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Nonlocal PT-Symmetric Integrable Equations of Fourth-Order Associated with so$(3, \mathbb{R})$

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Abstract: The paper aims to construct nonlocal PT-symmetric integrable equations of fourth-order, from nonlocal integrable reductions of a fourth-order integrable system associated with the Lie algebra so$(3, \mathbb{R})$. The nonlocalities involved are reverse-space, reverse-time, and reverse-spacetime. All of the resulting nonlocal integrable equations possess infinitely many symmetries and conservation laws.

Keywords: matrix spectral probems; nonlocal integrable reduction; PT-symmetry

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1. Introduction

Matrix spectral probems associated with matrix Lie algebras are used to construct and classify integrable equations [1–3], which possess infinitely many symmetries and conservation laws. Hamiltonian structures that guarantee the Liouville integrability can be established through the trace identity [4,5]. Associated with simple Lie algebras, the well-known integrable equations include the KdV equation, the nonlinear Schrödinger equation and the derivative nonlinear Schrödinger equation [6–8]. Nonlocal integrable equations have also been recently explored in soliton theory, including scalar equations [9,10] and vector generalizations (see, e.g., [11,12]).

In this paper, we will use the special orthogonal Lie algebra $\mathfrak{g} = \mathrm{so}(3, \mathbb{R})$ over the field of real numbers. This Lie algebra can be presented by all 3×3 trace-free, skew-symmetric real matrices, and a basis can be chosen as follows:

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

whose corresponding structure equations are

$$[e_1, e_2] = e_3, \; [e_2, e_3] = e_1, \; [e_3, e_1] = e_2. \tag{2}$$

There is another representation su(2) of this Lie algebra, which uses 2×2 complex matrices, in the study of soliton surfaces [13,14]. The special orthogonal Lie algebra so$(3, \mathbb{R})$ and the special linear algebra sl$(2, \mathbb{R})$ are the only two three-dimensional real Lie algebras, whose derived algebra is equal to itself. The special linear algebra sl$(2, \mathbb{R})$ has been frequently used to study integrable equations [2]. An important fact is that the two Lie algebras sl(2) and so(3) are not isomorphic over the field of real numbers, but they are isomorphic over the field of complex numbers. We will be concentrated on the field of real numbers, and so, if we only consider real potentials, integrable equations associated with those two Lie algebras cannot be transformed into each other, which reflects this subtle difference between the two Lie algebras.

The corresponding matrix loop algebra that we will use is

$$\tilde{\mathfrak{g}} = \widetilde{\mathrm{so}}(3,\mathbb{R}) = \{M \in \mathrm{so}(3,\mathbb{R}) \,|\, \text{entries of } M \text{ - Laurent series in } \lambda\}, \tag{3}$$

where λ is a spectral parameter. Take two spectral matrices, U and V, from a given matrix loop algebra, for example, $\widetilde{\mathrm{so}}(3,\mathbb{R})$, and then the zero curvature equation

$$U_t - V_x + i[U,V] = 0 \tag{4}$$

will present integrable equations. Here, and thereafter, the subscripts denote the partial derivatives with respect to the independent variables. This zero curvature equation is the compatibility condition of the following two matrix spectral problems:

$$-i\phi_x = U\phi, \ -i\phi_t = V\phi, \tag{5}$$

which is often used to establish inverse scattering transforms to solve associated integrable equations. The matrix loop algebra $\widetilde{\mathrm{so}}(3,\mathbb{R})$ has been recently used to generate integrable equations [15,16].

More generally, if we start from non-semisimple Lie algebras, matrix spectral problems can yield so-called integrable couplings [17], and the variational identity [18] is a powerful tool for furnishing their Hamiltonian structures and hereditary recursion operators in block matrix form [19]. Integrable couplings possess rich mathematical structures and need further investigation. Based on the perturbation-type loop algebras of $\widetilde{\mathrm{so}}(3,\mathbb{R})$, we can present integrable couplings [20].

