Darboux Transformations for Bi-integrable Couplings of the AKNS System

Yu-Juan Zhang and Wen-Xiu Ma

Abstract We construct Darboux transformations for bi-integrable couplings of soliton equations. Then we apply the resulting theory to a kind of bi-integrable couplings of the AKNS systems. Particularly, we present exact one-soliton-like solutions for the bi-integrable couplings of the nonlinear Schrödinger equations.

Keywords Non-semisimple Lie algebra · AKNS bi-integrable couplings · Nonlinear Schrödinger equation · Soliton-like solutions

1 Introduction

Integrable systems usually possess linear representations, e.g., Lax representations associated with matrix loop algebras. Simple matrix loop algebras generate integrable systems, and semisimple matrix loop algebras generate separated integrable systems. Integrable couplings [1, 2] are a kind of integrable systems which are associated with non-semisimple matrix loop algebras [3]. Particularly, by enlarging semisimple matrix loop algebras to non-semisimple matrix loop algebras, we obtain Lax pairs for integrable couplings. This is based on a fact that every non-semisimple Lie algebra possesses a semi-direct sum decomposition of a semisimple Lie algebra and a solvable Lie algebra [4], i.e., let $\mathfrak{g}$ denote a non-semisimple Lie algebra.

$$\bar{\mathfrak{g}} = \mathfrak{g} \in\!\!\!\!+ \mathfrak{g}_c, \ \ \mathfrak{g} \text{ - semisimple}, \ \ \mathfrak{g}_c \text{ - solvable}, \tag{1}$$

where the subscript c indicates a contribution to the construction of coupling systems. The notion of semi-direct sum means that the two Lie subalgebras $\mathfrak{g}$ and $\mathfrak{g}_c$ satisfy

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$$[\mathfrak{g}, \mathfrak{g}_c] \subseteq \mathfrak{g}_c, \tag{2}$$

where $[\mathfrak{g}, \mathfrak{g}_c] = \{[A, B] \mid A \in \mathfrak{g},\ B \in \mathfrak{g}_c\}$, with $[\cdot, \cdot]$ denoting the Lie bracket of $\bar{\mathfrak{g}}$. Obviously, $\mathfrak{g}_c$ is an ideal Lie sub-algebra of $\bar{\mathfrak{g}}$.

An integrable coupling of a given integrable system

$$u_t = K(u) \tag{3}$$

is a triangular integrable system of the following form [1, 5, 6]:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = T(u, u_1). \end{cases} \tag{4}$$

Let A_1 and A_2 be square matrices of the same order. Then the 2×2 block matrices

$$M_1(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \end{bmatrix}, \tag{5}$$

define an enlarged Lie algebra $\bar{\mathfrak{g}}$ with the following semidirect sum decomposition:

$$\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c, \quad \mathfrak{g} = \{M_1(A_1, 0)\}, \quad \mathfrak{g}_c = \{M_1(0, A_2)\}, \tag{6}$$

which can be used to generate integrable couplings. Moreover, the variational identity is applied to construct the Hamiltonian structures of integrable couplings [5, 7].

A bi-integrable coupling [8] of a given integrable system (3) is an enlarged triangular integrable system of the following form:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = T_1(u, u_1), \\ u_{2,t} = T_2(u, u_1, u_2). \end{cases} \tag{7}$$

Similarly, let A_1, A_2 and A_3 be square matrices of the same order. Then the 3×3 block matrices of the following type:

$$M_2(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & A_2 \\ 0 & 0 & A_1 \end{bmatrix}, \tag{8}$$

define an enlarged Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c$ with

$$\mathfrak{g} = \{M_2(A_1, 0, 0)\}, \quad \mathfrak{g}_c = \{M_2(0, A_2, A_3)\}, \tag{9}$$

which can be used to generate bi-integrable couplings.

Furthermore, a tri-integrable coupling [6, 9] is of the form:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = T_1(u, u_1), \\ u_{2,t} = T_2(u, u_1, u_2), \\ u_{3,t} = T_2(u, u_1, u_2, u_3). \end{cases} \tag{10}$$

The 4×4 block matrices

$$M_3(A_1, A_2, A_3, A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1 & A_2 & A_3 \\ 0 & 0 & A_1 & A_2 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \tag{11}$$

defining an enlarged Lie algebra $\bar{\mathfrak{g}} = \mathfrak{g} \Subset \mathfrak{g}_c$ with

$$\mathfrak{g} = \{M_3(A_1, 0, 0, 0)\}, \quad \mathfrak{g}_c = \{M_3(0, A_2, A_3, A_4)\},$$

produce Lax pair matrices for tri-integrable couplings.

There are many approaches for solving integrable systems, for example, the homogeneous balance method [10], the Hirota bilinear method [11], the bilinear neural network method [12–15], the transformed rational function method [16], the Darboux transformation [17–19] and the inverse scattering transformation [20]. The Darboux transformation is a pretty systematic and direct approach, and it relies on Lax pairs involving a spectral parameter.

Darboux transformations for integrable couplings have been solved in [21]. In this paper, we construct Darboux transformations for bi-integrable couplings. This paper is organized as follows: In Sect. 2, we present a procedure for constructing Darboux transformations for spectral problems associated with bi-integrable couplings, thereby giving a formula of Darboux transformations of bi-integrable couplings. In Sect. 3, we apply this formula to a kind of bi-integrable couplings of the AKNS hierarchy, and compute exact solutions for the bi-integrable coupling system of the nonlinear Schrödinger equations. At the end, we will give a concluding remark.

2 Darboux Transformations of Bi-integrable Couplings

To define the spectral problems of bi-integrable couplings, we denote $\bar{u} = (u^T, u_1^T, u_2^T)^T$ as the potential functions, where u, u_1, u_2 being N dimensional column vectors. Set $\bar{U}$ and $\bar{V}^{[m]}$ being elements in a non-semisimple matrix loop algebra defined by

$$\bar{U}(\bar{u},\lambda)=\begin{bmatrix} U(u,\lambda) & U_1(u_1,\lambda) & U_2(u_2,\lambda) \\ 0 & U(u,\lambda) & U_1(u_1,\lambda) \\ 0 & 0 & U(u,\lambda) \end{bmatrix}, \tag{12}$$

$$\bar{V}^{[m]}(\bar{u},\lambda)=\begin{bmatrix} V^{[m]}(u,\lambda) & V_1^{[m]}(u,u_1,\lambda) & V_2^{[m]}(u,u_1,u_2,\lambda) \\ 0 & V^{[m]}(u,\lambda) & V_1^{[m]}(u,u_1,\lambda) \\ 0 & 0 & V^{[m]}(u,\lambda) \end{bmatrix}, \tag{13}$$

in which U, U_1, U_2 are $N \times N$ matrices depending on the spectral parameter λ. Set $\bar{\phi}=(\chi^T,\psi^T,\phi^T)^T$ as the enlarged eigenfunction, with χ, ψ, ϕ being N dimensional column vectors. Then the spectral problems of bi-integrable couplings (7) are defined as:

$$\begin{cases} \bar{\phi}_x = \bar{U}(\bar{u},\lambda)\bar{\phi}, \\ \bar{\phi}_{t_m} = \bar{V}^{[m]}(\bar{u},\lambda)\bar{\phi}, \end{cases} \tag{14}$$

where m is a positive integer, indicating the hierarchy. Furthermore, we assume

$$U=\lambda J+P, \quad U_i=\lambda J_i+P_i, \quad i=1,2,$$

where J and J_i being $N \times N$ diagonal matrices, P and P_i being $N \times N$ matrices consisting of dependent variables, which have zero diagonal elements, $V^{[m]}$, $V_1^{[m]}$, $V_2^{[m]}$ being $N \times N$ polynomial matrices of λ:

$$V^{[m]}=\sum_{j=0}^{m} V_j\lambda^{m-j}, \quad V_i^{[m]}=\sum_{j=0}^{m} V_{i,j}\lambda^{m-j}, \quad i=1,2.$$

