Yufeng Zhang* and Wen-Xiu Ma*

A Study on Rational Solutions to a KP-like Equation

Abstract: A KP-like nonlinear differential equation is introduced through a generalised bilinear equation which possesses the same bilinear form as the standard KP bilinear equation. By symbolic computation, nine classes of rational solutions to the resulting KP-like equation are generated from a search for polynomial solutions to the corresponding generalised bilinear equation. Three generalised bilinear differential operators adopted are associated with the prime number $p=3$.

Keywords: Generalised Bilinear Form; KP Equation; Rational Solution.

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1 Introduction

Hirota bilinear equations [1] and generalised bilinear equations [2] generate diverse nonlinear equations of mathematical physics, among which are the KdV equation, the Boussinesq equation, the Toda lattie equation, and the KP equation. In recent years, there has been a renewed and growing interest in rational solutions to nonlinear differential equations (see, e.g., [3, 4]). Particularly, rogue wave solutions draw big attention of mathematicians and physicists worldwide, and such rational solutions could be used to describe significant nonlinear wave phenomena in both oceanography [5, 6] and nonlinear optics [7, 8]. It is very natural and interesting for us to study rational solutions to nonlinear differential equations generated from generalised bilinear equations.

Rational solutions to integrable equations (see [1, 9, 10]) have been considered systematically on the basis of the Wronskian formulation, the Casoratian formulation and the Pfaffian formulation. The celebrated examples include the KdV equation and the Boussinesq equation in (1 + 1)-dimensions, the KP equation in (2 + 1)-dimensions, and the Toda lattice equation in (0 + 1)-dimensions (see, e.g., [11–14]). Attempts have been made to find rational solutions to the nonintegrable (3 + 1)-dimensional KP I [15, 16] and KP II [17] by direct approaches including the tanh-function method [18], the tanh-coth function method [19] and the $\frac{G'}{G}$-expansion method [20]. Rational solutions to the (3 + 1)-dimensional KP II can also be generated from rational solutions to the good Boussinesq equation by a transformation of dependent variables [17]. Moreover, bilinear Bäcklund transformations are used to construct rational solutions to (3 + 1)-dimensional generalised KP equations (see, e.g., [21]).

In this article, we introduce a KP-like nonlinear differential equation in terms of a generalised bilinear differential equation of KP type using three generalised bilinear differential operators $D_{3,x}$, $D_{3,t}$ and $D_{3,y}$. We will search for polynomial solutions to the corresponding generalised bilinear equation by Maple symbolic computation and generate nine classes of rational solutions to the resulting KP-like equation. Three particular rational solutions will be plotted to exhbit different distributions of singularities. A few concluding remarks will be given at the end of the article.

2 A KP-like Equation

Let us begin with a generalised bilinear differential equation of KP type:

$$(D_{3,t}D_{3,x}+D_{3,x}^4+D_{3,y}^2)f\cdot f=2f_{xt}f-2f_xf_t+6f_{xx}^2 + 2f_{yy}f-2f_y^2=0. \tag{1}$$

This bilinear equation has the same bilinear form as the standard bilinear KP equation [1]. The bilinear differential operators adopted above are a kind of generalised bilinear differential operators associated with the prime number $p=3$, which are introduced in various research [2, 22, 23]:

***Corresponding authors: Yufeng Zhang,** College of Sciences, China University of Mining and Technology, Xuzhou 221116, PR China, E-mail: mathzhang@126.com; and **Wen-Xiu Ma:** Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA, E-mail: mawx@cas.usf.edu