In this paper, based on the zero curvature formulation, we would first like to recall an application of $\mathrm{so}(3,\mathbb{R})$ to integrable equations [15], with a slightly modified spectral matrix from the one in [15]. We will then make three pairs of nonlocal integrable reductions for an associated integrable system to present scalar nonlocal PT-symmetric integrable equations of fourth-order, which possess the Liouville integrability, i.e., possess infinitely many commuting symmetries and conservation laws. Let $\delta = \pm 1$. The presented scalar nonlocal integrable equations are the nonlocal reverse-space integrable equation

$$\begin{aligned} i\delta p_t = {} & p^*_{xxxx}(-x,t) - \tfrac{5}{2}(p^*(-x,t))^2 p^*_{xx}(-x,t) - \tfrac{5}{2}p^*(-x,t)(p^*_x(-x,t))^2 \\ & -\tfrac{3}{2}p^2 p^*_{xx}(-x,t) - 3pp_x p^*_x(-x,t) - pp_{xx}p^*(-x,t) \\ & +\tfrac{1}{2}p_x^2 p^*(-x,t) + \tfrac{3}{8}p^*(-x,t)[p^2 + (p^*(-x,t))^2]^2, \end{aligned} \tag{6}$$

where p^* denotes the complex conjugate of p, the nonlocal reverse-time integrable equation

$$\begin{aligned} i\delta p_t = {} & p_{xxxx}(x,-t) - \tfrac{5}{2}(p(x,-t))^2 p_{xx}(x,-t) - \tfrac{5}{2}p(x,-t)(p_x(x,-t))^2 \\ & -\tfrac{3}{2}p^2 p_{xx}(x,-t) - 3pp_x p_x(x,-t) - pp_{xx}p(x,-t) \\ & +\tfrac{1}{2}p_x^2 p(x,-t) + \tfrac{3}{8}p(x,-t)[p^2 + (p(x,-t))^2]^2, \end{aligned} \tag{7}$$

and the nonlocal reverse-spacetime integrable equation

$$\begin{aligned} i\delta p_t = {} & -p_{xxxx}(-x,-t) + \tfrac{5}{2}(p(-x,-t))^2 p_{xx}(-x,-t) + \tfrac{5}{2}p(-x,-t)(p_x(-x,-t))^2 \\ & +\tfrac{3}{2}p^2 p_{xx}(-x,-t) + 3pp_x p_x(-x,-t) + pp_{xx}p(-x,-t) \\ & -\tfrac{1}{2}p_x^2 p(-x,-t) - \tfrac{3}{8}p(-x,-t)[p^2 + (p(-x,-t))^2]^2. \end{aligned} \tag{8}$$

Those nonlinear integrable equations share the PT symmetry, i.e., they are all invariant under the parity-time transformation ($x \to -x,\ t \to -t,\ i \to -i$). Note that PT-symmetric non-Hermitian physics has been the subject of intense study and broad interest over the past decades in both classical optics and quantum mechanics (see, e.g., [21]).

2. Matrix Spectral Problems and Bi-Hamiltonian Structure

2.1. Matrix Spectral Problems

We consider a pair of matrix spectral problems:

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{9}$$

where

$$U = U(u,\lambda) = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \tag{10}$$

and

$$V = V(u,\lambda) = \sum_{l=0}^{4} \begin{bmatrix} 0 & -c_l & -a_l \\ c_l & 0 & -b_l \\ a_l & b_l & 0 \end{bmatrix} \lambda^l \tag{11}$$

are two matrices in the matrix loop algebra $\widetilde{\text{so}}(3,\mathbb{R})$. In the above matrix spectral problems, λ is a spectral parameter, $u = (p,q)^T$ is a potential, $\phi = (\phi_1,\phi_2,\phi_3)^T$ is a column eigenfunction, and a_l, b_l, c_l are determined by