To construct a Darboux transformation for (14), we rewrite the spectral problems (14) as follows:

$$\begin{cases} \bar{\phi}_x = \bar{U}\bar{\phi} = \left(\lambda\bar{J}+\bar{P}\right)\bar{\phi}, \\ \bar{\phi}_{t_m} = \bar{V}^{[m]}\bar{\phi} = \sum_{j=0}^{m}\bar{V}_j\lambda^{m-j}\bar{\phi}. \end{cases} \tag{15}$$

where

$$\bar{J}=\begin{bmatrix} J & J_1 & J_2 \\ 0 & J & J_1 \\ 0 & 0 & J \end{bmatrix}, \quad \bar{P}=\begin{bmatrix} P & P_1 & P_2 \\ 0 & P & P_1 \\ 0 & 0 & P \end{bmatrix}, \quad \bar{V}_j=\begin{bmatrix} V_j & V_{1,j} & V_{2,j} \\ 0 & V_j & V_{1,j} \\ 0 & 0 & V_j \end{bmatrix}. \tag{16}$$

Assume that $\bar{D}=\bar{D}(x,t,\lambda)$ is a Darboux matrix, that is to say, $\bar{\phi}'=\bar{D}\bar{\phi}$ satisfies the same form as the spectral problems (15), i.e.,

$$\begin{cases} \bar{\phi}'_x = \bar{U}'\bar{\phi}' = \left(\lambda\bar{J}+\bar{P}'\right)\bar{\phi}', \\ \bar{\phi}'_{t_m} = \bar{V}^{[m]'}\bar{\phi}' = \sum_{j=0}^{m}\bar{V}'_j\lambda^{m-j}\bar{\phi}', \end{cases} \tag{17}$$

where $\bar{P}^{'}$ is the new potential matrix. Therefore $\bar{\phi} \to \bar{\phi}^{'}$, $\bar{P} \to \bar{P}^{'}$ form a Darboux transformation of the spectral problems (14).

In particular, we consider a Darboux matrix of the form $\bar{D}(\lambda) = \lambda\bar{I} - \bar{S}$, where $\bar{I} = diag(I, I, I)$, with I being the $N \times N$ identity matrix. A similar calculation as in [21] shows that $\lambda\bar{I} - \bar{S}$ is a Darboux matrix of the enlarged spectral problems (14) if and only if $\bar{S}$ satisfies

$$\bar{S}_x = [\bar{J}\bar{S} + \bar{P}, \bar{S}], \tag{18}$$

$$\bar{S}_{t_m} = [\sum_{j=0}^{m} \bar{V}_j \bar{S}^{m-j}, \bar{S}]. \tag{19}$$

Moreover, the potentials satisfy

$$\bar{P}^{'} = \bar{P} + [\bar{J}, \bar{S}], \tag{20}$$

where

$$\bar{S} = \begin{bmatrix} S & S_1 & S_2 \\ 0 & S & S_1 \\ 0 & 0 & S \end{bmatrix}, \quad \bar{P}^{'} = \begin{bmatrix} P^{'} & P_1^{'} & P_2^{'} \\ 0 & P^{'} & P_1^{'} \\ 0 & 0 & P^{'} \end{bmatrix}. \tag{21}$$

Introduce $\bar{S} = \bar{H}\bar{\Lambda}\bar{H}^{-1}$ as in the general Darboux transformation theory [17], where

$$\bar{H} = \begin{bmatrix} H & H_1 & H_2 \\ 0 & H & H_1 \\ 0 & 0 & H \end{bmatrix}, \quad \bar{\Lambda} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{bmatrix}, \tag{22}$$

with Λ, H, H_1, H_2 being all $N \times N$ matrices. Then substitute these choices into the expression of $\bar{S}$ in (21), we obtain

$$\begin{cases} S = H\Lambda H^{-1}, \\ S_1 = -H\Lambda H^{-1}H_1H^{-1} + H_1\Lambda H^{-1}, \\ S_2 = H\Lambda H^{-1}H_1H^{-1}H_1H^{-1} - H_1\Lambda H^{-1}H_1H^{-1} - H\Lambda H^{-1}H_2H^{-1} + H_2\Lambda H^{-1}, \end{cases} \tag{23}$$

i.e.,

$$\begin{cases} S = H\Lambda H^{-1}, \\ S_1 = -SH_1H^{-1} + H_1\Lambda H^{-1}, \\ S_2 = -S_1H_1H^{-1} - SH_2H^{-1} + H_2\Lambda H^{-1}. \end{cases} \tag{24}$$

Now let us introduce N eigenvalues $\lambda_1, \cdots, \lambda_N$, and set $\Lambda = diag(\lambda_1, \cdots, \lambda_N)$. Denote the corresponding eigenfunctions by $(\bar{\phi}^{(1)}, \cdots, \bar{\phi}^{(N)})$, where $\bar{\phi}^{(i)} = (\chi^{(i)T}, \psi^{(i)T}, \phi^{(i)T})^T$, $i = 1, \cdots, N$. $\bar{\phi}^{(i)}$ satisfies the spectral problems (14), and thus

$$\begin{cases}\chi_x^{(i)}=(\lambda J+P)\chi^{(i)}+(\lambda J_1+P_1)\psi^{(i)}+(\lambda J_2+P_2)\phi^{(i)},\\ \psi_x^{(i)}=(\lambda J+P)\psi^{(i)}+(\lambda J_1+P_1)\phi^{(i)},\\ \phi_x^{(i)}=(\lambda J+P)\phi^{(i)},\end{cases}$$

$$\begin{cases}\chi_{t_m}^{(i)}=\sum_{j=0}^m V_j\lambda^{m-j}\chi^{(i)}+\sum_{j=0}^m V_{1,j}\lambda^{m-j}\psi^{(i)}+\sum_{j=0}^m V_{2,j}\lambda^{m-j}\phi^{(i)},\\ \psi_{t_m}^{(i)}=\sum_{j=0}^m V_j\lambda^{m-j}\psi^{(i)}+\sum_{j=0}^m V_{1,j}\lambda^{m-j}\phi^{(i)},\\ \phi_{t_m}^{(i)}=\sum_{j=0}^m V_j\lambda^{m-j}\phi^{(i)},\end{cases}$$

where $i=1,\cdots,N$. Now set

$$H=[\phi^{(1)},\cdots,\phi^{(N)}],\quad H_1=[\psi^{(1)},\cdots,\psi^{(N)}],\quad H_2=[\chi^{(1)},\cdots,\chi^{(N)}],\tag{25}$$

where H, H_1, H_2 satisfy

$$\begin{cases}H_x=JH\Lambda+PH,\\ H_{1x}=JH_1\Lambda+PH_1+J_1H\Lambda+P_1H,\\ H_{2x}=JH_2\Lambda+PH_2+J_1H_1\Lambda+P_1H_1+J_2H\Lambda+P_2H;\end{cases}\tag{26}$$

$$\begin{cases}H_{t_m}=\sum_{j=0}^m V_jH\Lambda^{m-j},\\ H_{1t_m}=\sum_{j=0}^m V_jH_1\Lambda^{m-j}+\sum_{j=0}^m V_{1j}H\Lambda^{m-j},\\ H_{2t_m}=\sum_{j=0}^m V_jH_2\Lambda^{m-j}+\sum_{j=0}^m V_{1j}H_1\Lambda^{m-j}+\sum_{j=0}^m V_{2j}H\Lambda^{m-j}.\end{cases}\tag{27}$$

Sum up the above discussions, we obtain the following theorem:

Theorem 1 *Let $\bar{H}$ and $\bar{\Lambda}$ be defined by (22). Then $\bar{H}$ is invertible if and only if H is invertible. When H is invertible, then $\bar{S}=\bar{H}\bar{\Lambda}\bar{H}^{-1}$ can be represented as in (21) with S, S_1, S_2 defined in (23), then $\bar{D}=\lambda\bar{I}-\bar{S}$ is a Darboux matrix of the enlarged spectral problems (14), which leads to the Bäcklund transformation for the bi-integrable coupling (7):*

$$\begin{cases}P^{[1]}=P^{[0]}+[J,S],\\ P_1^{[1]}=P_1^{[0]}+[J,S_1]+[J_1,S],\\ P_2^{[1]}=P_2^{[0]}+[J,S_2]+[J_1,S_1]+[J_2,S],\end{cases}\tag{28}$$

where $P^{[0]}$, $P_1^{[0]}$ and $P_2^{[0]}$ are a giving seed solution.