$$\begin{aligned} D_{p,x}^m D_{p,t}^n f\cdot f &= \left(\frac{\partial}{\partial x}+\alpha_p\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}+\alpha_p\frac{\partial}{\partial t'}\right)^n f(x,t)f(x',t')|_{x'=x,t'=t} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j\frac{\partial^{m-i}}{\partial x^{m-i}}\frac{\partial^i}{\partial x'^i}\frac{\partial^{n-j}}{\partial t^{n-j}}\frac{\partial^j}{\partial t'^j}f(x,t)f(x',t')|_{x'=x,t'=t} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j\frac{\partial^{m+n-i-j}f(x,t)}{\partial x^{m-i}\partial t^{n-j}}\frac{\partial^{i+j}f(x,t)}{\partial x^i\partial t^j},\ m,\ n\ge 0, \end{aligned} \tag{2}$$

where α_p^s is computed under the rule of

$$\alpha_p^s=(-1)^{r_p(s)},\ s=r_p(s) \bmod p. \tag{3}$$

Note that α_p is a symbol and we do not have

$$\alpha_p^i\alpha_p^j=\alpha_p^{i+j},\ i,j\ge 0,$$

when p is a prime number greater than 2.

When $p=3$, we have

$$\alpha_3=-1,\ \alpha_3^2=1,\ \alpha_3^3=1,\ \alpha_3^4=-1,\ \alpha_3^5=1,\ \alpha_3^6=1,\ \alpha_3^7=-1,\ \alpha_3^8=1,\ \alpha_3^9=1,$$

and, thus, we obtain

$$D_{3,t}D_{3,x}f\cdot f=2f_{xt}f-2f_xf_t,\ D_{3,x}^4f\cdot f=6f_{xx}^2,\ D_{3,y}^2f\cdot f=2f_{yy}f-2f_y^2.$$

When $p=2$, which corresponds to the Hirota case, we obtain

$$\begin{cases} D_{2,t}D_{2,x}f\cdot f=2f_{xt}f-2f_xf_t,\\ D_{2,x}^4f\cdot f=2f_{xxxx}f-8f_{xxx}f_x+6f_{xx}^2,\\ D_{2,y}^2f\cdot f=2f_{yy}f-2f_y^2,\end{cases}$$

and so, the standard bilinear KP equation reads [1]:

$$\begin{aligned}(D_{2,t}D_{2,x}+D_{2,x}^4+D_{2,y}^2)f\cdot f&=2f_{xt}f-2f_xf_t+2f_{xxxx}f\\ &-8f_{xxx}f_x+6f_{xx}^2+2f_{yy}f-2f_y^2=0,\end{aligned} \tag{4}$$

and the KP equation:

$$u_{xt}+6uu_{xx}+6u_x^2+u_{xxxx}+u_{yy}=0, \tag{5}$$

through the transformation $u=2(\ln f)_{xx}$.

Bell polynomial theories (see, e.g., [22–24]) suggest a dependent variable transformation

$$u=2(\ln f)_x, \tag{6}$$

to transform bilinear equations to nonlinear equations. From the generalised bilinear (1), we obtain, through (6), a KP-like nonlinear differential equation

$$u_{xt}+3u_xu_{xx}+3uu_x^2+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}+u_{yy}=0. \tag{7}$$

Actually under the transformation (6), we have the following equality:

$$\left[\frac{(D_{3,t}D_{3,x}+D_{3,x}^4+D_{3,y}^2)f\cdot f}{f^2}\right]_x=u_{xt}+3u_xu_{xx}+3uu_x^2+\frac{3}{2}u^3u_x+\frac{3}{2}u^2u_{xx}+u_{yy}. \tag{8}$$

Therefore, if f solves (1), then $u=2(\ln f)_x$ will present a solution to the KP-like (7). The KP-like (7) has more terms and higher nonlinearity than the standard KP (5). Comparing their bilinear counterparts, we see a different phenomenon that the generalised bilinear KP (1) is much simpler than the standard bilinear KP (4).

Resonant solutions to generalised bilinear equations have been analyzed in terms of the two kinds of transcendental functions: exponential functions and trigonometric functions [22, 23, 25]. In the following section, we generate rational solutions to the KP-like (7), on the basis of a search for polynomial solutions to the generalised bilinear (1).