$$\begin{aligned}
&a_0 = -1,\ b_0 = c_0 = 0;\\
&b_1 = -p,\ c_1 = -q,\ a_1 = 0;\\
&b_2 = iq_x,\ c_2 = -ip_x,\ a_2 = \tfrac{1}{2}(p^2+q^2);\\
&b_3 = -p_{xx} + \tfrac{1}{2}p^3 + \tfrac{1}{2}pq^2,\ c_3 = -q_{xx} + \tfrac{1}{2}p^2q + \tfrac{1}{2}q^3,\ a_3 = i(p_xq - pq_x);\\
&b_4 = i(q_{xxx} - \tfrac{3}{2}p^2q_x - \tfrac{3}{2}q^2q_x),\ c_4 = i(-p_{xxx} + \tfrac{3}{2}p^2p_x + \tfrac{3}{2}p_xq^2),\\
&a_4 = pp_{xx} + qq_{xx} - \tfrac{1}{2}p_x^2 - \tfrac{1}{2}q_x^2 - \tfrac{3}{8}(p^2+q^2)^2.
\end{aligned}$$

The coefficients a_l, b_l, c_l are determined by the recursion relation

$$b_{l+1} = -ic_{l,x} + pa_l,\ c_{l+1} = ib_{l,x} + qa_l,\ a_{l+1,x} = i(pc_{l+1} - qb_{l+1}),\ l \geq 0. \tag{12}$$

under the integration condition

$$a_l|_{u=0} = 0,\ l \geq 1, \tag{13}$$

i.e., take the constant of integration as zero, which implies that

$$b_l|_{u=0} = c_l|_{u=0} = 0,\ l \geq 1. \tag{14}$$

Such a matrix in $\widetilde{\text{so}}(3,\mathbb{R})$

$$W = ae_1 + be_2 + ce_3 = \sum_{l=0}^{\infty} \begin{bmatrix} 0 & -c_l & -a_l \\ c_l & 0 & -b_l \\ a_l & b_l & 0 \end{bmatrix} \lambda^{-l} \tag{15}$$

solves the stationary zero curvature equation

$$W_x = i[U,W]. \tag{16}$$

This equation reads

$$a_x = i(pc - qb),\ b_x = i(-\lambda c + qa),\ c_x = i(\lambda b - pa),$$

which leads to the recursion relation (12). The solution W is the starting point for generating a soliton hierarchy [4,5].

It is direct to see that the zero curvature equation with the above pair of matrix spectral matrices

$$U_t - V_x + i[U, V] = 0, \tag{17}$$

presents a fourth-order integrable system:

$$\begin{cases} p_t = -i[-q_{xxxx} + \frac{5}{2}q^2 q_{xx} + \frac{5}{2}qq_x^2 + \frac{3}{2}p^2 q_{xx} \\ \qquad +3pp_x q_x + pp_{xx}q - \frac{1}{2}p_x^2 q - \frac{3}{8}q(p^2+q^2)^2], \\ q_t = i[-p_{xxxx} + \frac{5}{2}p^2 p_{xx} + \frac{5}{2}pp_x^2 + \frac{3}{2}p_{xx}q^2 \\ \qquad +3p_x q q_x + pqq_{xx} - \frac{1}{2}pq_x^2 - \frac{3}{8}p(p^2+q^2)^2]. \end{cases} \tag{18}$$

2.2. Bi-Hamiltonian Structure

We can apply the trace identity [4] with our spectral matrix iU:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(W\frac{\partial U}{\partial \lambda})\,dx = \lambda^{-\gamma}\frac{\lambda}{\partial \lambda}\lambda^{\gamma}\,\mathrm{tr}(W\frac{\partial U}{\partial u}), \tag{19}$$

where the constant γ is determined by [19]

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle| \tag{20}$$

to construct the following bi-Hamiltonian structure [22] for the integrable system (18):

$$u_t = J\frac{\delta \mathcal{H}_2}{\delta u} = M\frac{\delta \mathcal{H}_1}{\delta u}, \tag{21}$$

where the Hamiltonian pair, J and M, is given by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ M = i\begin{bmatrix} -\partial + q\partial^{-1}q & -q\partial^{-1}p \\ -p\partial^{-1}q & -\partial + p\partial^{-1}p \end{bmatrix}, \tag{22}$$

and the Hamiltonian functionals, $\mathcal{H}_1$ and $\mathcal{H}_2$, are defined by