Proof As we did in the Darboux transformation for the integrable coupling case in Ref. [21], we need to prove that for this choice of $\bar{H}$, the two conditions (18) and (19) are satisfied.

First, let us prove the Eq. (18). It is equivalent to prove

$$\begin{cases}S_x=[JS+P,S],\\ S_{1x}=[JS_1+J_1S+P_1,S]+[JS+P,S_1],\\ S_{2x}=[JS_2+J_1S_1+J_2S+P_2,S]+[JS_1+J_1S+P_1,S_1]+[JS+P,S_2].\end{cases}\tag{29}$$

From Eq. (24), we obtain

$$\begin{cases} S_x = H_x \Lambda H^{-1} - H \Lambda H^{-1} H_x H^{-1}, \\ S_{1x} = -S_x H_1 H^{-1} - S H_{1x} H^{-1} + S H_1 H^{-1} H_x H^{-1} + H_{1x} \Lambda H^{-1} \\ \quad -H_1 \Lambda H^{-1} H_x H^{-1}, \\ S_{2x} = -S_{1x} H_1 H^{-1} - S_1 H_{1x} H^{-1} + S_1 H_1 H^{-1} H_x H^{-1} - S_x H_2 H^{-1} - S H_{2x} H^{-1} \\ \quad +S H_2 H^{-1} H_x H^{-1} + H_{2x} \Lambda H^{-1} - H_2 \Lambda H^{-1} H_x H^{-1}, \end{cases} \tag{30}$$

by using (26). Then a tedious calculation can show that the right hand sides of (30) are equal to the right hand sides of (29), respectively. Thus we proved Eq. (18).

Second, let us prove Eq. (19). We compute that

$$\bar{S}^n = \begin{bmatrix} S^n & T_n & M_n \\ 0 & S^n & T_n \\ 0 & 0 & S^n \end{bmatrix}, \tag{31}$$

where

$$M_n = T_n^{(2)} + \sum_{k=1}^{n-1} T_k S_1 S^{n-1-k}, \quad T_n = T_n^{(1)} = \sum_{k=1}^{n} T_{nk}^{(1)}, \quad T_n^{(2)} = \sum_{k=1}^{n} T_{nk}^{(2)}, \tag{32}$$

$$T_{nk}^{(1)} = S^{n-k} S_1 S^{k-1}, \quad T_{nk}^{(2)} = S^{n-k} S_2 S^{k-1}, \tag{33}$$

which tell

$$T_{11}^{(1)} = S_1, \quad T_{11}^{(2)} = S_2. \tag{34}$$

Equation (19) is equivalent to

$$\bar{S}_{t_m} = \sum_{j=0}^{m} [\bar{V}_j \bar{S}^{m-j}, \bar{S}], \tag{35}$$

and thus, we need to prove

$$\begin{cases} S_{t_m} = \sum_{j=0}^m [V_j S^{m-j}, S], \\ S_{1t_m} = \sum_{j=0}^m ([V_j S^{m-j}, S_1] + [V_{1j} S^{m-j}, S] \\ +[V_j T_{m-j}, S]), \\ S_{2t_m} = \sum_{j=0}^m ([V_j S^{m-j}, S_2] + [V_j T_{m-j}, S_1] + [V_{1j} S^{m-j}, S_1] \\ +[V_j M^{m-j}, S] + [V_{1j} T_{m-j}, S] + [V_{2j} S^{m-j}, S]). \end{cases} \tag{36}$$

From Eq. (24), we obtain

$$\begin{cases} S_{t_m} = H_{t_m}\Lambda H^{-1} - H\Lambda H^{-1}H_{t_m}H^{-1}, \\ S_{1t_m} = -S_{t_m}H_1H^{-1} - SH_{1t_m}H^{-1} + SH_1H^{-1}H_{t_m}H^{-1} + H_{1t_m}\Lambda H^{-1} \\ \qquad -H_1\Lambda H^{-1}H_{t_m}H^{-1}, \\ S_{2t_m} = -S_{1t_m}H_1H^{-1} - S_1H_{1t_m}H^{-1} + S_1H_1H^{-1}H_{t_m}H^{-1} - S_{t_m}H_2H^{-1} \\ \qquad -SH_{2t_m}H^{-1} + SH_2H^{-1}H_{t_m}H^{-1} + H_{2t_m}\Lambda H^{-1} - H_2\Lambda H^{-1}H_{t_m}H^{-1}, \end{cases} \tag{37}$$

by using (27). Then a tedious calculation can show that the right hand sides of (37) are equal to the right hand sides of (36), respectively. Thus we proved Eq. (19).

Finally, a simple computation

$$[\bar{J}, \bar{S}] = \begin{bmatrix} [J, S] & [J, S_1] + [J_1, S] & [J, S_2] + [J_1, S_1] + [J_2, S] \\ 0 & [J, S] & [J, S_1] + [J_1, S] \\ 0 & 0 & [J, S] \end{bmatrix}.$$

Therefore, $\bar{P}'$ and $\bar{P} + [\bar{J}, \bar{S}]$ have the same matrix form, which tells the transformation (20), i.e., $\bar{P}' = \bar{P} + [\bar{J}, \bar{S}]$. The proved transformation (20) generates Bäcklund transformation presented in (28). This completes the proof. □

Specially, for the AKNS systems, we have $N = 2$, so that all the sub-matrix in (12) and (13), as well as J, J_1, J_2 and P are 2×2 matrices. We assume $J_1 = J_2 = J$, and

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & q_1 \\ r_1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & q_2 \\ r_2 & 0 \end{bmatrix}, \tag{38}$$

$$\chi = (\chi_1, \chi_2)^T, \quad \psi = (\psi_1, \psi_2)^T, \quad \phi = (\phi_1, \phi_2)^T, \tag{39}$$

$$u = (q, r)^T, \quad u_1 = (q_1, r_1)^T, \quad u_2 = (q_2, r_2)^T. \tag{40}$$

Take two arbitrary constants λ_1 and λ_2, and denote $\phi_{jk} = \phi_j(\lambda_k)$, $\psi_{jk} = \psi_j(\lambda_k)$, $\chi_{jk} = \chi_j(\lambda_k)$, $j, k = 1, 2$, and set

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad H = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad H_1 = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}, \quad H_2 = \begin{bmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{bmatrix}. \tag{41}$$

Then we obtain the associated Bäcklund transformation

$$\begin{cases} q^{[1]} = P^{[1]}[1, 2], \quad r^{[1]} = P^{[1]}[2, 1], \\ q_1^{[1]} = P_1^{[1]}[1, 2], \quad r_1^{[1]} = P_1^{[1]}[2, 1], \\ q_2^{[1]} = P_2^{[1]}[1, 2], \quad r_2^{[1]} = P_2^{[1]}[2, 1]; \end{cases} \tag{42}$$

concretely,

$$\begin{cases} q^{[1]} = q^{[0]} + 2\,\frac{\phi_{11}\phi_{12}(\lambda_1-\lambda_2)}{\phi_{11}\phi_{22}-\phi_{12}\phi_{21}}, \quad r^{[1]} = r^{[0]} + 2\,\frac{\phi_{21}\phi_{22}(\lambda_1-\lambda_2)}{\phi_{11}\phi_{22}-\phi_{12}\phi_{21}}, \\ q_1^{[1]} = q_1^{[0]} + \frac{2(\lambda_1-\lambda_2)\left(\phi_{11}{}^2\phi_{12}\phi_{22}-\phi_{11}{}^2\phi_{12}\psi_{22}+\phi_{11}{}^2\phi_{22}\psi_{12}-\phi_{11}\phi_{12}{}^2\phi_{21}+\phi_{11}\phi_{12}{}^2\psi_{21}-\phi_{12}{}^2\phi_{21}\psi_{11}\right)}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\ r_1^{[1]} = r_1^{[0]} + \frac{2(\lambda_1-\lambda_2)\left(\phi_{11}\phi_{21}\phi_{22}{}^2+\phi_{11}\phi_{22}{}^2\psi_{21}-\phi_{12}\phi_{21}{}^2\phi_{22}-\phi_{12}\phi_{21}{}^2\psi_{22}+\phi_{21}{}^2\phi_{22}\psi_{12}-\phi_{21}\phi_{22}{}^2\psi_{11}\right)}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^2}, \\ q_2^{[1]} = q_2^{[0]} + \frac{2(\lambda_1-\lambda_2)\zeta_1}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3}, \quad r_2^{[1]} = r_2^{[0]} + \frac{2(\lambda_1-\lambda_2)\zeta_2}{(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})^3}, \end{cases} \tag{43}$$