3 Rational Solutions

We apply the computer algebra system Maple to search for polynomial solutions to the generalised bilinear KP (1). A direct Maple symbolic computation with

$$f=\sum_{i=0}^{3}\sum_{j=0}^{2}\sum_{k=0}^{2}c_{i,j,k}x^it^jy^k \tag{9}$$

presents 15 classes of polynomial solutions to (1). Those solutions, in turn, lead to nine classes of rational solutions to the KP-like (7) through the transformation (6). We list those classess of rational solutions as follows. The first class of rational solutions to (7) reads

$$u_1=\frac{2p}{q} \tag{10}$$

with

$$p=c_{2,0,0}\Big(8tc_{0,0,2}{}^{6}-6tc_{0,0,2}{}^{3}c_{0,1,1}{}^{2}c_{2,0,0}+tc_{0,1,1}{}^{4}c_{2,0,0}{}^{2}+8xc_{0,0,2}{}^{5}c_{2,0,0}-2xc_{0,0,2}{}^{2}c_{0,1,1}{}^{2}c_{2,0,0}{}^{2}-4yc_{0,0,2}{}^{4}c_{0,1,1}c_{2,0,0}+yc_{0,0,2}c_{0,1,1}{}^{3}c_{2,0,0}{}^{2}$$
$$+4c_{0,0,2}{}^{5}c_{1,0,0}-c_{0,0,2}{}^{2}c_{0,1,1}{}^{2}c_{1,0,0}c_{2,0,0}\Big),$$
$$q=4t^{2}c_{0,0,2}{}^{7}-t^{2}c_{0,0,2}{}^{4}c_{0,1,1}{}^{2}c_{2,0,0}+8txc_{0,0,2}{}^{6}c_{2,0,0}-6txc_{0,0,2}{}^{3}c_{0,1,1}{}^{2}c_{2,0,0}{}^{2}+txc_{0,1,1}{}^{4}c_{2,0,0}{}^{3}+4tyc_{0,0,2}{}^{5}c_{0,1,1}c_{2,0,0}-tyc_{0,0,2}{}^{2}c_{0,1,1}{}^{3}c_{2,0,0}{}^{2}$$
$$+4x^{2}c_{0,0,2}{}^{5}c_{2,0,0}{}^{2}-x^{2}c_{0,0,2}{}^{2}c_{0,1,1}{}^{2}c_{2,0,0}{}^{3}-4xyc_{0,0,2}{}^{4}c_{0,1,1}c_{2,0,0}{}^{2}+xyc_{0,0,2}c_{0,1,1}{}^{3}c_{2,0,0}{}^{3}+4y^{2}c_{0,0,2}{}^{6}c_{2,0,0}-y^{2}c_{0,0,2}{}^{3}c_{0,1,1}{}^{2}c_{2,0,0}{}^{2}$$
$$+4tc_{0,0,1}c_{0,0,2}{}^{4}c_{0,1,1}c_{2,0,0}-tc_{0,0,1}c_{0,0,2}c_{0,1,1}{}^{3}c_{2,0,0}{}^{2}+4tc_{0,0,2}{}^{6}c_{1,0,0}-tc_{0,0,2}{}^{3}c_{0,1,1}{}^{2}c_{1,0,0}c_{2,0,0}+4xc_{0,0,2}{}^{5}c_{1,0,0}c_{2,0,0}-xc_{0,0,2}{}^{2}c_{0,1,1}{}^{2}c_{1,0,0}c_{2,0,0}{}^{2}$$
$$+4yc_{0,0,1}c_{0,0,2}{}^{5}c_{2,0,0}-yc_{0,0,1}c_{0,0,2}{}^{2}c_{0,1,1}{}^{2}c_{2,0,0}{}^{2}+c_{0,0,1}{}^{2}c_{0,0,2}{}^{4}c_{2,0,0}+c_{0,0,1}c_{0,0,2}{}^{3}c_{0,1,1}c_{1,0,0}c_{2,0,0}+c_{0,0,2}{}^{5}c_{1,0,0}{}^{2}-12c_{0,0,2}{}^{4}c_{2,0,0}{}^{3}.$$

