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A STUDY ON LUMP SOLUTIONS TO A (2+1)-DIMENSIONAL COMPLETELY GENERALIZED HIROTA-SATSUMA-ITO EQUATION

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Abstract. We aim to generalize the (2+1)-dimensional Hirota-Satsuma-Ito (HSI) equation, passing the three-soliton test, to a new one which still has diverse solution structures. We add all second-order derivative terms to the HSI equation but demand the existence of lump solutions. Such lump solutions are formulated in terms of the coefficients, except two, in the resulting generalized HSI equation. As an illustrative example, a special completely generalized HSI equation is given, together with a lump solution, and three 3d-plots and contour plots of the lump solution are made to elucidate the characteristics of the presented lump solutions.

1. **Introduction.** Lump solutions are analytical rational function solutions which are localized in all directions in space, originated from solving integrable equations in (2+1)-dimensions (see, e.g., [19, 20, 36]). Taking long wave limits of N-soliton solutions, one can work out specific lumps [34]. Many integrable equations in (2+1)-dimensions exhibit the strickingly high richness of lump solutions (see, e.g., [19, 20]), which can be used to describe various wave phenomena in sciences. Those equations include the KPI equation [21], whose special lump solutions are generated from N-soliton solutions [30], the three-dimensional three-wave resonant interaction [11], the BKP equation [5, 42], the Davey-Stewartson equation II [34], the Ishimori-I equation [10], and the KP equation with a self-consistent source [47].

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In soliton theory, the Hirota bilinear method provides us with a working approach to soliton solutions, historically developed for nonlinear integrable equations [9]. Soliton solutions are analytic and exponentially localized in all directions in space and time. Let a polynomial P define a bilinear differential equation

$$P(D_x, D_y, D_t)f \cdot f = 0,$$

in (2+1)-dimensions, where D_x, D_y and D_t are Hirota's bilinear derivatives [9] (but could also be generalized bilinear derivatives [22]). The associated partial differential equation with a dependent variable u is usually determined by one of the logarithmical transformations:

$$u = 2(\ln f)_x,\ \ u = 2(\ln f)_{xx}.$$

On the basis of bilinear forms, an important step in constructing lump solutions is to find positive quadratic function solutions to bilinear equations [19, 20]. Then through the mentioned logarithmical transformations, one presents lump solutions to nonlinear differential equations (see, e.g., [19] for the case of Hirota bilinear equations and [20] for the case of generalized bilinear equations).

In this paper, we would like to generalize the (2+1)-dimensional Hirota-Satsuma-Ito (HSI) equation to a new one which still has diverse solution structures. Our analysis will be based on the Hirota bilinear formulation (see, e.g., [19, 20, 18, 1] for other equations). We will add all second-order derivative terms to the original HSI bilinear equation while requiring the existence of lump solutions. Via symbolic computations with Maple, we will determine lump solutions in terms of the coefficients, except two, in the resulting completely generalized HSI equation. As an illustrative example, a special completely generalized HSI equation will be presented, together with a lump solution, and three 3d-plots and three contour plots of the lump solution will be made via the Maple plot tool, to shed light on the characteristic of the presented lump solutions. A few concluding remarks will be given in the final section.

2. **Lump solutions.** The Hirota-Satsuma shallow water wave equation reads [9]

$$u_t = u_{xxt} + 3uu_t - 3u_x v_t - u_x,\ \ v_x = -u, \tag{1}$$

which possesses a Hirota bilinear form

$$(D_t D_x^3 - D_t D_x - D_x^2) f \cdot f = 0, \tag{2}$$

under the logarithmic transformations $u = 2(\ln f)_{xx}$ and $v = -2(\ln f)_x$. An integrable $(2+1)$-dimensional extension of this Hirota-Satsuma equation is defined by

$$u_{xxxt} + 3(u_x u_t)_x + u_{yt} + u_{xx} = 0, \tag{3}$$

which passes the Hirota three-soliton test [7], and has a Hirota bilinear form under the logarithmic transformation $u = 2(\ln f)_x$:

$$(D_x^3 D_t + D_y D_t + D_x^2) f \cdot f = 0. \tag{4}$$

We refer the interested readers to [8, 23] for plenty of examples of or supporting details on the Hirota three-soliton test. The nonlinear equation (3) is called the (2+1)-dimensional Hirota-Satsuma-Ito (HSI) equation [7].