with

$$\begin{aligned} \zeta_1 = {} & \chi_{11}\phi_{11}\phi_{12}{}^2\phi_{21}\phi_{22} - \chi_{11}\phi_{12}{}^3\phi_{21}{}^2 - \chi_{12}\phi_{11}{}^3\phi_{22}{}^2 + \chi_{12}\phi_{11}{}^2\phi_{12}\phi_{21}\phi_{22} - \chi_{21}\phi_{11}{}^2\phi_{12}{}^2\phi_{22} \\ & +\chi_{21}\phi_{11}\phi_{12}{}^3\phi_{21} + \chi_{22}\phi_{11}{}^3\phi_{12}\phi_{22} - \chi_{22}\phi_{11}{}^2\phi_{12}{}^2\phi_{21} - \phi_{11}{}^3\phi_{12}\phi_{22}{}^2 + \phi_{11}{}^3\phi_{12}\phi_{22}\psi_{22} \\ & -\phi_{11}{}^3\phi_{12}\psi_{22}{}^2 - \phi_{11}{}^3\phi_{22}{}^2\psi_{12} + \phi_{11}{}^3\phi_{22}\psi_{12}\psi_{22} + 2\,\phi_{11}{}^2\phi_{12}{}^2\phi_{21}\phi_{22} - \phi_{11}{}^2\phi_{12}{}^2\phi_{21}\psi_{22} \\ & -\phi_{11}{}^2\phi_{12}{}^2\phi_{22}\psi_{21} + 2\,\phi_{11}{}^2\phi_{12}{}^2\psi_{21}\psi_{22} + \phi_{11}{}^2\phi_{12}\phi_{21}\phi_{22}\psi_{12} + \phi_{11}{}^2\phi_{12}\phi_{21}\psi_{12}\psi_{22} \\ & -2\,\phi_{11}{}^2\phi_{12}\phi_{22}\psi_{12}\psi_{21} - \phi_{11}{}^2\phi_{21}\phi_{22}\psi_{12}{}^2 - \phi_{11}\phi_{12}{}^3\phi_{21}{}^2 + \phi_{11}\phi_{12}{}^3\phi_{21}\psi_{21} \\ & -\phi_{11}\phi_{12}{}^3\psi_{21}{}^2 + \phi_{11}\phi_{12}{}^2\phi_{21}\phi_{22}\psi_{11} - 2\,\phi_{11}\phi_{12}{}^2\phi_{21}\psi_{11}\psi_{22} + \phi_{11}\phi_{12}{}^2\phi_{22}\psi_{11}\psi_{21} \\ & +2\,\phi_{11}\phi_{12}\phi_{21}\phi_{22}\psi_{11}\psi_{12} - \phi_{12}{}^3\phi_{21}{}^2\psi_{11} + \phi_{12}{}^3\phi_{21}\psi_{11}\psi_{21} - \phi_{12}{}^2\phi_{21}\phi_{22}\psi_{11}{}^2, \\ \zeta_2 = {} & \chi_{11}\phi_{11}\phi_{21}\phi_{22}{}^3 - \chi_{11}\phi_{12}\phi_{21}{}^2\phi_{22}{}^2 - \chi_{12}\phi_{11}\phi_{21}{}^2\phi_{22}{}^2 + \chi_{12}\phi_{12}\phi_{21}{}^3\phi_{22} - \chi_{21}\phi_{11}{}^2\phi_{22}{}^3 \\ & +\chi_{21}\phi_{11}\phi_{12}\phi_{21}\phi_{22}{}^2 + \chi_{22}\phi_{11}\phi_{12}\phi_{21}{}^2\phi_{22} - \chi_{22}\phi_{12}{}^2\phi_{21}{}^3 - \phi_{11}{}^2\phi_{21}\phi_{22}{}^3 - \phi_{11}{}^2\phi_{22}{}^3\psi_{21} \\ & +2\,\phi_{11}\phi_{12}\phi_{21}{}^2\phi_{22}{}^2 + \phi_{11}\phi_{12}\phi_{21}{}^2\phi_{22}\psi_{22} - \phi_{11}\phi_{12}\phi_{21}{}^2\psi_{22}{}^2 + \phi_{11}\phi_{12}\phi_{21}\phi_{22}{}^2\psi_{21} \\ & +2\,\phi_{11}\phi_{12}\phi_{21}\phi_{22}\psi_{21}\psi_{22} - \phi_{11}\phi_{12}\phi_{22}{}^2\psi_{21}{}^2 - \phi_{11}\phi_{21}{}^2\phi_{22}{}^2\psi_{12} + \phi_{11}\phi_{21}{}^2\phi_{22}\psi_{12}\psi_{22} \\ & +\phi_{11}\phi_{21}\phi_{22}{}^3\psi_{11} - 2\,\phi_{11}\phi_{21}\phi_{22}{}^2\psi_{12}\psi_{21} + \phi_{11}\phi_{22}{}^3\psi_{11}\psi_{21} - \phi_{12}{}^2\phi_{21}{}^3\phi_{22} - \phi_{12}{}^2\phi_{21}{}^3\psi_{22} \\ & +\phi_{12}\phi_{21}{}^3\phi_{22}\psi_{12} + \phi_{12}\phi_{21}{}^3\psi_{12}\psi_{22} - \phi_{12}\phi_{21}{}^2\phi_{22}{}^2\psi_{11} - 2\,\phi_{12}\phi_{21}{}^2\phi_{22}\psi_{11}\psi_{22} \\ & +\phi_{12}\phi_{21}\phi_{22}{}^2\psi_{11}\psi_{21} - \phi_{21}{}^3\phi_{22}\psi_{12}{}^2 + 2\,\phi_{21}{}^2\phi_{22}{}^2\psi_{11}\psi_{12} - \phi_{21}\phi_{22}{}^3\psi_{11}{}^2. \end{aligned}$$

i.e.,

$$\begin{cases} q^{[1]} = q^{[0]} + \frac{2(\lambda_1-\lambda_2)}{|H|}\phi_{11}\phi_{12}, \quad r^{[1]} = r^{[0]} + \frac{2(\lambda_1-\lambda_2)}{|H|}\phi_{21}\phi_{22}, \\ q_1^{[1]} = q_1^{[0]} + \frac{2(\lambda_1-\lambda_2)}{|H|^2}\left(\zeta_{11}-\zeta_{12}\right), \quad r_1^{[1]} = r_1^{[0]} + \frac{2(\lambda_1-\lambda_2)}{|H|^2}\left(\zeta_{13}-\zeta_{14}\right), \\ q_2^{[1]} = q_2^{[0]} - \frac{2(\lambda_1-\lambda_2)}{|H|^3}\left(\zeta_{21}+\zeta_{22}+\zeta_{23}+\zeta_{24}\right), \\ r_2^{[1]} = r_2^{[0]} - \frac{2(\lambda_1-\lambda_2)}{|H|^3}\left(\zeta_{25}+\zeta_{26}+\zeta_{27}+\zeta_{28}\right), \end{cases} \tag{44}$$

with

$$\zeta_{11} = \begin{vmatrix} \phi_{11} & 0 & 0 & 0 \\ 0 & \phi_{11} & \phi_{11} & 0 \\ 0 & -\psi_{12} & \phi_{12} & \phi_{12} \\ 0 & 0 & \psi_{22} & \phi_{22} \end{vmatrix}, \qquad \zeta_{12} = \begin{vmatrix} \phi_{12} & 0 & 0 & 0 \\ 0 & \phi_{12} & \phi_{12} & 0 \\ 0 & -\psi_{11} & \phi_{11} & \phi_{11} \\ 0 & 0 & \psi_{21} & \phi_{21} \end{vmatrix},$$