The second class of rational solutions to (7) reads

$$u_2=-\frac{8p}{q} \tag{11}$$

with

$$p=\left(2tc_{0,0,2}c_{0,1,1}{}^{2}-8xc_{0,0,2}{}^{3}+4yc_{0,0,2}{}^{2}c_{0,1,1}-c_{1,0,0}c_{0,1,1}{}^{2}\right)c_{0,0,2}{}^{2},$$
$$q=t^{2}c_{0,1,1}{}^{4}c_{0,0,2}-8c_{0,0,2}{}^{3}xtc_{0,1,1}{}^{2}+4c_{0,1,1}{}^{3}tyc_{0,0,2}{}^{2}+16c_{0,0,2}{}^{5}x^{2}$$
$$-16c_{0,0,2}{}^{4}xyc_{0,1,1}+4c_{0,0,2}{}^{3}y^{2}c_{0,1,1}{}^{2}+tc_{0,1,1}{}^{4}c_{1,0,0}$$
$$+4c_{1,0,0}xc_{0,1,1}{}^{2}c_{0,0,2}{}^{2}+4c_{0,0,0}c_{0,1,1}{}^{2}c_{0,0,2}{}^{2}+2atc_{0,1,1}+2ayc_{0,0,2},$$

where $-768c_{0,0,2}{}^{8}+c_{0,1,1}{}^{6}c_{1,0,0}{}^{2}+2ac_{0,1,1}{}^{3}c_{1,0,0}+a^{2}=0$.

The third class of rational solutions to (7) reads

$$u_3=\frac{2p}{q} \tag{12}$$

with

$$p=c_{1,1,0}{}^{2}\left(tc_{1,1,0}{}^{2}-2xc_{1,0,1}{}^{2}+yc_{1,0,1}c_{1,1,0}+c_{1,0,0}c_{1,1,0}\right),$$
$$q=txc_{1,1,0}{}^{4}-x^{2}c_{1,0,1}{}^{2}c_{1,1,0}{}^{2}+xyc_{1,0,1}c_{1,1,0}{}^{3}+tc_{0,1,0}c_{1,1,0}{}^{3}$$
$$+xc_{1,0,0}c_{1,1,0}{}^{3}+yc_{0,1,0}c_{1,0,1}c_{1,1,0}{}^{2}+c_{0,1,0}{}^{2}c_{1,0,1}{}^{2}$$
$$+c_{0,1,0}c_{1,0,0}c_{1,1,0}{}^{2}-12c_{1,0,1}{}^{4}.$$

The fourth class of rational solutions to (7) reads

$$u_4=-\frac{2c_{0,0,1}{}^{2}}{tc_{0,1,0}{}^{2}-xc_{0,0,1}{}^{2}+yc_{0,0,1}c_{0,1,0}+c_{0,0,0}c_{0,1,0}}. \tag{13}$$

The fifth class of rational solutions to (7) reads

$$u_5=\frac{2p}{q} \tag{14}$$

with

$$p=108x^{2}c_{3,0,0}{}^{2}+72xc_{2,0,0}c_{3,0,0}-c_{0,0,1}{}^{2}+12c_{2,0,0}{}^{2},$$
$$q=36x^{3}c_{3,0,0}{}^{2}+36x^{2}c_{2,0,0}c_{3,0,0}+1296tc_{3,0,0}{}^{2}-xc_{0,0,1}{}^{2}$$
$$+12xc_{2,0,0}{}^{2}+36yc_{0,0,1}c_{3,0,0}+36c_{0,0,0}c_{3,0,0}.$$

