1a}$$

$$i u_{2,t} + u_{2,xx} + 2\left(|u_2|^2 + |u_1|^2\right)u_2 = 0, \tag{1b}$$

are a physically and mathematically significant nonlinear model [15]. In optical fibers, the Manakov system (1) can be used to describe the propagation of two optical fields in the Kerr or Kerr-like media [16]. The initial value problems of system (1) both with vanishing and nonvanishing boundary conditions can be solved by the IST method [5,15,17]. The *N*-soliton solution to the Manakov system has been obtained by the Riemann–Hilbert problem approach [18]. The bright multi-soliton solution have been obtained by the Hirota method [19]. It has been shown that the collisions with complete or partial switching of energy between bright solitons can take pace in Manakov system [20,21].

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In this paper, we consider the following coherently-coupled NLS system

$$i u_{1,t} + u_{1,xx} + 2\left(|u_1|^2 + 2|u_2|^2\right)u_1 - 2u_2^2 u_1^* = 0, \tag{2a}$$

$$i u_{2,t} + u_{2,xx} + 2\left(|u_2|^2 + 2|u_1|^2\right)u_2 - 2u_1^2 u_2^* = 0, \tag{2b}$$

which can be used to describe the simultaneous propagation of polarized optical waves in an isotropic medium [22,23]. Compared with system (1), the additional last terms represent the coherent coupling that governs the energy exchange between two axes of the fiber [22,23]. For system (2), many exact solutions have been obtained such as the bright multi-soliton solutions [24–28] and rogue wave solutions [29] by the Hirota bilinear method and the Darboux transformation method. In Ref. [25], it has been shown that system (2) admits both degenerate and non-degenerate solitons in which the latter can take single-hump, double-hump and flat-top profiles. Moreover, in Refs. [25,26], the collision mechanisms of bright solitons in system (2) have been revealed, namely, the collisions among degenerate solitons alone or among non-degenerate solitons alone are elastic; the collision of a degenerate soliton with a non-degenerate soliton can undergo nontrivial behavior. Furthermore, the integrability and bright soliton solutions for N-coupled version of system (2) have also been studied extensively in Refs. [30–32].

Recently, Ref. [33] has studied a similar coupled NLS system by the Riemann–Hilbert method. Keeping in mind the features of system (2), we have found that the coupled system (2) is different from that in Ref. [33]. The main reason for this is that the potential matrices in linear spectral problem of two system have different forms (This issues can be seen in Ref. [24]). To the best of our knowledge, the Riemann–Hilbert problem of system (2) associated with a 4×4 matrix spectral problem has not been investigated.

In this paper, we will apply the Riemann–Hilbert approach to investigate the coherently-coupled NLS system (2). In Section 2, firstly we study the analyticity of the scattering eigenfunctions for the 4×4 Lax pair; Secondly, the inverse problem is formulated as a matrix Riemann–Hilbert problem associated with analytic eigenfunctions. By considering the asymptotic behavior of the eigenfunction for large values of the scattering parameters, we present the reconstruction expressions of the potentials; Finally, we discuss the involution properties and the time evolution of the scattering data. In Section 3, we present the N-soliton solution of the coherently-coupled NLS system (2) from a specific Riemann–Hilbert problem with vanishing scattering coefficients. In Section 4, we give our conclusions.

2. Riemann–Hilbert problem

In this section, we will consider the scattering and inverse scattering transforms, and formulate a Riemann–Hilbert problem on the real line for the coherently-coupled NLS system (2).

The Lax pair associated with Eqs. (2a) and (2b) can be written as the $4\times$ 4 Ablowitz–Kaup–Newell–Segur form

$$Y_x = (-i\zeta\Lambda + Q)Y, \tag{3a}$$

$$Y_t = \left(-2i\zeta^2\Lambda + 2\zeta Q + i\Lambda\left(Q_x - Q^2\right)\right)Y, \tag{3b}$$

with

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & u_1 & u_2 \\ 0 & 0 & -u_2 & u_1 \\ -u_1^* & u_2^* & 0 & 0 \\ -u_2^* & -u_1^* & 0 & 0 \end{pmatrix},$$

where ζ is a spectral parameter, $Y(x, t, \zeta)$ is a matrix function, and the asterisk denotes the complex conjugate. One can check that the compatibility condition $Y_{xt} = Y_{tx}$ is equivalent to Eqs. (2a) and (2b).