$$\zeta_{13}=\begin{vmatrix}\phi_{22}&0&0&0\\0&\phi_{22}&\phi_{22}&0\\0&-\psi_{21}&\phi_{21}&\phi_{21}\\0&0&\psi_{11}&\phi_{11}\end{vmatrix},\qquad \zeta_{14}=\begin{vmatrix}\phi_{21}&0&0&0\\0&\phi_{21}&\phi_{21}&0\\0&-\psi_{22}&\phi_{22}&\phi_{22}\\0&0&\psi_{12}&\phi_{12}\end{vmatrix},$$

$$\zeta_{21}=\begin{vmatrix}\phi_{11}&0&0&0&0&0\\0&\phi_{11}&0&\psi_{21}&\phi_{21}&\phi_{21}\\0&0&\phi_{11}&\phi_{11}&0&0\\0&0&-\psi_{12}&\phi_{12}&\phi_{12}&0\\0&\phi_{12}&0&\psi_{22}&\phi_{22}&\phi_{22}\\0&0&0&\chi_{22}&\phi_{22}&\psi_{22}\end{vmatrix},\qquad \zeta_{22}=\begin{vmatrix}\phi_{11}&0&0&0&0&0\\0&\phi_{11}&\psi_{11}&0&0&0\\0&0&\phi_{12}&0&0&\phi_{22}\\0&0&\chi_{12}&\psi_{12}&0&0\\0&0&0&\phi_{11}&\phi_{21}&2\psi_{21}\\0&\phi_{12}&\psi_{12}&\phi_{12}&\phi_{22}&2\psi_{22}\end{vmatrix},$$

$$\zeta_{23}=\begin{vmatrix}\phi_{12}&0&0&0&0&0\\0&\phi_{12}&0&\psi_{22}&\phi_{22}&\phi_{22}\\0&0&\phi_{12}&\phi_{12}&0&0\\0&0&-\psi_{11}&\phi_{11}&\phi_{11}&0\\0&\phi_{11}&0&\psi_{21}&\phi_{21}&\phi_{21}\\0&0&0&\chi_{21}&\phi_{21}&\psi_{21}\end{vmatrix},\qquad \zeta_{24}=\begin{vmatrix}\phi_{12}&0&0&0&0&0\\0&\phi_{12}&\phi_{11}&0&0&0\\0&0&\phi_{11}&0&0&\phi_{21}\\0&0&\chi_{11}&\psi_{11}&0&0\\0&0&0&\phi_{12}&\phi_{22}&2\psi_{22}\\0&\psi_{12}&\psi_{11}&\phi_{11}&\phi_{21}&2\psi_{21}\end{vmatrix},$$

$$\zeta_{25}=\begin{vmatrix}\phi_{22}&0&0&0&0&0\\0&\phi_{22}&0&\psi_{12}&\phi_{12}&\phi_{12}\\0&0&\phi_{22}&\phi_{22}&0&0\\0&0&-\psi_{21}&\phi_{21}&\phi_{21}&0\\0&\phi_{21}&0&\psi_{11}&\phi_{11}&\phi_{11}\\0&0&0&\chi_{11}&\phi_{11}&\psi_{11}\end{vmatrix},\qquad \zeta_{26}=\begin{vmatrix}\phi_{22}&0&0&0&0&0\\0&\phi_{22}&\psi_{22}&0&0&0\\0&0&\phi_{21}&0&0&\phi_{11}\\0&0&\chi_{21}&\psi_{21}&0&0\\0&0&0&\phi_{22}&\phi_{12}&2\psi_{12}\\0&\phi_{21}&\psi_{21}&\phi_{21}&\phi_{11}&2\psi_{11}\end{vmatrix},$$

$$\zeta_{27}=\begin{vmatrix}\phi_{21}&0&0&0&0&0\\0&\phi_{21}&0&\psi_{11}&\phi_{11}&\phi_{11}\\0&0&\phi_{21}&\phi_{21}&0&0\\0&0&-\psi_{22}&\phi_{22}&\phi_{22}&0\\0&\phi_{22}&0&\psi_{12}&\phi_{12}&\phi_{12}\\0&0&0&\chi_{12}&\phi_{12}&\psi_{12}\end{vmatrix},\qquad \zeta_{28}=\begin{vmatrix}\phi_{21}&0&0&0&0&0\\0&\phi_{21}&\phi_{22}&0&0&0\\0&0&\phi_{22}&0&0&\phi_{12}\\0&0&\chi_{22}&\psi_{22}&0&0\\0&0&0&\phi_{21}&\phi_{11}&2\psi_{11}\\0&\psi_{21}&\psi_{22}&\phi_{22}&\phi_{12}&2\psi_{12}\end{vmatrix}.$$

To obtain the second Darboux transformation, we do this procedure again. Precisely, we derive new eigenfunctions from the first Darboux transformation:

$$\begin{aligned}\bar{\phi}^{[1]}=\bar{\phi}'=\bar{D}\bar{\phi}=(\lambda\bar{I}-\bar{S})\bar{\phi}&=\begin{bmatrix}\lambda I-S&S_1&S_2\\0&\lambda I-S&S_1\\0&0&\lambda I-S\end{bmatrix}\begin{bmatrix}\chi\\\psi\\\phi\end{bmatrix}\\&=\begin{bmatrix}(\lambda I-S)\chi+S_1\psi+S_2\phi\\(\lambda I-S)\psi+S_1\phi\\(\lambda I-S)\phi\end{bmatrix}=\begin{bmatrix}\chi^{[1]}\\\psi^{[1]}\\\phi^{[1]}\end{bmatrix},\end{aligned}\tag{45}$$

which is the new eigenfunction we need to use in the second Darboux transformation. Similarly, $\chi^{[1]}, \psi^{[1]}, \phi^{[1]}$ are two component vectors which we denoted by $\chi^{[1]}=(\chi_1^{[1]},\chi_2^{[1]})^T$, $\psi^{[1]}=(\psi_1^{[1]},\psi_2^{[1]})^T$, $\phi^{[1]}=(\phi_1^{[1]},\phi_2^{[1]})^T$. Assume λ_3 and λ_4 are another two arbitrary constants which are different from λ_1 and λ_2, and denote

$$\phi(\lambda_k) = \begin{pmatrix} \phi_{1k} \\ \phi_{2k} \end{pmatrix} = \begin{pmatrix} \phi_1(\lambda_k) \\ \phi_2(\lambda_k) \end{pmatrix}, \quad \psi(\lambda_k) = \begin{pmatrix} \psi_{1k} \\ \psi_{2k} \end{pmatrix} = \begin{pmatrix} \psi_1(\lambda_k) \\ \psi_2(\lambda_k) \end{pmatrix}, \tag{46}$$

$$\chi(\lambda_k) = \begin{pmatrix} \chi_{1k} \\ \chi_{2k} \end{pmatrix} = \begin{pmatrix} \chi_1(\lambda_k) \\ \chi_2(\lambda_k) \end{pmatrix}, \quad k = 3, 4. \tag{47}$$

Set

$$\tilde{\Lambda} = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix}, \tag{48}$$

$$\begin{cases} \tilde{H} = \left(\phi^{[1]}(\lambda_3), \phi^{[1]}(\lambda_4)\right) = ((\lambda_3 I - S)\phi(\lambda_3), (\lambda_4 I - S)\phi(\lambda_4)), \\ \tilde{H}_1 = \left(\psi^{[1]}(\lambda_3), \psi^{[1]}(\lambda_4)\right) \\ \quad = ((\lambda_3 I - S)\psi(\lambda_3) + S_1\phi(\lambda_3), (\lambda_4 I - S)\psi(\lambda_4) + S_1\phi(\lambda_4)), \\ \tilde{H}_2 = \left(\chi^{[1]}(\lambda_3), \chi^{[1]}(\lambda_4)\right) \\ \quad = ((\lambda_3 I - S)\chi(\lambda_3) + S_1\psi(\lambda_3) + S_2\phi(\lambda_3), (\lambda_4 I - S)\chi(\lambda_4) + S_1\psi(\lambda_4) + S_2\phi(\lambda_4)). \end{cases}$$