The sixth class of rational solutions to (7) reads

$$u_6=\frac{2c_{1,2,0}}{xc_{1,2,0}+c_{0,2,0}}. \tag{15}$$

The seventh class of rational solutions to (7) reads

$$u_7=\frac{2(6byc_{3,0,0}+3c_{3,0,0}x^{2}+2c_{2,0,0}x+c_{1,0,0})}{6bxyc_{3,0,0}+x^{3}c_{3,0,0}+2byc_{2,0,0}+x^{2}c_{2,0,0}+xc_{1,0,0}+c_{0,0,0}},$$
$$b^{2}=3. \tag{16}$$

The eighth class of rational solutions to (2.7) reads

$$u_8=-\frac{2p}{q} \tag{17}$$

with

$$p=4xybc_{2,1,0}{}^{2}-2x^{3}c_{2,1,0}{}^{2}+2ybc_{1,1,0}c_{2,1,0}-3x^{2}c_{1,1,0}c_{2,1,0}$$
$$-2xc_{0,1,0}c_{2,1,0}-xc_{1,1,0}{}^{2}-c_{0,1,0}c_{1,1,0},$$
$$q=x^{4}c_{2,1,0}{}^{2}+2x^{3}c_{1,1,0}c_{2,1,0}+2x^{2}c_{0,1,0}c_{2,1,0}+x^{2}c_{1,1,0}{}^{2}$$
$$-12y^{2}c_{2,1,0}{}^{2}+2xc_{0,1,0}c_{1,1,0}+c_{0,1,0}{}^{2},\ b^{2}=3.$$

The ninth class of rational solutions to (7) reads

$$u_9=-\frac{4p}{q} \tag{18}$$

with

$$p=x^{4}bc_{0,1,1}c_{3,1,0}+72x^{3}ybc_{3,1,0}{}^{2}-18x^{5}c_{3,1,0}{}^{2}-x^{2}bc_{0,1,1}c_{1,1,0}$$
$$-2xybc_{0,1,1}{}^{2}-36y^{2}bc_{0,1,1}c_{3,1,0}+x^{3}c_{0,1,1}{}^{2}-24x^{3}c_{1,1,0}c_{3,1,0}$$
$$+36x^{2}yc_{0,1,1}c_{3,1,0}+648xy^{2}c_{3,1,0}{}^{2}-2xbc_{0,1,0}c_{0,1,1}$$
$$-36ybc_{0,1,0}c_{3,1,0}-18x^{2}c_{0,1,0}c_{3,1,0}-6xc_{1,1,0}{}^{2}$$
$$-6yc_{0,1,1}c_{1,1,0}-6c_{0,1,0}c_{1,1,0},$$
$$q=12x^{6}c_{3,1,0}{}^{2}-x^{4}c_{0,1,1}{}^{2}+24x^{4}c_{1,1,0}c_{3,1,0}-48x^{3}yc_{0,1,1}c_{3,1,0}$$
$$-1296x^{2}y^{2}c_{3,1,0}{}^{2}+24x^{3}c_{0,1,0}c_{3,1,0}+12x^{2}c_{1,1,0}{}^{2}$$
$$+24xyc_{0,1,1}c_{1,1,0}+12y^{2}c_{0,1,1}{}^{2}+24xc_{0,1,0}c_{1,1,0}$$
$$+24yc_{0,1,0}c_{0,1,1}+12c_{0,1,0}{}^{2},\ b^{2}=3.$$

The aforementioned eighth and ninth solutions are generated from

$$f = tx^2c_{2,1,0} + txc_{1,1,0} + 2btyc_{2,1,0} + x^2c_{2,0,0} + tc_{0,1,0} + \frac{xc_{1,1,0}c_{2,0,0}}{c_{2,1,0}} + 2byc_{2,0,0} + \frac{c_{0,1,0}c_{2,0,0}}{c_{2,1,0}},$$

$$f = tx^3c_{3,1,0} + \frac{1}{6}btx^2c_{0,1,1} + 6btxyc_{3,1,0} + x^3c_{3,0,0} + txc_{1,1,0} + tyc_{0,1,1} + \frac{bx^2c_{0,1,1}c_{3,0,0}}{6c_{3,1,0}} + 6bxyc_{3,0,0} + tc_{0,1,0} + \frac{xc_{1,1,0}c_{3,0,0}}{c_{3,1,0}} + \frac{c_{0,1,1}c_{3,0,0}y}{c_{3,1,0}} + \frac{c_{0,1,0}c_{3,0,0}}{c_{3,1,0}},$$

respectively.