2.1. Spectral analysis of the Lax pair

In our analysis, we always assume that potential functions $u_1(x, t)$ and $u_2(x, t)$ decay to zero sufficiently fast as $x\to\pm\infty$. Hence, we see from Eqs. (3a) and (3b) that $Y \propto E = e^{-i\zeta\Lambda x - 2i\zeta^2\Lambda t}$. By introducing the variable transformation

$$Y = J(x,t)e^{-i\zeta\Lambda x - 2i\zeta^2\Lambda t}, \tag{4}$$

we find that the original forms of Lax pair (3a) and (3b) become

$$J_x = -i\zeta[\Lambda, J] + QJ, \tag{5a}$$

$$J_t = -2i\zeta^2[\Lambda, J] + \widetilde{Q}J, \tag{5b}$$

where $[\Lambda, J] = \Lambda J - J\Lambda$ is the matrix commutator, and $\widetilde{Q} = 2\zeta Q + i\Lambda\left(Q_x - Q^2\right)$. Notice that the traces of both matrices Q and $\widetilde{Q}$ are equal to zero, i.e., $\mathrm{tr}(Q) = \mathrm{tr}(\widetilde{Q}) = 0$ ("tr" denotes the trace of a matrix), and the potential matrix Q is anti-Hermitian, i.e., $Q^\dagger = -Q$.

In what follows, we only consider the scattering Eq. (5a) and will treat time t as a dummy variable. Let us construct two Jost matrix solutions $J_\pm(x, \zeta)$ for Eq. (5a)

$$J_\pm = ([J_\pm]_1, [J_\pm]_2, [J_\pm]_3, [J_\pm]_4), \tag{6}$$

with the asymptotic condition

$$J_\pm \to I, \qquad x \to \pm\infty, \tag{7}$$

where $[J_\pm]_j\ (j = 1, 2, 3, 4)$ denotes the jth column of $J_\pm$, I is the 4×4 identity matrix, and the subscripts in $J_\pm$ refer to which end of the x-axis the boundary conditions are set.

Using the method of variation of parameters as well as the boundary condition (7), we can turn the scattering Eq. (5a) into Volterra integral equations

$$J_-(x,\zeta) = I + \int_{-\infty}^{x} e^{i\zeta\Lambda(y-x)}Q(y)J_-(y,\zeta)e^{i\zeta\Lambda(x-y)}dy, \tag{8a}$$

$$J_+(x,\zeta) = I - \int_{x}^{\infty} e^{i\zeta\Lambda(y-x)}Q(y)J_+(y,\zeta)e^{i\zeta\Lambda(x-y)}dy. \tag{8b}$$

It can be noted that the existence and uniqueness of the Jost solutions $J_\pm(x, \zeta)$ for integral Eqs. (8a) and (8b) can be proved according to the standard procedures [4]. Thus, $J_\pm(x, \zeta)$ allow analytical continuations off the real axis $\zeta \in \mathbb{R}$ as long as the integrals on their right hand sides converge. In view of the structure of the potential matrix Q, we find that $[J_-]_1$, $[J_-]_2$, $[J_+]_3$ and $[J_+]_4$ can be analytically continued to the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C}\ |\mathrm{Im}(z) > 0\}$. In a similar way, $[J_-]_3$, $[J_-]_4$, $[J_+]_1$ and $[J_+]_2$ are analytically continued to the upper half-plane $\mathbb{C}^- = \{z \in \mathbb{C}\ |\mathrm{Im}(z) < 0\}$.

Utilizing Abel's identity and the boundary condition (7), from Eq. (5a) we see that

$$\det J_\pm(x,\zeta) = 1. \tag{9}$$

Since $J_\pm(x, \zeta)E_1$ ($E_1 = e^{-i\zeta\Lambda x}$) are both solutions of Eq. (3a), they must be linearly related by a matrix $S(\zeta)$

$$J_- E_1 = J_+ E_1 S(\zeta), \qquad \zeta \in \mathbb{R}, \tag{10}$$

where $S(\zeta) \equiv (s_{ij})_{4\times4}$ is called the scattering matrix. It is obvious to verify that $\det S(\zeta) = 1$ from Eqs. (9) to (10). Furthermore, from Eq. (10) we can derive the integral expression of the scattering matrix

$$S(\zeta) = I + \int_{-\infty}^{+\infty} e^{i\zeta\Lambda x}Q(x)J_-(x,\zeta)e^{-i\zeta\Lambda x}dx. \tag{11}$$

According to the above analytical property of $J_-(x, \zeta)$, we immediately see that the scattering matrix elements s_{11}, s_{12}, s_{21} and s_{22} can be analytically extended to the upper half-plane $\mathbb{C}^+$, whereas s_{33}, s_{34}, s_{43} and s_{44} can be analytically extended to the lower half-plane $\mathbb{C}^-$. Other elements do not allow analytical extensions to $\mathbb{C}^\pm$.