$$\begin{aligned} \mathcal{H}_1 = \frac{1}{4}\int (&p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_x q_{xx} \\ &-\frac{3}{2}p^2 p_x q + \frac{3}{2}pq^2 q_x + \frac{3}{2}p^3 q_x - \frac{3}{2}p_x q^3)\,dx, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \mathcal{H}_2 = -\frac{i}{5}\int [&pp_{xxxx} + qq_{xxxx} - p_x p_{xxx} - q_x q_{xxx} + \frac{1}{2}p_{xx}^2 + \frac{1}{2}q_{xx}^2 \\ &-\frac{5}{2}p(p^2+q^2)p_{xx} - \frac{5}{2}q(p^2+q^2)q_{xx} - 5pqp_x q_x \\ &-\frac{5}{4}(p^2-q^2)p_x^2 + \frac{5}{4}(p^2-q^2)q_x^2 + \frac{5}{16}(p^2+q^2)^3]\,dx. \end{aligned} \tag{24}$$

There is more information on how to generate such Hamiltonian structures in the literature (see, e.g., [15]).

The bi-Hamiltonian structure (21) leads to infinitely many symmetries and conservation laws for the integrable system (18), which can often be generated through symbolic computation by computer algebra systems (see, e.g., [23,24]). The operator

$$\Phi = MJ^{-1} = i \begin{bmatrix} q\partial^{-1}p & -\partial + q\partial^{-1}q \\ \partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix}, \ \partial = \frac{\partial}{\partial x}$$

is a common hereditary [25] recursion operator [26] for the integrable system (18).

3. Nonlocal Integrable Equations by Reduction

3.1. Nonlocal Reverse-Space Reductions

Firstly, we consider two specific nonlocal reverse-space reductions for the spectral matrix

$$U^{\dagger}(-x,t,-\lambda^*) = -CU(x,t\lambda)C^{-1}, \ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix}, \ \delta = \pm 1, \tag{25}$$

where $\dagger$ denotes the Hermitian transpose. This pair of reductions leads to the potential reductions

$$p(x,t) = -\delta q^*(-x,t), \ \delta = \pm 1, \tag{26}$$

respectively. Under these potential reductions, one can have

$$a_l^*(-x,t) = (-1)^l a_l(x,t), \ b_l^*(-x,t) = (-1)^l \delta c_l(x,t), \ l \geq 1. \tag{27}$$

The results in (27) can be proved by the mathematical induction. Actually, under the induction assumption for $l = n$ and applying the recursion relation (12), one can compute

$$\begin{aligned} b_{n+1}^*(-x,t) &= ic_{n,x}^*(-x,t) + p^*(-x,t)a_n^*(-x,t) \\ &= (-1)^{n+1}\delta[ib_{n,x}(x,t) + q(x,t)a_n(x,t)] \\ &= (-1)^{n+1}\delta c_{n+1}(x,t), \\ a_{n+1,x}^*(-x,t) &= -i[p^*(-x,t)c_{n+1}^*(-x,t) - q^*(-x,t)b_{n+1}^*(-x,t)] \\ &= i(-1)^{n+1}[p(x,t)c_{n+1}(x,t) - q(x,t)b_{n+1}(x,t)] \\ &= (-1)^{n+1}a_{n+1,x}(x,t). \end{aligned}$$

Therefore, one obtains

$$V^{\dagger}(-x,t,-\lambda^*) = CV(x,t,\lambda)C^{-1}, \tag{28}$$

and further

$$((U_t - V_x + i[U,V])(-x,t,-\lambda^*))^{\dagger} = -C(U_t - V_x + i[U,V])(x,t,\lambda)C^{-1}. \tag{29}$$

This implies that both of the potential reductions in (26) are compatible with the zero curvature representation (17) of the integrable system (18).