A special solution of (11) with

$$c_{i,j,k} = 1 + ijk,\ 0 \le i \le 3,\ 0 \le j,\ k \le 2,$$

is given by

$$u = -\frac{8(2t - 8x + 4y - 1)}{32\sqrt{3}t + 32\sqrt{3}y + t^2 - 8tx + 4ty + 16x^2 - 16xy + 4y^2 - t + 4x - 2y + 4}. \tag{19}$$

A special solution of (12) with

$$c_{i,j,k} = 1 + i^2 + j^2 + k^2,\ 0 \le i \le 3,\ 0 \le j,\ k \le 2,$$

is given by

$$u = \frac{6(3t - 6x + 3y + 2)}{9tx - 9x^2 + 9xy + 6t + 6x + 6y - 100}. \tag{20}$$

A special solution of (14) with

$$c_{i,j,k} = 1 + ijk,\ 0 \le i \le 3,\ 0 \le j,\ k \le 2,$$

is given by

$$u = \frac{2(108x^2 + 72x + 11)}{36x^3 + 36x^2 + 1296t + 11x + 36y + 36}. \tag{21}$$

The solutions in (19), (20), and (21) with $t = 1$ are depicted in Figures 1–3, respectively.

4 Concluding Remarks

On the basis of the generalised bilinear formulation [2, 22, 23], we introduced a KP-like nonlinear differential equation through a generalised bilinear equation of KP type, and constructed nine classes of rational solutions to the resulting KP-like equation by symbolic computation. The basic tool is the generalised bilinear differential operators $D_{3,x}$ and $D_{3,t}$ introduced from three studies [2, 22, 23].

We remark that it is very interesting to see if there exists any Wronskian solutions, more generally, Pfaffian solutions, to the KP-like nonlinear (7). Are there stable solutions to its Cauchy problems (see, e.g., [26] for analysis on the Burgers type equations)? Moreover, a kind of generalised tri-linear differential equations was introduced in [27], together with resonant solutions in terms of exponential functions. Rational solutions to generalised tri-linear differential equations, which can be viewed as continuous functions of the extended complex variables, particularly