In order to present the formulation of a Riemann–Hilbert problem, we introduce the following matrix as a collection of columns in Jost solution $J_\pm(x, \zeta)$

$$\Phi^+(\zeta) = ([J_-]_1, [J_-]_2, [J_+]_3, [J_+]_4) = J_-(x,\zeta)H_1 + J_+(x,\zeta)H_2, \quad (12)$$

which is analytic in $\zeta \in \mathbb{C}^+$, where

$$H_1 = \mathrm{diag}(1,1,0,0), \qquad H_2 = \mathrm{diag}(0,0,1,1). \quad (13)$$

Furthermore, by considering the large-ζ asymptotic behavior of $\Phi^+(\zeta)$, we find that

$$\Phi^+(\zeta) \to I, \qquad \zeta \in \mathbb{C}^+ \to \infty. \quad (14)$$

Similarly, the Jost matrix solution

$$([J_-]_3, [J_-]_4, [J_+]_1, [J_+]_2) = J_+(x,\zeta)H_1 + J_-(x,\zeta)H_2, \quad (15)$$

is analytic in $\zeta \in \mathbb{C}^-$, and its large-ζ asymptotic is

$$([J_-]_3, [J_-]_4, [J_+]_1, [J_+]_2) \to I, \qquad \zeta \in \mathbb{C}^- \to \infty. \quad (16)$$

In what follows, we construct the analytical counterpart of $\Phi^+(\zeta)$ in $\mathbb{C}^-$ to formulate a Riemann–Hilbert problem. We consider the adjoint equation of scattering Eq. (5a)

$$K_x = -i\zeta[\Lambda, K] - KQ. \quad (17)$$

where $K(x, t)$ is the Hermitian of $J(x, t)$. It is easy to see that $J_\pm^{-1}(x,\zeta)$ satisfy adjoint Eq. (17), and have the boundary condition $J_\pm^{-1}(x,\zeta) \to I$ as $x \to \pm\infty$. We express the inverse Jost matrices $J_\pm^{-1}(x,\zeta)$ as a collection of rows

$$J_\pm^{-1}(x,\zeta) = \begin{pmatrix} [J_\pm^{-1}]^1 \\ [J_\pm^{-1}]^2 \\ [J_\pm^{-1}]^3 \\ [J_\pm^{-1}]^4 \end{pmatrix}, \quad (18)$$

where $[J_\pm^{-1}]^j$ $(j = 1, 2, 3, 4)$ denotes the j-th row of $J_\pm^{-1}(x,\zeta)$.

By similar spectral analysis used above, we can show that four rows $[J_-^{-1}]^1$, $[J_-^{-1}]^2$, $[J_+^{-1}]^3$, $[J_+^{-1}]^4$ can be analytically extended to the lower half-plane $\mathbb{C}^-$, whereas $[J_-^{-1}]^3$, $[J_-^{-1}]^4$, $[J_+^{-1}]^1$, $[J_+^{-1}]^2$ allow analytic extensions to the upper half-plane $\mathbb{C}^+$. By collecting the rows in $J_\pm^{-1}(x,\zeta)$, we can show that the matrix solution

$$\Phi^-(\zeta) = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_-^{-1}]^2 \\ [J_+^{-1}]^3 \\ [J_+^{-1}]^4 \end{pmatrix} = H_1 J_-^{-1} + H_2 J_+^{-1}, \quad (19)$$

is analytic for $\zeta \in \mathbb{C}^-$, and the other matrix solution

$$\begin{pmatrix} [J_-^{-1}]^3 \\ [J_-^{-1}]^4 \\ [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix} = H_1 J_+^{-1} + H_2 J_-^{-1}, \quad (20)$$

is analytic for $\zeta \in \mathbb{C}^+$. In the same way, by considering the large-ζ asymptotic behavior, we find that

$$\Phi^-(\zeta) \to I, \qquad \zeta \in \mathbb{C}^- \to \infty. \quad (21)$$

and

$$\begin{pmatrix} [J_-^{-1}]^3 \\ [J_-^{-1}]^4 \\ [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \end{pmatrix} \to I, \qquad \zeta \in \mathbb{C}^+ \to \infty. \quad (22)$$

By an inverse computation for Eq. (10), we have

$$E_1^{-1}J_-^{-1} = R(\zeta)E_1^{-1}J_+^{-1}, \qquad \zeta \in \mathbb{R}, \quad (23)$$

where $R(\zeta) \equiv (r_{ij})_{4\times 4} = S^{-1}(\zeta)$. Similarly, according to the above analytical properties of $J_\pm(x, \zeta)$, we show that the scattering matrix elements r_{11}, r_{12}, r_{21} and r_{22} can be analytically extended to the lower half-plane $\mathbb{C}^-$, whereas r_{33}, r_{34}, r_{43} and r_{44} can be analytically extended to the upper half-plane $\mathbb{C}^+$. Other elements do not allow analytical extensions to $\mathbb{C}^\pm$.