We would like to add all four other second-order derivative terms to the (2+1)-dimensional HSI equation to formulate a new one:

$$P(u) = u_{xxxt} + 3(u_x u_t)_x + \delta_1 u_{yt} + \delta_2 u_{xx} + \delta_3 u_{xy} + \delta_4 u_{xt} + \delta_5 u_{yy} + \delta_6 u_{tt} = 0, \tag{5}$$

which has at least diverse lump solutions. This equation is called a completely generalized Hirota-Satsuma-Ito (cgHSI) equation, due to an involvement of all second-order dissipative-type terms. It possesses a Hirota bilinear form under the logarithmic transformation $u = 2(\ln f)_x$:

$$B(f) = (D_x^3 D_t + \delta_1 D_y D_t + \delta_2 D_x^2 + \delta_3 D_x D_y + \delta_4 D_x D_t + \delta_5 D_y^2 + \delta_6 D_t^2) f \cdot f = 0. \quad (6)$$

Actually, we have the relation $P(u) = (\frac{B(f)}{f^2})_x$, under $u = 2(\ln f)_x$.

In what follows, we are going to search for lump solutions to the (2+1) -dimensional cgHSI equation (5), through symbolic computations with Maple. We start to determine positive quadratic solutions to the cgHSI bilinear equation (6):

$$f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9, \quad (7)$$

to present lump solutions to the cgHSI equation (5). Plugging this quadratic function f into the cgHSI bilinear equation (6) leads to a system of algebraic equations on the parameters $a_i,\ 1 \le i \le 9$, and the coefficients δ_i, $1 \le i \le 6$. It consists of ten complicated equations, each of which contains more than twenty terms. Though we have no clue about the existence of solutions, we finally determine, through conducting direct symbolic computations with Maple, a solution for the parameters and the coefficients:

$$\begin{cases} a_9 = -\dfrac{3(a_1^2 + a_5^2)(a_2^2 + a_6^2)(a_1 a_3 + a_5 a_7)}{(a_1 a_6 - a_2 a_5)^2 \delta_2 - (a_1 a_6 - a_2 a_5)(a_2 a_7 - a_3 a_6)\delta_4 + (a_2 a_7 - a_3 a_6)^2 \delta_6}, \\ \delta_1 = \dfrac{b_1}{(a_2^2 + a_6^2)(a_1 a_7 - a_3 a_5)},\ \delta_3 = \dfrac{b_2}{(a_2^2 + a_6^2)(a_1 a_7 - a_3 a_5)}, \end{cases} \quad (8)$$

where the involved two constants b_1 and b_2 are determined by

$$\begin{cases} b_1 = (a_1^2 + a_5^2)(a_1 a_6 - a_2 a_5)\delta_2 - (a_1^2 + a_5^2)(a_2 a_7 - a_3 a_6)\delta_4 \\ \qquad -(a_2^2 + a_6^2)(a_1 a_6 - a_2 a_5)\delta_5 + [(a_3^2 - a_7^2)(a_1 a_6 + a_2 a_5) \\ \qquad -2\, a_3 a_7 (a_1 a_2 - a_5 a_6)]\delta_6, \\ b_2 = [2\, a_1 a_5 (a_2 a_3 - a_6 a_7) - (a_1^2 - a_5^2)(a_2 a_7 + a_3 a_6)]\delta_2 \\ \qquad -(a_3^2 + a_7^2)(a_1 a_6 - a_2 a_5)\delta_4 - (a_2^2 + a_6^2)(a_2 a_7 - a_3 a_6)\delta_5 \\ \qquad +(a_3^2 + a_7^2)(a_2 a_7 - a_3 a_6)\delta_6, \end{cases} \quad (9)$$

and all other a_i's and δ_i's are arbitrary. Those formulas in (8) and (9) were made through a simplification process with the help of Maple.