Then

$$\begin{cases} \tilde{S} = \tilde{H}\tilde{\Lambda}\tilde{H}^{-1}, \\ \tilde{S}_1 = -\tilde{H}\tilde{\Lambda}\tilde{H}^{-1}\tilde{H}_1\tilde{H}^{-1} + \tilde{H}_1\tilde{\Lambda}\tilde{H}^{-1}, \\ \tilde{S}_2 = \tilde{H}\tilde{\Lambda}\tilde{H}^{-1}\tilde{H}_1\tilde{H}^{-1}\tilde{H}_1\tilde{H}^{-1} - \tilde{H}_1\tilde{\Lambda}\tilde{H}^{-1}\tilde{H}_1\tilde{H}^{-1} - \tilde{H}\tilde{\Lambda}\tilde{H}^{-1}\tilde{H}_2\tilde{H}^{-1} + \tilde{H}_2\tilde{\Lambda}\tilde{H}^{-1}, \end{cases} \tag{49}$$

Such that $\lambda\bar{I} - \bar{\tilde{S}}$ is the second Darboux matrix, where

$$\bar{\tilde{S}} = \begin{bmatrix} \tilde{S} & \tilde{S}_1 & \tilde{S}_2 \\ 0 & \tilde{S} & \tilde{S}_1 \\ 0 & 0 & \tilde{S} \end{bmatrix}. \tag{50}$$

Starting from the first DT solutions $P^{[1]}$, $P_1^{[1]}$ and $P_2^{[1]}$, we obtain the second Darboux transformation solutions:

$$P^{[2]} = P^{[1]} + [J, \tilde{S}], \quad P_1^{[2]} = P_1^{[1]} + [J, \tilde{S}_1] + [J_1, \tilde{S}], \tag{51}$$

$$P_2^{[2]} = P_2^{[1]} + [J, \tilde{S}_2] + [J_1, \tilde{S}_1] + [J_2, \tilde{S}]. \tag{52}$$

3 Applications to Bi-integrable Couplings of the AKNS System

In this section, we construct a hierarchy of AKNS bi-integrable couplings and then apply the resulting Darboux transformation theory to the construction of the one-soliton-like solutions of the AKNS bi-integrable coupling system, particularly, we present the one-soliton-like solution of the bi-integrable couplings of the nonlinear Schrödinger equation.

3.1 *A Hierarchy of the Bi-integrable Couplings of the AKNS System*

First, we construct a hierarchy of AKNS bi-integrable couplings. We assume that $\bar{\phi} = (\chi^T, \psi^T, \phi^T)^T = (\chi_1, \chi_2, \psi_1, \psi_2, \phi_1, \phi_2)^T$ is the eigenfunction, and $\bar{u} = (q, r, q_1, r_1, q_2, r_2)^T$ is the potential. The spatial spectral problem is defined in the first equation of (14) and the Eq. (12), with $J_2 = J_1 = J$, and

$$U(u,\lambda) = \begin{bmatrix} -\lambda & q \\ r & \lambda \end{bmatrix}, \quad U_1(u_1,\lambda) = \begin{bmatrix} -\lambda & q_1 \\ r_1 & \lambda \end{bmatrix}, \quad U_2(u_2,\lambda) = \begin{bmatrix} -\lambda & q_2 \\ r_2 & \lambda \end{bmatrix}, \tag{53}$$

where we denote $u = (q, r)^T$, $u_1 = (q_1, r_1)^T$, $u_2 = (q_2, r_2)^T$. In addition, we introduce

$$\bar{W}(\bar{u},\lambda) = \begin{bmatrix} W(u,\lambda) & W_1(u,u_1,\lambda) & W_2(u,u_1,u_2,\lambda) \\ 0 & W(u,\lambda) & W_1(u,u_1,\lambda) \\ 0 & 0 & W(u,\lambda) \end{bmatrix}, \tag{54}$$

$$W(u,\lambda) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \quad W_1(u,u_1,\lambda) = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}, \quad W_2(\bar{u},\lambda) = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix}. \tag{55}$$

Then the stationary zero curvature equation $\bar{W}_x = [\bar{U}, \bar{W}]$ results in

$$W_x = [U, W], \tag{56}$$
$$W_{1x} = [U, W_1] + [U_1, W], \tag{57}$$
$$W_{2x} = [U, W_2] + [U_1, W_1] + [U_2, W]. \tag{58}$$

i.e.,

$$\begin{cases} a_x = qc - rb, \\ b_x = -2\lambda b - 2qa, \\ c_x = 2\lambda c + 2ra; \end{cases} \tag{59}$$

$$\begin{cases} e_x = qg - rf + q_1 c - r_1 b, \\ f_x = -2\lambda f - 2qe - 2\lambda b - 2q_1 a, \\ g_x = 2\lambda g + 2re + 2\lambda c + 2r_1 a; \end{cases} \tag{60}$$

$$\begin{cases} e'_x = qg' - rf' + q_1 g - r_1 f + q_2 c - r_2 b, \\ f'_x = -2\lambda f' - 2qe' - 2\lambda f - 2q_1 e - 2\lambda b - 2q_2 a, \\ g'_x = 2\lambda g' + 2re' + 2\lambda g + 2r_1 e + 2\lambda c + 2r_2 a. \end{cases} \tag{61}$$

Assume that m is a positive integer, and set

$$W = \sum_{i\geq 0} W_i \lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix} \lambda^{-i}, \tag{62}$$

$$W_1 = \sum_{i\geq 0} W_{1,i} \lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} e_i & f_i \\ g_i & -e_i \end{bmatrix} \lambda^{-i}, \tag{63}$$

$$W_2 = \sum_{i\geq 0} W_{2,i} \lambda^{-i} = \sum_{i\geq 0} \begin{bmatrix} e'_i & f'_i \\ g'_i & -e'_i \end{bmatrix} \lambda^{-i}. \tag{64}$$

Substituting them into the Eqs. (56), (57) and (58), and comparing the coefficients of λ, we obtain the recursion relations:

$$\begin{cases} a_0 = \alpha, \quad b_0 = 0, \quad c_0 = 0, \\ a_{i,x} = qc_i - rb_i, \\ b_{i+1} = -\frac{1}{2} b_{i,x} - qa_i, \\ c_{i+1} = \frac{1}{2} c_{i,x} - ra_i, \end{cases} \quad i \geq 0; \tag{65}$$

$$\begin{cases} e_0 = \beta, \quad f_0 = 0, \quad g_0 = 0, \\ e_{i,x} = qg_i - rf_i + q_1 c_i - r_1 b_i, \\ f_{i+1} = -\frac{1}{2} f_{i,x} - qe_i - q_1 a_i - b_{i+1}, \\ g_{i+1} = \frac{1}{2} g_{i,x} - re_i - r_1 a_i - c_{i+1}, \end{cases} \quad i \geq 0; \tag{66}$$

$$\begin{cases} e'_0 = \gamma, \quad f'_0 = 0, \quad g'_0 = 0, \\ e'_{i,x} = qg'_i - rf'_i + q_1 g_i - r_1 f_i + q_2 c_i - r_2 b_i, \\ f'_{i+1} = -\frac{1}{2} f'_{i,x} - qe'_i - q_1 e_i - q_2 a_i - f_{i+1} - b_{i+1}, \\ g'_{i+1} = \frac{1}{2} g'_{i,x} - re'_i - r_1 e_i - r_2 a_i - g_{i+1} - c_{i+1}, \end{cases} \quad i \geq 0. \tag{67}$$

where α, β and γ are arbitrary constants, real or complex numbers. In addition, choose the constants of integration to be zero:

$$a_i|_{u=0} = 0, \quad e_i|_{(u,u_1)=0} = 0, \quad e_i^{'}|_{(u,u_1,u_2)=0} = 0, \quad i \geq 1. \tag{68}$$

This way we can define $b_i, c_i, f_i, g_i, f_i^{'}, g_i^{'}$, and $a_i, e_i, e_i^{'}$ from (65), (66) and (67) uniquely.