Hence, we have constructed two matrix functions $\Phi^-(\zeta)$ and $\Phi^+(\zeta)$ which are analytic in $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. On the real line, they are related by

$$\Phi^-(\zeta)\Phi^+(\zeta) = G(\zeta), \qquad \zeta \in \mathbb{R}, \quad (24)$$

where the jump matrix $G(\zeta)$ takes the form

$$G(\zeta) = E\begin{pmatrix} 1 & 0 & r_{13} & r_{14} \\ 0 & 1 & r_{23} & r_{24} \\ s_{31} & s_{32} & \sum_{j=1}^4 r_{j3}s_{3j} & \sum_{j=1}^4 r_{j4}s_{3j} \\ s_{41} & s_{42} & \sum_{j=1}^4 r_{j3}s_{4j} & \sum_{j=1}^4 r_{j4}s_{4j} \end{pmatrix}E^{-1}. \quad (25)$$

In fact, by use of $R(\zeta) = S^{-1}(\zeta)$ and $\det S(\zeta) = 1$, the jump matrix $G(\zeta)$ can be simplified as

$$G(\zeta) = E\begin{pmatrix} 1 & 0 & r_{13} & r_{14} \\ 0 & 1 & r_{23} & r_{24} \\ s_{31} & s_{32} & 1 & 0 \\ s_{41} & s_{42} & 0 & 1 \end{pmatrix}E^{-1}. \quad (26)$$

Hence, Eq. (24) determines a matrix Riemann–Hilbert problem on the real ζ-line for the coherently-coupled system (2). The canonical normalization condition for this Riemann–Hilbert problem has been obtained from Eqs. (14) to (21)

$$\Phi^\pm(\zeta) \to I, \qquad \zeta \in \mathbb{C}^\pm \to \infty. \quad (27)$$

If this Riemann–Hilbert problem can be solved from the given scattering data $\{s_{31}, s_{32}, s_{41}, s_{42}, r_{13}, r_{14}, r_{23}, r_{24}\}$, then the potential Q can be reconstructed from the following asymptotic expansion

$$\Phi^\pm(x,\zeta) = I + \zeta^{-1}\Phi_1^\pm(x) + O(\zeta^{-2}), \qquad \zeta \to \infty. \quad (28)$$

Substituting this expansion into Eq. (5a) and comparing the same power about ζ, we have

$$Q = i[\Lambda, \Phi_1^+] = 2i\begin{pmatrix} 0 & 0 & (\Phi_1^+)_{13} & (\Phi_1^+)_{14} \\ 0 & 0 & (\Phi_1^+)_{23} & (\Phi_1^+)_{24} \\ -(\Phi_1^+)_{31} & -(\Phi_1^+)_{32} & 0 & 0 \\ -(\Phi_1^+)_{41} & -(\Phi_1^+)_{42} & 0 & 0 \end{pmatrix}. \quad (29)$$

Furthermore, the potentials $u_1(x, t)$ and $u_2(x, t)$ can be reconstructed by

$$u_1 = 2i(\Phi_1^+)_{13} = 2i(\Phi_1^+)_{24}, \qquad u_2 = 2i(\Phi_1^+)_{14} = -2i(\Phi_1^+)_{23}, \quad (30)$$

$$u_1^* = 2i(\Phi_1^+)_{31} = 2i(\Phi_1^+)_{42}, \qquad u_2^* = -2i(\Phi_1^+)_{32} = 2i(\Phi_1^+)_{41}. \quad (31)$$

From Eqs. (30) and (31), it follows that

$$\left(\Phi_1^+\right)_{13} = \left(\Phi_1^+\right)_{24} = -\left(\Phi_1^+\right)^*_{31} = -\left(\Phi_1^+\right)^*_{42}, \tag{32}$$

$$\left(\Phi_1^+\right)_{14} = -\left(\Phi_1^+\right)_{23} = \left(\Phi_1^+\right)^*_{32} = -\left(\Phi_1^+\right)^*_{41}. \tag{33}$$

In general, the Riemann–Hilbert problem (24) is not regular and does not have a unique solution because $\det\Phi^+(\zeta)$ and $\det\Phi^-(\zeta)$ may have zero roots at certain discrete locations $\zeta_k \in \mathbb{C}^+$ and $\bar{\zeta}_k \in \mathbb{C}^-$.