First, when one takes

$$\delta_1 = 1,\ \delta_2 = 1, \delta_3 = \delta_4 = \delta_5 = \delta_6 = 0, \quad (10)$$

one recovers the original (2+1)-dimensional HSI equation (3), and obtains a system of two algebraic equations on the parameters:

$$\begin{cases} (a_1^2 + a_5^2)(a_1 a_6 - a_2 a_5) = (a_2^2 + a_6^2)(a_1 a_7 - a_3 a_5), \\ 2 a_1 a_5 (a_2 a_3 - a_6 a_7) = (a_1^2 - a_5^2)(a_2 a_7 + a_3 a_6). \end{cases}$$

Solving this system to get the expressions for a_2 and a_6 and substituting them into the expression for a_9 in (8) give rise to

$$\begin{cases} a_2 = -\dfrac{a_1^2a_3 + 2a_1a_5a_7 - a_3a_5^2}{a_3^2 + a_7^2}, \\ a_6 = \dfrac{a_1^2a_7 - 2a_1a_3a_5 - a_5^2a_7}{a_3^2 + a_7^2}, \\ a_9 = -\dfrac{3(a_1^2 + a_5^2)(a_3^2 + a_7^2)(a_1a_3 + a_5a_7)}{(a_1a_7 - a_3a_5)^2}. \end{cases} \tag{11}$$

It is now easy to know that

$$a_1a_6 - a_2a_5 = \frac{(a_1^2 + a_5^2)(a_1a_7 - a_3a_5)}{a_3^2 + a_7^2},$$

and hence, the two conditions of

$$a_1a_3 + a_5a_7 < 0,\ a_1a_7 - a_3a_5 \neq 0 \tag{12}$$

guarantee that $u = 2(\ln f)_x$ with (7) and (11) presents a class of lump solutions to the (2+1)-dimensional HSI equation (3).

Secondly, taking

$$\delta_2 = \delta_6 = 1,\ \delta_4 = -2, \tag{13}$$

we have a compact expression for a_9:

$$a_9 = -\frac{3(a_1^2 + a_5^2)(a_2^2 + a_6^2)(a_1a_3 + a_5a_7)}{[(a_1 - a_3)a_6 - a_2(a_5 - a_7)]^2}, \tag{14}$$

from (8). It then follows that the function f in (7) is positive, if one requires

$$a_1a_3 + a_5a_7 < 0,\ a_1a_6 - a_2a_5 \neq a_3a_6 - a_2a_7. \tag{15}$$

Together with

$$a_1a_6 - a_2a_5 \neq 0, \tag{16}$$

the conditions in (15) ensure that the function f defined by (7) with (8) and (9) yields a class of lump solutions:

$$u = 2(\ln f)_x = \frac{2f_x}{f} \tag{17}$$

to the (2+1)-dimensional cgHSI equation (5) generated with (13).

Further taking

$$a_1 = -1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 2,\ a_5 = -2,\ a_6 = 3,\ a_7 = 2,\ a_8 = 8,\ \delta_5 = \frac{1}{2}, \tag{18}$$

we obtain

$$a_9 = 36,\ \delta_1 = -4,\ \delta_3 = -1, \tag{19}$$

and thus, we have the following special cgHSI equation:

$$u_{xxxt} + 3(u_xu_t)_x - 4u_{yt} + 2u_{xx} - u_{xy} - 2u_{xt} + \frac{1}{2}u_{yy} + u_{tt} = 0, \tag{20}$$

which, through (17), possesses a Hirota bilinear form

$$(D_x^3D_t - 4D_yD_t + 2D_x^2 - D_xD_y - 2D_xD_t + \frac{1}{2}D_y^2 + D_t^2)f \cdot f = 0. \tag{21}$$