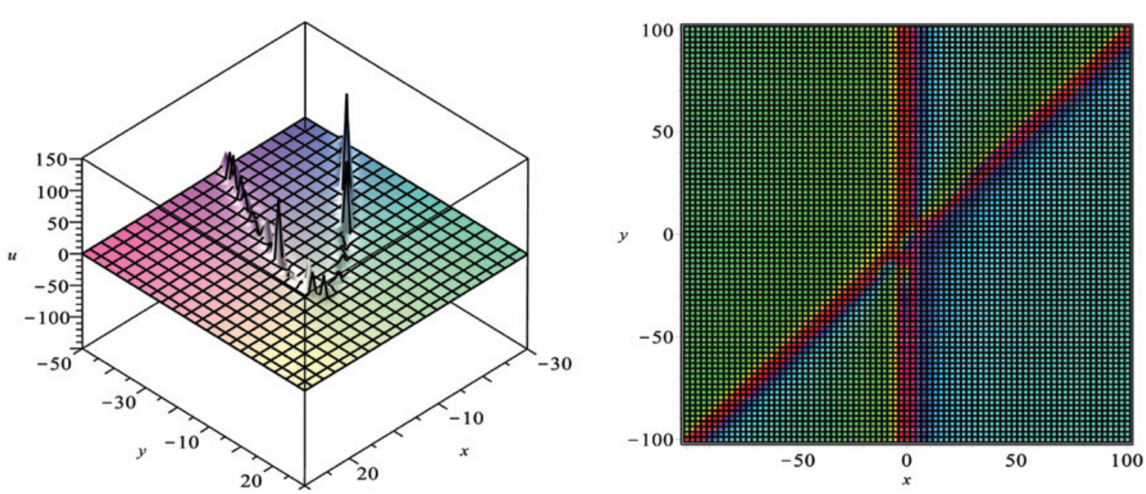

Figure 1: Pictures of (19) with $t = 1$: 3d plot (left) and density plot (right).

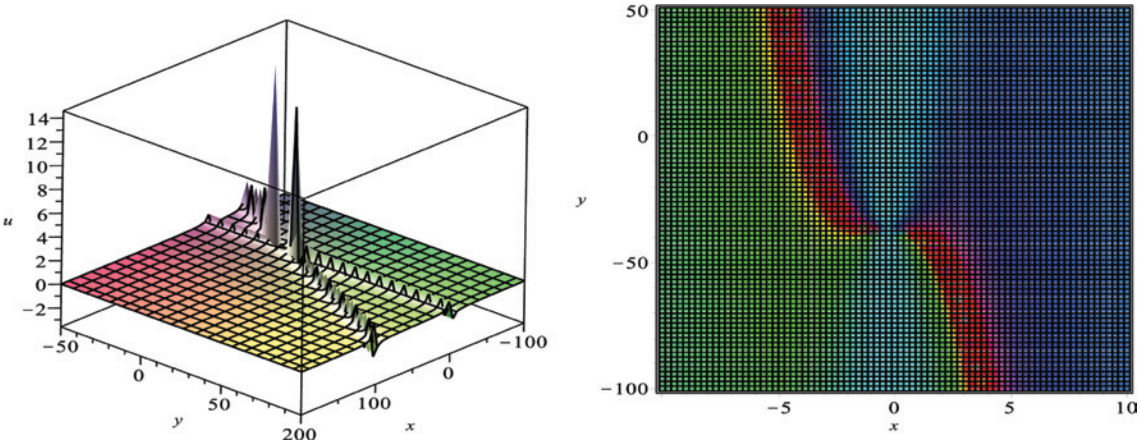

Figure 2: Pictures of (20) with $t=1$: 3d plot (left) and density plot (right).

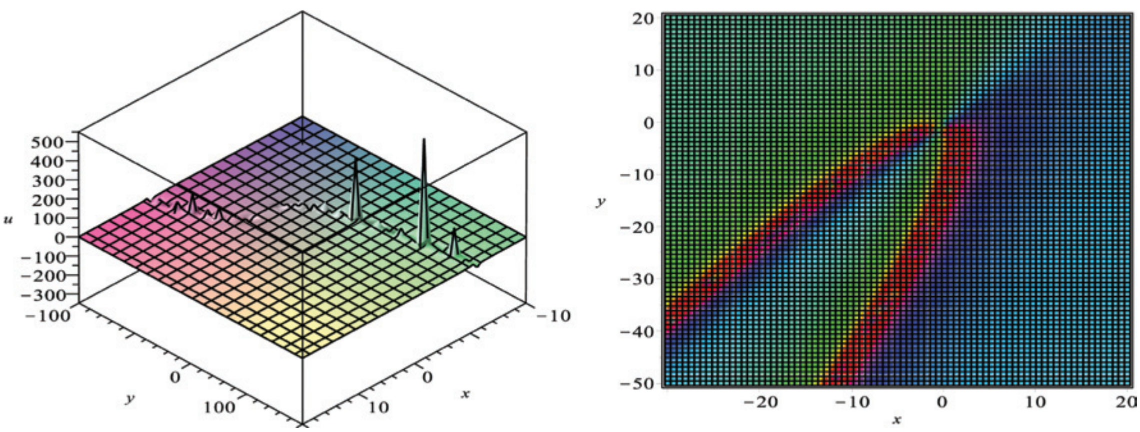

Figure 3: Pictures of (21) with $t=1$: 3d plot (left) and density plot (right).

rogue wave solutions, will be another extremely interesting topic. Higher-order rogue wave solutions will be linked to a wide variety of mathematical topics including generalised Wronskian solutions [28, 29] and generalised Darboux transformations [30]. In addition, higher dimensional generalisations, especially (3 + 1)-dimensional ones and discrete cases (see, e.g., [21, 31]), would be a good topic for future research.