2.2. Riemann–Hilbert problem

In this subsection, we solve the Riemann–Hilbert problem by considering the second order zeros of determinants of eigenfunction matrices. Let $\Phi^+(\zeta)$ and $\Phi^-(\zeta)$ each have but one second-order zero ζ_1 and $\bar{\zeta}_1$, respectively:

$$\det\Phi^+(\zeta) = (\zeta-\zeta_1)^2\varphi(\zeta), \qquad \det\Phi^-(\zeta) = (\zeta-\bar{\zeta}_1)^2\bar{\varphi}(\zeta), \tag{34}$$

where $\varphi(\zeta_1)\neq 0$ and $\bar{\varphi}(\bar{\zeta}_1)\neq 0$. Suppose the geometric multiplicities of ζ_1 and $\bar{\zeta}_1$ are the same and equal to its algebraic multiplicity. Thus, in the kernels of the matrices $\Phi^+(\zeta_1)$ and $\Phi^-(\bar{\zeta}_1)$, vectors $v_{j,1}$ and $\bar{v}_{j,1}$ $(j=1,2)$ satisfy

$$\Phi^+(\zeta_1)v_{j,1} = 0, \qquad \bar{v}_{j,1}\Phi^-\left(\bar{\zeta}_1\right) = 0, \qquad (j=1,2). \tag{35}$$

We introduce two new matrices to cancel the zeros ζ_1 of $\Phi^+(\zeta)$ and $\bar{\zeta}_1$ of $\Phi^-(\zeta)$

$$\widehat{\Phi}^+(\zeta) = \Phi^+(\zeta)\chi_1^{-1}(\zeta), \qquad \widehat{\Phi}^-(\zeta) = \chi_1(\zeta)\Phi^-(\zeta), \tag{36}$$

with

$$\chi_1(\zeta) = I - \frac{\zeta_1-\bar{\zeta}_1}{\zeta-\bar{\zeta}_1}P_1, \qquad \chi_1^{-1}(\zeta) = I + \frac{\zeta_1-\bar{\zeta}_1}{\zeta-\zeta_1}P_1, \tag{37}$$

where P_1 is a projector matrix and expressible in the form

$$P_1 = \sum_{j,k=1}^{2} v_{j,1}K_{jk}^{-1}v_{k,1}^\dagger, \qquad K_{jk} = v_{j,1}^\dagger v_{k,1}. \tag{38}$$

It is easy to see that $\det\chi_1(\zeta) = (\zeta-\zeta_1)^2/\left(\zeta-\bar{\zeta}_1\right)^2$. Then, $\widehat{\Phi}^\pm(\zeta)$ is the unique solution to the following regular Riemann–Hilbert problem

$$\widehat{\Phi}^-(\zeta)\widehat{\Phi}^+(\zeta) = \chi_1(\zeta)G(\zeta)\chi_1^{-1}(\zeta), \qquad \zeta\in\mathbb{R}, \tag{39}$$

where $\widehat{\Phi}^-(\zeta)$ and $\widehat{\Phi}^+(\zeta)$ are nondegenerate and analytic in the domains $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively, and $\widehat{\Phi}^\pm(\zeta)\to I$ as $\zeta\to\infty$. By the matrix $\chi_1(\zeta)$, it has been shown that a Riemann–Hilbert problem with zeros is reduced to another one without zeros.

In a general case, we can consider the case of N pairs of second-order zeros $\{\zeta_k\}_{k=1}^N$ and $\{\bar{\zeta}_k\}_{k=1}^N$. The geometric multiplicities of ζ_k and $\bar{\zeta}_k$ are the same and equal to 2, and the null vectors $\{v_{1,k}, v_{2,k}\}_{k=1}^N$ and $\{\bar{v}_{1,k}, \bar{v}_{2,k}\}_{k=1}^N$ from the respective kernels:

$$\Phi^+(\zeta_k)v_{j,k} = 0, \bar{v}_{j,k}\Phi^-(\bar{\zeta}_k) = 0, (k=1,2,\ldots,N; j=1,2). \tag{40}$$

By repeating the above process of Eq. (36), we have the following regular Riemann–Hilbert problem:

$$\widetilde{\Phi}^-(\zeta)\widetilde{\Phi}^+(\zeta) = \Gamma(\zeta)G(\zeta)\Gamma^{-1}(\zeta), \zeta\in\mathbb{R}, \tag{41}$$

with

$$\Gamma(\zeta) = I - \sum_{i,j=1}^{N}\sum_{m,l=1}^{2}\frac{v_{m,i}\left(M^{-1}\right)_{im,jl}\bar{v}_{l,j}}{\zeta-\bar{\zeta}_k},$$

$$\Gamma^{-1}(\zeta) = I + \sum_{i,j=1}^{N}\sum_{m,l=1}^{2}\frac{v_{l,j}\left(M^{-1}\right)_{jl,im}\bar{v}_{m,i}}{\zeta-\zeta_j}, \tag{42}$$

where the element in matrix M is given by

$$M_{im,jl} = \frac{\bar{v}_{m,i}v_{l,j}}{\zeta_j-\bar{\zeta}_i}, \qquad (1\le m,l\le 2; 1\le i,j\le N). \tag{43}$$

To this stage, the following scattering data are needed to solve the nonregular Riemann–Hilbert problem (24)

$$\{s_{3j}(\zeta), s_{4j}(\zeta), r_{j3}(\zeta), r_{j4}(\zeta),\ \zeta\in\mathbb{R}; \zeta_k, \bar{\zeta}_k, v_{j,k}, \bar{v}_{j,k}, (j=1,2; 1\le k\le N)\}. \tag{44}$$