Obviously, the reduced nonlocal reverse-space fourth-order integrable equations read

$$\begin{aligned} i\delta p_t = {} & p_{xxxx}^*(-x,t) - \tfrac{5}{2}(p^*(-x,t))^2 p_{xx}^*(-x,t) - \tfrac{5}{2}p^*(-x,t)(p_x^*(-x,t))^2 \\ & -\tfrac{3}{2}p^2 p_{xx}^*(-x,t) - 3pp_x p_x^*(-x,t) - pp_{xx}p^*(-x,t) \\ & +\tfrac{1}{2}p_x^2 p^*(-x,t) + \tfrac{3}{8}p^*(-x,t)[p^2 + (p^*(-x,t))^2]^2, \end{aligned} \tag{30}$$

where p^* denotes the complex conjugate of p. They are PT-symmetric, and present two nonlocal reverse-space PT-symmetric integrable equations of fourth-order associated with Lax pairs from the Lie algebra so(3,$\mathbb{R}$). They inherit Hamiltonian structure from the integrable system (18). The infinitely many symmetries and conservation laws for the integrable system (18) are reduced to infinitely many ones for the two scalar nonlocal integrable equations in (30).

3.2. Nonlocal Reverse-Time Reductions

Secondly, we consider two specific nonlocal reverse-time reductions for the spectral matrix

$$U^T(x,-t,-\lambda) = -CU(x,t\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix},\ \delta = \pm 1, \tag{31}$$

where T stands for the transpose of a matrix. They yield the potential reductions

$$p(x,t) = -\delta q(x,-t),\ \delta = \pm 1. \tag{32}$$

Under these potential reductions, we can have

$$a_l(x,-t) = (-1)^l a_l(x,t),\ b_l(x,-t) = (-1)^l \delta c_l(x,t),\ l \geq 1. \tag{33}$$

We can prove these results by the mathematical induction. In fact, under the induction assumption for $l = n$ and using the recursion relation (12), one can make the following computation:

$$\begin{aligned} b_{n+1}(x,-t) &= -ic_{n,x}(x,-t) + p(x,-t)a_n(x,-t) \\ &= (-1)^{n+1}\delta[ib_{n,x}(x,t) + q(x,t)a_n(x,t)] \\ &= (-1)^{n+1}\delta c_{n+1}(x,t), \\ a_{n+1,x}(x,-t) &= i[p(x,-t)c_{n+1}(x,-t) - q(x,-t)b_{n+1}(x,-t)] \\ &= i(-1)^{n+1}[p(x,t)c_{n+1}(x,t) - q(x,t)b_{n+1}(x,t)] \\ &= (-1)^{n+1}a_{n+1,x}(x,t). \end{aligned}$$

Therefore, we obtain

$$V^T(x,-t,-\lambda) = CV(x,t,\lambda)C^{-1}, \tag{34}$$

and further

$$((U_t - V_x + i[U,V])(x,-t,-\lambda))^T = C(U_t - V_x + i[U,V])(x,t,\lambda)C^{-1}. \tag{35}$$

This implies that both of the potential reductions in (32) are compatible with the zero curvature representation (17) of the integrable system (18).

Evidently, the reduced nonlocal reverse-time fourth-order integrable equations read

$$\begin{aligned} i\delta p_t = {} & p_{xxxx}(x,-t) - \tfrac{5}{2}(p(x,-t))^2 p_{xx}(x,-t) - \tfrac{5}{2}p(x,-t)(p_x(x,-t))^2 \\ & -\tfrac{3}{2}p^2 p_{xx}(x,-t) - 3pp_x p_x(x,-t) - pp_{xx}p(x,-t) \\ & +\tfrac{1}{2}p_x^2 p(x,-t) + \tfrac{3}{8}p(x,-t)[p^2 + (p(x,-t))^2]^2, \end{aligned} \tag{36}$$

which are PT-symmetric. They present two nonlocal reverse-space PT-symmetric integrable equations of fourth-order associated with Lax pairs from the Lie algebra so(3,$\mathbb{R}$). Both equations inherit Hamiltonian structures from the integrable system (18). The infinitely many symmetries and conservation laws for the integrable system (18) are reduced to infinitely many ones for the two scalar nonlocal integrable equations in (36).