The corresponding lump solution, defined by (17), for the special cgHSI equation (20) reads

$$u = -\frac{4(6\,t - 5\,x + 7\,y + 18)}{(2\,t - x + y + 2)^2 + (2\,t - 2\,x + 3\,y + 8)^2 + 36}. \tag{22}$$

Three 3d-plots and contour plots of this lump solution are made with the Maple plot tool, to shed some light on the characteristics of the presented lump solutions, in Figure 1.

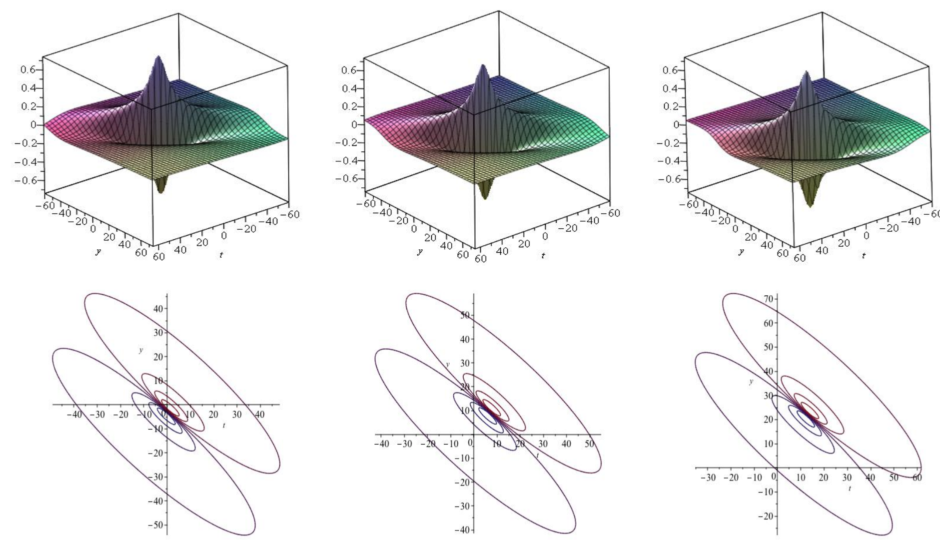

FIGURE 1. Profiles of u when $x = 0, 25, 50$: 3d plots (top) and contour plots (bottom)

3. **Concluding remarks.** We have studied a (2+1)-dimensional completely generalized Hirota-Satsuma-Ito (cgHSI) equation, via symbolic computations with Maple. The results enrich the context of lumps and solitons, adding a new example of (2+1)-dimensional nonlinear equations which possess lump structures. An illustrative example of the resulting cgHSI equation and its lump solution were presented with Maple, together with three 3d-plots and contour plots of the lump solution.

All the exact solutions presented in the last section add helpful insights into the existing theories on soliton solutions and dromion-type solutions, built through many powerful solution techniques such as the Hirota perturbation approach, the Riemann-Hilbert approach, the Wronskian technique, symmetry reductions and symmetry constraints (see, e.g., [14]-[24]).

Recent studies also demonstrate the remarkable richness of lump solutions to linear partial differential equations [25] and nonlinear partial differential equations in (2+1)-dimensions (see, e.g., [48]-[33]) and in (3+1)-dimensions (see, e.g., [26]-[35]). There exist abundant interaction solutions for many integrable equations in (2+1)-dimensions as well, including lump-soliton interaction solutions (see, e.g., [43]-[45]) and lump-kink interaction solutions (see, e.g., [37]-[12]). Diversity of lump and interaction solutions supplements exact solution structures formulated from different kinds of combinations and yields various Lie-Bäcklund symmetries, which can also be used to determine conservation laws by symmetries and adjoint symmetries [28].

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