In what follows, using the symmetry relation $Q^\dagger = -Q$, we deduce the involution properties in the scattering matrix as well as in the Jost solutions. The Hermitian equation of the spectral Eq. (5a) is

$$(J_\pm^\dagger)_x = -i\zeta^*\left[\Lambda, J_\pm^\dagger\right] - J_\pm^\dagger Q, \tag{45}$$

from which we obviously see that $J_\pm^\dagger$ satisfy the adjoint Eq. (17). Recalling that $J_\pm^{-1}$ also satisfy the adjoint Eq. (17) and the boundary conditions at $x\to\pm\infty$, we obtain the following involution property

$$J_\pm^{-1}(x,\zeta) = J_\pm^\dagger(x,\zeta^*), \tag{46}$$

which in turn leads to

$$(\Phi^+)^\dagger(\zeta^*) = \Phi^-(\zeta). \tag{47}$$

Furthermore, from this property as well as the relation $J_-E = J_+ES$, we obtain the involution property of the scattering matrix

$$S^\dagger(\zeta^*) = S^{-1}(\zeta) = R(\zeta). \tag{48}$$

Due to the involution property in Eq. (48) and the definitions of $\Phi^\pm(\zeta)$, it is shown that if ζ_k is a zero of $\det\Phi^+(\zeta)$, then $\bar{\zeta}_k = \zeta_k^*$ is a zero of $\det\Phi^-(\zeta)$. Furthermore, we can derive the relationship between each pair of $v_{j,k}(x,t)$ and $\bar{v}_{j,k}(x,t)$ from Eq. (40)

$$\bar{v}_{j,k}(x,t) = v_{j,k}^\dagger(x,t), \qquad (j=1,2; 1\le k\le N). \tag{49}$$

where the spatial evolution for vectors $v_{j,k}$ can be determined by taking the x-derivative of Eq. (40)

$$v_{j,k} = e^{-i\zeta_k\Lambda x}p_{j,k}, \qquad \bar{v}_{j,k} = p_{j,k}^\dagger e^{i\zeta_k^*\Lambda x}, \qquad (j=1,2; 1\le k\le N), \tag{50}$$

where $p_{j,k}$ are column constant vectors.

2.3. Scattering data evolution

In this subsection, we investigate the time evolution of the scattering data (44). Using the vanishing conditions of the potentials and taking the t-derivative of Eq. (5b), we get

$$S_t = -2i\zeta^2[\Lambda, S], \tag{51}$$

from which the time evolution of the scattering data (44) can be written as

$$s_{3j}(t;\zeta) = s_{3j}(0;\zeta)e^{4i\zeta^2 t}, \qquad s_{4j}(t;\zeta) = s_{4j}(0;\zeta)e^{4i\zeta^2 t}, \quad (j=1,2). \tag{52}$$

Now we can derive the time evolution for vectors $v_{j,k}$ by taking the t-derivative of Eq. (40)

$$\frac{\partial v_{jk}}{\partial t} + 2i\zeta_k^2\Lambda v_{jk} = 0. \tag{53}$$

Therefore, recalling the spatial dependence (50), we present the expressions of v_{jk} and $\bar{v}_{j,k}$

$$v_{j,k} = e^{-i\zeta_k\Lambda x - 2i\zeta_k^2\Lambda t}p_{j,k}, \qquad \bar{v}_{j,k} = p_{j,k}^\dagger e^{i\zeta_k^*\Lambda x + 2i\zeta_k^{*2}\Lambda t}, \quad (j=1,2; 1\le k\le N). \tag{54}$$