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References

[1] R. Hirota, Cambridge University Press, Cambridge, MA 2004.
[2] W. X. Ma, Stud. Nonlinear Sci. **2**, 140 (2011).
[3] A. J. Tacchella, Geom. Phys. **61**, 1319 (2011).
[4] A. M. Wazwaz, Comput. Fluids **86**, 357 (2013).
[5] P. Müller, C. Garrett, and A. Osborne, Oceanography **18**, 66 (2005).
[6] C. Kharif, E. Pelinovsky, and A. Slunyaev, Rogue Waves in the Ocean, Springer-Verlag, Berlin 2009.
[7] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, Nature **450**, 1054 (2007).

[8] N. Akhmediev, J. M. Dudley, D. R. Solli, and S. K. Turitsyn, J. Opt. **15**, 060201 (2013).
[9] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, UK 1991.
[10] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, Theory of Solitons: The Inverse Scattering Method, Consultants Bureau/A Division of Plenum Publishing, New York 1984.
[11] W. X. Ma and Y. You, Trans. Amer. Math. Soc. **357**, 1753 (2005).
[12] W. X. Ma, C. X. Li, and J. S. He. Nonlinear Anal. TMA **70**, 4245 (2009).
[13] F. Gesztesy and W. Schweiger. Rep. Math. Phys. **30**, 205 (1991).
[14] W. X. Ma and Y. You. Chaos Soliton. Fract. **22**, 395 (2004).
[15] M. Khalfallah, Comm. Nonlinear Sci. Numer. Simul. **14**, 1169 (2009).
[16] D. I. Sinelshchikov, Comm. Nonlinear Sci. Numer. Simul. **15**, 3235 (2010).
[17] W. X. Ma, Comm. Nonlinear Sci. Numer. Simul. **16**, 2663 (2011).
[18] W. A. Malfliet, Amer. J. Phys. **60**, 650 (1992).
[19] A. M. Wazwaz, Appl. Math. Comput. **188**, 1930 (2007).
[20] M. L. Wang, X. Z. Li, and J. L. Zhang, Phys. Lett. A **372**, 417 (2008).
[21] W. X. Ma and A. Abdeljabbar, Appl. Math. Lett. **12**, 1500 (2007).
[22] W. X. Ma, J. Phys.: Conf. Ser. **411**, 012021 (2013).
[23] W. X. Ma, Rep. Math. Phys. **72**, 41 (2013).
[24] C. Gilson, F. Lambert, J. Nimmo, and R. Willox, Proc. R. Soc. Lond. A **452**, 223, 1996.
[25] H. C. Zheng, W. X. Ma, and X. Gu, Appl. Math. Comput. **220**, 226 (2013).
[26] N. A. Kudryashov and D. I. Sinelshchikov, Nonlinear Dyn. **76**, 561 (2014).
[27] W. X. Ma, Front. Math. China **8**, 1139 (2013).
[28] W. X. Ma, Phys. Lett. A **301**, 35 (2002).
[29] W. X. Ma, Chaos Soliton. Fract. **19**, 163 (2004).
[30] B. L. Guo, L. M. Ling, and Q. P. Liu, Phys. Rev. E **85**, 026607 (2012).
[31] Y.C. Hon, Y. F. Zhang, and J. Q. Mei, Mod. Phys. Lett. B **24**, 2713 (2010).