Now, take into account the temporal spectral problem defined in the second equation of (14) and the Eq. (13), with

$$V^{[m]} = \begin{bmatrix} a^{[m]} & b^{[m]} \\ c^{[m]} & -a^{[m]} \end{bmatrix} = (\lambda^m W)_+ = \sum_{i=0}^{m} W_i \lambda^{m-i}, \quad m \geq 0,$$
$$V_1^{[m]} = \begin{bmatrix} e^{[m]} & f^{[m]} \\ g^{[m]} & -e^{[m]} \end{bmatrix} = (\lambda^m W_1)_+ = \sum_{i=0}^{m} W_{1,i} \lambda^{m-i}, \quad m \geq 0,$$
$$V_2^{[m]} = \begin{bmatrix} e^{[m]'} & f^{[m]'} \\ g^{[m]'} & -e^{[m]'} \end{bmatrix} = (\lambda^m W_2)_+ = \sum_{i=0}^{m} W_{2,i} \lambda^{m-i}, \quad m \geq 0.$$

Then the enlarged zero curvature equations: $\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0$, i.e.,

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \tag{69}$$
$$U_{1t_m} - V_{1x}^{[m]} + [U, V_1^{[m]}] + [U_1, V^{[m]}] = 0, \tag{70}$$
$$U_{2t_m} - V_{2x}^{[m]} + [U, V_2^{[m]}] + [U_1, V_1^{[m]}] + [U_2, V^{[m]}] = 0, \tag{71}$$

together with the recursion relations (65), (66) and (67), generate the enlarged hierarchy of AKNS bi-integrable couplings:

$$\bar{u}_{t_m} = \begin{bmatrix} q \\ r \\ q_1 \\ r_1 \\ q_2 \\ r_2 \end{bmatrix}_{t_m} = \bar{K}_m(\bar{u}) = \begin{bmatrix} -2b_{m+1} \\ 2c_{m+1} \\ -2(f_{m+1} + b_{m+1}) \\ 2(g_{m+1} + c_{m+1}) \\ -2(f_{m+1}^{'} + f_{m+1} + b_{m+1}) \\ 2(g_{m+1}^{'} + g_{m+1} + c_{m+1}) \end{bmatrix} \tag{72}$$

$$= \bar{\Phi}^m \begin{bmatrix} -2b_1 \\ 2c_1 \\ -2(f_1 + b_1) \\ 2(g_1 + c_1) \\ -2(f_1^{'} + f_1 + b_1) \\ 2(g_1^{'} + g_1 + c_1) \end{bmatrix} = \bar{\Phi}^m \begin{bmatrix} 2\alpha q \\ -2\alpha r \\ 2(\beta q + \alpha q_1) \\ -2(\beta r + \alpha r_1) \\ 2(\gamma q + \beta q_1 + \alpha q_2) \\ -2(\gamma r + \beta r_1 + \alpha r_2) \end{bmatrix}, \tag{73}$$

where the enlarged hereditary recursion operator $\bar{\Phi}$ reads

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 & 0 \\ \Phi_1 - \Phi & \Phi & 0 \\ \Phi_2 - \Phi_1 & \Phi_1 - \Phi & \Phi \end{bmatrix}, \tag{74}$$

with Φ, Φ_1 and Φ_2 being defined by

$$\Phi = \begin{bmatrix} -\frac{1}{2}\partial + q\partial^{-1}r & q\partial^{-1}q \\ -r\partial^{-1}r & \frac{1}{2}\partial - r\partial^{-1}q \end{bmatrix},$$
$$\Phi_1 = \begin{bmatrix} q_1\partial^{-1}r + q\partial^{-1}r_1 & q_1\partial^{-1}q + q\partial^{-1}q_1 \\ -(r_1\partial^{-1}r + r\partial^{-1}r_1) & -(r_1\partial^{-1}q + r\partial^{-1}q_1) \end{bmatrix},$$
$$\Phi_2 = \begin{bmatrix} q_2\partial^{-1}r + q_1\partial^{-1}r_1 + q\partial^{-1}r_2 & q_2\partial^{-1}q + q_1\partial^{-1}q_1 + q\partial^{-1}q_2 \\ -(r_2\partial^{-1}r + r_1\partial^{-1}r_1 + r\partial^{-1}r_2) & -(r_2\partial^{-1}q + r_1\partial^{-1}q_1 + r\partial^{-1}q_2) \end{bmatrix}.$$

The first few equations are computed as follows:

$$\bar{u}_{t_1} = \bar{K}_1(\bar{u}) = \begin{bmatrix} -\alpha q_x \\ -\alpha r_x \\ -\beta q_x - \alpha(q_{1x} - q_x) \\ -\beta r_x - \alpha(r_{1x} - r_x) \\ -\gamma q_x - \beta(q_{1x} - q_x) - \alpha(q_{2x} - q_{1x}) \\ -\gamma r_x - \beta(r_{1x} - r_x) - \alpha(r_{2x} - r_{1x}) \end{bmatrix}, \tag{75}$$

$$\bar{u}_{t_2} = \bar{K}_2(\bar{u}) = \begin{bmatrix} \alpha K_{2,1} \\ -\alpha \tilde{K}_{2,1} \\ \beta K_{2,1} + \alpha\left(K_{2,2} - 2K_{2,1}\right) \\ -\beta \tilde{K}_{2,1} - \alpha\left(\tilde{K}_{2,2} - 2\tilde{K}_{2,1}\right) \\ \gamma K_{2,1} + \beta\left(K_{2,2} - 2K_{2,1}\right) + \alpha\left(K_{2,3} - 2K_{2,2} + K_{2,1}\right) \\ -\gamma \tilde{K}_{2,1} - \beta\left(\tilde{K}_{2,2} - 2\tilde{K}_{2,1}\right) - \alpha\left(\tilde{K}_{2,3} - 2\tilde{K}_{2,2} + \tilde{K}_{2,1}\right) \end{bmatrix}, \tag{76}$$

with

$$K_{2,1} = \frac{1}{2}q_{xx} - q^2 r, \quad K_{2,2} = \frac{1}{2}q_{1xx} - q^2 r_1 - 2qrq_1,$$
$$K_{2,3} = \frac{1}{2}q_{2xx} - q^2 r_2 - 2qrq_2 - q_1^2 r - 2q_1 r_1 q;$$
$$\tilde{K}_{2,1} = \frac{1}{2}r_{xx} - qr^2, \quad \tilde{K}_{2,2} = \frac{1}{2}r_{1xx} - r^2 q_1 - 2qrr_1,$$
$$\tilde{K}_{2,3} = \frac{1}{2}r_{2xx} - r^2 q_2 - 2qrr_2 - r_1^2 q - 2q_1 r_1 r;$$

$$\bar{u}_{t_3} = \bar{K}_3(\bar{u}) = \begin{bmatrix} \alpha K_{3,1} \\ \alpha \tilde{K}_{3,1} \\ \beta K_{3,1} + \alpha\left(K_{3,2} - 3K_{3,1}\right) \\ \beta \tilde{K}_{3,1} + \alpha\left(\tilde{K}_{3,2} - 3\tilde{K}_{3,1}\right) \\ \gamma K_{3,1} + \beta\left(K_{3,2} - 3K_{3,1}\right) + \alpha\left(K_{3,3} - 3K_{3,2} + 3K_{3,1}\right) \\ \gamma \tilde{K}_{3,1} + \beta\left(\tilde{K}_{3,2} - 3\tilde{K}_{3,1}\right) + \alpha\left(\tilde{K}_{3,3} - 3\tilde{K}_{3,2} + 3\tilde{K}_{3,1}\right) \end{bmatrix}, \tag{77}$$

with

$$\begin{aligned} K_{3,1} &= -\frac{1}{4}q_{xxx} + \frac{3}{2}qrq_x, \quad K_{3,2} = -\frac{1}{4}q_{1xxx} + \frac{3}{2}qrq_{1x} + \frac{3}{2}(q_1r + qr_1)q_x, \\ K_{3,3} &= -\frac{1}{4}q_{2xxx} + \frac{3}{2}qrq_{2x} + \frac{3}{2}(q_2r + qr_2 + q_1r_1)q_x + \frac{3}{2}(q_1r + qr_1)q_{1x}, \\ \tilde{K}_{3,1} &= -\frac{1}{4}r_{xxx} + \frac{3}{2}qrr_x, \quad \tilde{K}_{3,2} = -\frac{1}{4}r_{1xxx} + \frac{3}{2}qrr_{1x} + \frac{3}{2}(q_1r + qr_1)r_x, \\ \tilde{K}_{3,3} &= -\frac{1}{4}r_{2xxx} + \frac{3}{2}qrr_{2x} + \frac{3}{2}(q_2r + qr_2 + q_1r_1)r_x + \frac{3}{2}(q_1r + qr_1)r_{1x}. \end{aligned}$$