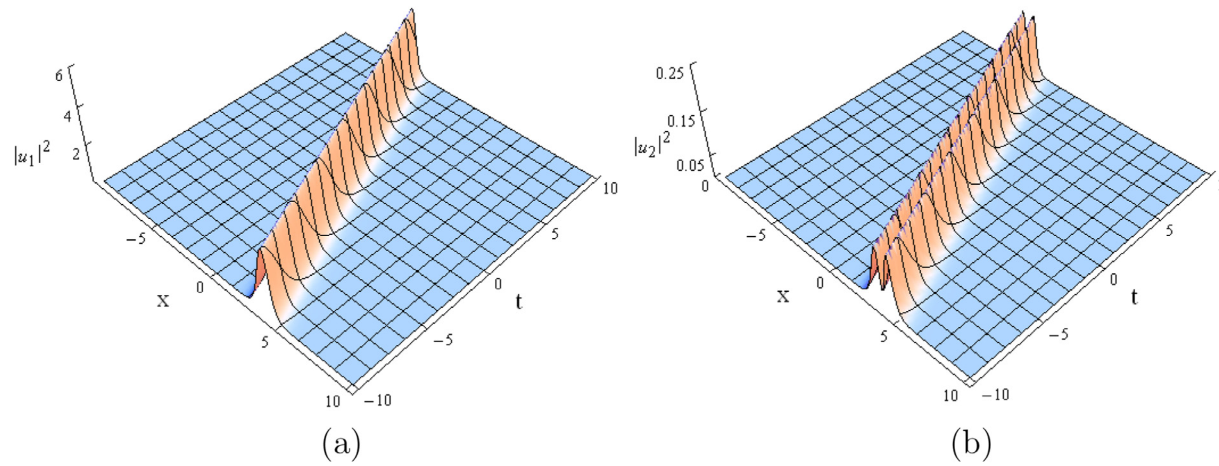

Fig. 1. A non-degenerate soliton via solution (57): (a) a single-hump non-degenerate soliton in the u_1 component; (b) a double-hump non-degenerate soliton in the u_2 component.

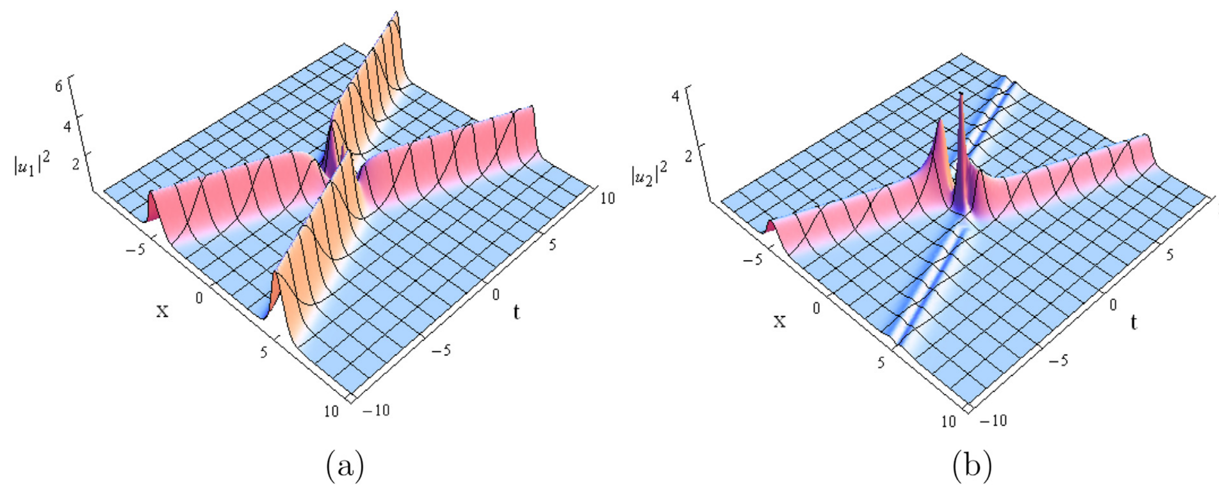

Fig. 2. Elastic collision of non-degenerate solitons.

3. *N*-soliton solution

It is well known that the soliton solution corresponds to the vanishing of scattering data $\{s_{3j}, s_{4j}, r_{j3}, r_{j4}\}_{j=1}^{2}$. For this case, the jump matrix $G(\zeta)$ is a 4×4 identity matrix, and corresponding solutions $u_1(x, t)$ and $u_2(x, t)$ are called the reflectionless potentials. According to the solution to the regular Riemann–Hilbert problem (41) and the asymptotic expression of $\Gamma(\zeta)$ in Eq. (42) as $\zeta\to\infty$, we can derive the matrix function Φ_1^+ in the expansion (28)

$$\Phi_1^+(x,t) = -\sum_{i,j=1}^{N}\sum_{m,l=1}^{2} v_{m,i}\big(M^{-1}\big)_{im,jl}\bar{v}_{l,j}, \tag{55}$$

where the matrix M has been given in Eq. (43).

Thus, from Eq. (30) the N-soliton solution of system (2) can be written explicitly as

$$u_1 = 2i\big(\Phi_1^+\big)_{13} = -2i\left(\sum_{i,j=1}^{N}\sum_{m,l=1}^{2} v_{m,i}\big(M^{-1}\big)_{im,jl}\bar{v}_{l,j}\right)_{13}, \tag{56a}$$

$$u_2 = 2i\big(\Phi_1^+\big)_{14} = -2i\left(\sum_{i,j=1}^{N}\sum_{m,l=1}^{2} v_{m,i}\big(M^{-1}\big)_{im,jl}\bar{v}_{l,j}\right)_{14}. \tag{56b}$$

By letting $p_{1,k} = (\alpha_k, \beta_k, \gamma_k, \delta_k)^T$ and $p_{2,k} = (-\beta_k, \alpha_k, -\delta_k, \gamma_k)^T$ with α_k, β_k, γ_k and δ_k as complex constants, one can easily check the identities in Eqs. (32) and (33).