3.3. Nonlocal Reverse-Spacetime Reductions

Thirdly, we consider two specific nonlocal reverse-spacetime reductions for the spectral matrix

$$U^T(-x,-t,\lambda) = CU(x,t\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ \delta & 0 & 0 \end{bmatrix},\ \delta = \pm 1, \tag{37}$$

where T stands for the transpose of a matrix again. They generate the potential reductions

$$p(x,t) = \delta q(-x,-t),\ \delta = \pm 1. \tag{38}$$

Under these potential reductions, one can have

$$a_l(-x,-t) = a_l(x,t),\ b_l(-x,-t) = \delta c_l(x,t),\ l \geq 1. \tag{39}$$

We can prove these formulas by the mathematical induction. Actually, under the induction assumption for $l = n$ and applying the recursion relation (12), one can compute

$$\begin{aligned} b_{n+1}(-x,-t) &= -ic_{n,x}(-x,-t) + p(-x,-t)a_n(-x,-t) \\ &= i\delta b_{n,x}(x,t) + \delta q(x,t)a_n(x,t) \\ &= \delta c_{n+1}(x,t), \\ a_{n+1,x}(-x,-t) &= -i[p(-x,-t)c_{n+1}(-x,-t) - q(-x,-t)b_{n+1}(-x,-t)] \\ &= i[p(x,t)c_{n+1}(x,t) - q(x,t)b_{n+1}(x,t)] \\ &= a_{n+1,x}(x,t). \end{aligned}$$

Therefore, one obtains

$$V^T(-x,-t,-\lambda) = CV(x,t,\lambda)C^{-1}, \tag{40}$$

and further

$$((U_t - V_x + i[U,V])(-x,-t,-\lambda))^T = -C(U_t - V_x + i[U,V])(x,t,\lambda)C^{-1}. \tag{41}$$

This implies that both of the potential reductions in (38) are compatible with the zero curvature representation (17) of the integrable system (18).

It is now easy to see that the reduced nonlocal reverse-time fourth-order integrable equations read

$$\begin{aligned} i\delta p_t = &-p_{xxxx}(-x,-t) + \tfrac{5}{2}(p(-x,-t))^2 p_{xx}(-x,-t) + \tfrac{5}{2}p(-x,-t)(p_x(-x,-t))^2 \\ &+\tfrac{3}{2}p^2 p_{xx}(-x,-t) + 3pp_x p_x(-x,-t) + pp_{xx}p(-x,-t) \\ &-\tfrac{1}{2}p_x^2 p(-x,-t) - \tfrac{3}{8}p(-x,-t)[p^2 + (p(-x,-t))^2]^2, \end{aligned} \tag{42}$$

which are PT-symmetric. They give rise to two nonlocal reverse-spacetime PT-symmetric integrable equations of fourth-order associated with Lax pairs from the Lie algebra so(3,$\mathbb{R}$). Both equations inherit Hamiltonian structures from the integrable system (18). The infinitely many symmetries and conservation laws for the integrable system (18) are reduced to infinitely many ones for those two scalar nonlocal integrable equations in (42).

4. Conclusions and Remarks

We have constructed three pairs of scalar nonlocal PT-symmetric integrable equations of the fourth-order, based on the zero curvature formulation, associated with the Lie algebra so$(3,\mathbb{R})$. The presented nonlocal integrable equations inherit bi-Hamiltonian structures and infinitely many commuting symmetries and conservation laws. In each pair of nonlocal reductions, the two resulting nonlocal integrable equations have only a sign difference. This feature for integrable equations associated with $so(3,\mathbb{R})$ is different from the one for integrable equations associated with $sl(2,\mathbb{R})$. In the case of sl(2,$\mathbb{R}$), there are two inequivalent focusing and defocusing integrable reductions.

There are many other interesting problems for both local and nonlocal integrable equations associated with the special orthogonal Lie algebras. For instance, how can one establish general structures of Darboux transformations associated with the special orthogonal Lie algebras, based on general formulations of Darboux transformations associated with the special linear Lie algebras [27,28]? We also do not know how to formulate Riemann–Hilbert problems and construct inverse scattering transforms to solve integrable equations associated with $so(3,\mathbb{R})$. The eigenfunctions of matrix spectral problems associated with those two kinds of Lie algebras exhibit a very different feature of analyticity with respect to the spectral parameter.

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