3.2 *One-Soliton-Like Solutions to the Bi-integrable Couplings of the Nonlinear Schrödinger Equation*

Let us consider the $\bar{K}_2$ system, i.e., we set $m = 2$ in the AKNS bi-integrable coupling hierarchy (73). The corresponding integrable coupling system (73) of the nonlinear Schrödingier (NLS) equations reads as follows:

$$\begin{cases} q_t = \alpha\left(\frac{1}{2}q_{xx} - q^2r\right), \\ r_t = -\alpha\left(\frac{1}{2}r_{xx} - r^2q\right), \\ q_{1t} = \alpha\left(\frac{1}{2}q_{1xx} - q^2r_1 - 2qrq_1 - q_{xx} + 2q^2r\right) + \beta\left(\frac{1}{2}q_{xx} - q^2r\right), \\ r_{1t} = -\alpha\left(\frac{1}{2}r_{1xx} - r^2q_1 - 2qrr_1 - r_{xx} + 2qr^2\right) - \beta\left(\frac{1}{2}r_{xx} - qr^2\right), \\ q_{2t} = \alpha\left(\frac{1}{2}q_{2xx} - q^2r_2 - 2qrq_2 - q_1^2r - 2q_1r_1q - q_{1xx} + 2q^2r_1 + 4qrq_1 + \frac{1}{2}q_{xx} - q^2r\right) \\ \quad +\beta\left(\frac{1}{2}q_{1xx} - q^2r_1 - 2qrq_1 - q_{xx} + 2q^2r\right) + \gamma\left(\frac{1}{2}q_{xx} - q^2r\right). \\ r_{2t} = -\alpha\left(\frac{1}{2}r_{2xx} - r^2q_2 - 2qrr_2 - r_1^2q - 2q_1r_1r - r_{1xx} + 2r^2q_1 + 4qrr_1 + \frac{1}{2}r_{xx} - qr^2\right) \\ \quad -\beta\left(\frac{1}{2}r_{1xx} - r^2q_1 - 2qrr_1 - r_{xx} + 2qr^2\right) - \gamma\left(\frac{1}{2}r_{xx} - qr^2\right). \end{cases} \tag{78}$$

Starting from the zero seed solution, by solving the corresponding linear systems in (14), we obtain the eigenfunctions:

$$\begin{cases} \chi_1 = \frac{1}{2}\lambda\left(\beta^2\lambda^3t^2 - 2\beta\lambda^2tx + 2\gamma\lambda t + \lambda x^2 - 2x\right)\mathrm{e}^{\lambda(\alpha\lambda t - x)}, \\ \chi_2 = \frac{1}{2}\lambda\left(\beta^2\lambda^3t^2 - 2\beta\lambda^2tx - 2\gamma\lambda t + \lambda x^2 + 2x\right)\mathrm{e}^{-\lambda(\alpha\lambda t - x)}, \\ \psi_1 = \lambda(\beta\lambda t - x)\,\mathrm{e}^{\lambda(\alpha\lambda t - x)}, \quad \psi_2 = -\lambda(\beta\lambda t - x)\,\mathrm{e}^{-\lambda(\alpha\lambda t - x)}, \\ \phi_1 = \mathrm{e}^{\lambda(\alpha\lambda t - x)}, \quad \phi_2 = \mathrm{e}^{-\lambda(\alpha\lambda t - x)}. \end{cases} \tag{79}$$

Substituting (79) into the associated Bäcklund transformation (43), we obtain the one-soliton-like solution of the integrable coupling system defined by (78):

$$\begin{aligned} q &= -(\lambda_1 - \lambda_2)e^{\alpha({\lambda_1}^2 + {\lambda_2}^2)t - (\lambda_1 + \lambda_2)x}\mathrm{sech}\xi, \\ r &= (\lambda_1 - \lambda_2)e^{-\alpha({\lambda_1}^2 + {\lambda_2}^2)t + (\lambda_1 + \lambda_2)x}\mathrm{sech}\xi, \\ q_1 &= -\frac{1}{2}(\lambda_1 - \lambda_2)\rho_1\,\mathrm{sech}^2\xi, \\ r_1 &= -\frac{1}{2}(\lambda_1 - \lambda_2)\rho_2\,\mathrm{sech}^2\xi, \\ q_2 &= -\frac{1}{4}(\lambda_1 - \lambda_2)\rho_3\,\mathrm{sech}^3\xi, \\ r_2 &= \frac{1}{4}(\lambda_1 - \lambda_2)\rho_4\,\mathrm{sech}^3\xi, \end{aligned}$$

where

$$\begin{aligned} \xi &= (\lambda_1 - \lambda_2)[\alpha(\lambda_1 + \lambda_2)t - x], \\ \rho_1 &= (2\beta{\lambda_2}^2t - 2\lambda_2x + 1)e^{2\lambda_1(\alpha\lambda_1t - x)} + (2\beta{\lambda_1}^2t - 2\lambda_1x + 1)e^{2\lambda_2(\alpha\lambda_2t - x)}, \\ \rho_2 &= (2\beta{\lambda_2}^2t - 2\lambda_2x - 1)e^{-2\lambda_1(\alpha\lambda_1t - x)} + (2\beta{\lambda_1}^2t - 2\lambda_1x - 1)e^{-2\lambda_2(\alpha\lambda_2t - x)}, \\ \rho_3 &= (\mu_1 + \delta_1)\,\mathrm{e}^{3\epsilon_2 - \epsilon_1} + (\mu_2 + \delta_2)\,\mathrm{e}^{3\epsilon_1 - \epsilon_2} + (\delta_1 - \mu_1 + \delta_2 - \mu_2 + 4 + 8\nu)\,\mathrm{e}^{\epsilon_1 + \epsilon_2}, \\ \rho_4 &= (\mu_1 - \delta_1)\,\mathrm{e}^{-3\epsilon_2 + \epsilon_1} + (\mu_2 - \delta_2)\,\mathrm{e}^{-(3\epsilon_1 - \epsilon_2)} + (-\delta_1 - \mu_1 - \delta_2 - \mu_2 + 4 + 8\nu) \\ &\quad \mathrm{e}^{-(\epsilon_1 + \epsilon_2)}), \end{aligned}$$

with

$$\begin{aligned} &\mu_i = 2\beta^2{\lambda_i}^4t^2 - 4\beta{\lambda_i}^3xt + 2{\lambda_i}^2x^2 + 1, \quad \delta_i = 2\beta{\lambda_i}^2t + 2\gamma{\lambda_i}^2t - 4\lambda_ix, \\ &\epsilon_i = \alpha{\lambda_i}^2t - \lambda_ix, \quad i = 1, 2, \\ &\nu = \beta^2{\lambda_1}^2{\lambda_2}^2t^2 - \beta{\lambda_1}^2\lambda_2xt - \beta\lambda_1{\lambda_2}^2xt + \lambda_1\lambda_2x^2. \end{aligned}$$

4 Conclusion and Comments

We have successfully constructed a kind of Darboux transformation for bi-integrable couplings. An application was made to the presented bi-integrable couplings of the AKNS system of integrable models. In particular, exact solutions were generated for the bi-integrable couplings of the nonlinear Schrödinger equations. The Darboux transformation for tri-integrable couplings can be similarly constructed. It is expected that physical applications could be presented to these integrable couplings of soliton equations in the future.

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