When $N = 1$, we get the one-soliton solution from the above formulas as

$$u_1 = \frac{2\zeta_{1I}\sqrt{\kappa_1\kappa_2}e^{\theta_1-\theta_1^*}\cosh(\xi_1+\eta)}{\sqrt{ab}\big[2\cosh^2(\xi_1+\eta)-1\big]+\frac{\rho}{2}}, \tag{57a}$$

$$u_2 = \frac{2\zeta_{1I}\sqrt{\mu_1\mu_2}e^{\theta_1-\theta_1^*}\cosh(\xi_1^*+\nu)}{\sqrt{ab}\big[2\cosh^2(\xi_1+\nu)-1\big]+\frac{\rho}{2}}, \tag{57b}$$

with

$$\xi_1 = \theta_1+\theta_1^*, \qquad \theta_1 = -i\zeta_1 x - 2i\zeta_1^2 t, \qquad e^{2\eta} = \frac{\kappa_1}{\kappa_2}, \qquad e^{2\nu} = \frac{\mu_1}{\mu_2},$$
$$\kappa_1 = (\alpha_1^2+\beta_1^2)(\alpha_1^*\gamma_1^*+\beta_1^*\delta_1^*), \quad \kappa_2 = (\gamma_1^{*2}+\delta_1^{*2})(\alpha_1\gamma_1+\beta_1\delta_1),$$
$$\mu_1 = (\alpha_1^2+\beta_1^2)(\alpha_1^*\delta_1^*-\beta_1^*\gamma_1^*), \qquad \mu_2 = (\gamma_1^{*2}+\delta_1^{*2})(\alpha_1\delta_1-\beta_1\gamma_1),$$
$$a = |\alpha_1|^4+|\beta_1|^4+\alpha_1^2\beta_1^{*2}+\beta_1^2\alpha_1^{*2}, \qquad b = |\gamma_1|^4+|\delta_1|^4+\gamma_1^2\delta_1^{*2}+\delta_1^2\gamma_1^{*2},$$
$$\rho = (|\alpha_1|^2+|\beta_1|^2)(|\gamma_1|^2+|\delta_1|^2)+(\alpha_1\beta_1^*-\alpha_1^*\beta_1)(\gamma_1\delta_1^*-\delta_1\gamma_1^*).$$

The intensity profiles of the presented one-soliton solution (57a) and (57b) have two kinds of shapes, i.e., the single- and double-hump solitons. Moreover, these solitons can vary their profile from a single hump to a double hump through a flat-top profile. The reason for this is that this one-soliton solution (57a) or

(57b) corresponds the Riemann-Hilbert problem (39) with the second-order zero. By analyzing the form of solution (57), we find that the one-soliton solution (57) is the same as that (the non-degenerate case) obtained in Refs. [24,25,28]. When the parameters are chosen as $\gamma_1 = \pm i\delta_1$ or $\beta_1 = \pm i$, solution (57) only has the single-hump soliton which is referred to as the degenerate case in Refs. [25,28].

For illustrative purpose, we plot the single- and double-hump solitons from solution (57a) and (57b), as shown in Fig. 1(a) and (b) for the parameters $\alpha_1 = 0.5,\ \beta_1 = 1 - 2i,\ \gamma_1 = 2i,\ \delta_1 = 4$ and $\zeta_1 = 0.1 - i$. For $N = 2$, two-soliton solution (56a) and (56b) can describe the collision dynamics of between two single-hump solitons, two double-hump solitons, or single- and double-hump solitons. In three kinds of collisions, under certain parametric conditions the interacting solitons can undergo shape-preserving or shape-changing behaviors between two components. Since the explicit expressions of the two-soliton solution is fairly complicated, we just show an illustrative example in Fig. 2 for the parameters $\alpha_1 = 0.5,\ \beta_1 = 1 - 2i,\ \gamma_1 = 2i,\ \delta_1 = 4,\ \alpha_2 = 1,\ \beta_2 = 2,\ \gamma_2 = 1 + i,\ \delta_2 = 1,\ \zeta_1 = 0.1 - i$ and $\zeta_2 = -0.1 - i$.

4. Conclusions

By using the Riemann–Hilbert approach, we have studied an integrable coherently-coupled nonlinear Schrödinger system associated with a 4×4 matrix spectral problem. By analyzing the spectral problem and considering the second order zeros of determinants of eigenfunction matrices, we have presented the Riemann–Hilbert problem. Moreover, from the identity jump matrix corresponding to the reflectionless state, we have generated the N-soliton solution to the coherently-coupled nonlinear Schrödinger system.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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