Modern Physics Letters B
Vol. 33, No. 5 (2019) 1950055 (10 pages)

DOI: 10.1142/S0217984919500556

General matrix exponent solutions to the coupled derivative nonlinear Schrödinger equation on half-line

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Received 30 October 2018
Accepted 6 December 2018
Published 18 January 2019

Generalized matrix exponential solutions to the coupled derivative nonlinear Schrödinger equation (DNLSE) are obtained by the inverse scattering transformation (IST). The resulting solutions involve six matrices, which satisfy the coupled Sylvester equations. Several kinds of explicit solutions including soliton, complexiton, and Matveev solutions are deduced from the generalized matrix exponential solutions by choosing different kinds of the six involved matrices through Mathematica symbolic computations.

Keywords: The Chen–Lee–Liu equation; inverse scattering transform; the coupled Sylvester equation.

1. Introduction

It is well known that the nonlinear Schrödinger equation (NLSE) is one of the most important soliton equations, and arises from a wide variety of fields, such as quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics.[1–3] To study the effect of higher-order perturbations, various modifications and generations of the NLSE have been proposed and studied.[3–10] Among them, there are three celebrated equations with derivative-type nonlinearities which are

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called the derivative nonlinear Schrödinger equation (DNLSE). One is the Kaup–Newell equation[4]

$$iq_t + q_{xx} + i(|q|^2 q)_x = 0\,,$$

which is called DNLSE I. The second type is the Chen–Lee–Liu equation[5,6]

$$iq_t + q_{xx} - i|q|^2 q_x = 0\,, \tag{1.1}$$

which is called DNLSE II. The last one takes the form

$$iq_t + q_{xx} - i|q|^2 q_x + \frac{1}{2} q^3 q^{*^2} = 0\,,$$

which is called the Gerjikov–Ivanov (GI) equation or DNLSE III.[7,8] In the above equation, i is the imaginary unit and the symbol $*$ denotes complex conjugate.

In this paper, we would like to consider the coupled DNLSE II

$$\begin{cases} iq_t + q_{xx} - iqrq_x = 0\,, \\ ir_t - r_{xx} - iqrr_x = 0\,, \end{cases} \tag{1.2}$$

with the help of a spectral problem, the Sylvester equation and inverse scattering transformation (IST). If $r = q^*$, Eq. (1.2) becomes the Chen–Lee–Liu equation (1.1). The Sylvester equation

$$AM - MB = C, \tag{1.3}$$

is one of the most well-known matrix equations. It appears frequently in many areas of applied mathematics and plays a cetral role, in particular, in systems and control theory, signal processing, filtering, model reduction, image restoration, and so on. In recent years, it has been used to solve soliton equations sucessfully.[11,12] The method based on the Sylvester equation is also known as the Cauchy matrix approach,[12–16] which is actually a by-product of the direct linearization approach first proposed by Fokas and Ablowitz in 1981[13] and developed to discrete integrable systems by Nijhoff *et al.* in early 1980s.[14] IST is one of the most powerful tools we have for studying soliton equations. It can be used to solve not only standard soliton equations but also non-standard soliton equations such as equations with self-consistent sources,[17] non-isospectral equations,[18,19] and equations with step-like finite-gap backgrounds.[20] Recently, Ablowitz and Musslimani developed the IST for the integrable nonlocal nonlinear Schrödinger equation.[21] It is interesting that the IST together with the Sylvester can also get the matrix exponential solutions to soliton equation on half-line.[22,23] Furthermore, through choosing the special matrices, the resulting generalized matrix exponential solutions can be reduced to solitons, complexitons and Matveev solutions.[23]

Equation (1.2) has already been proved to be integrable in the Liouville sense by means of the trace identity.[24] Its various kinds of solutions have been found by the Wronskian technique and Darboux transformation.[25,26] In this paper, we will construct its solitons, complexitons and Matveev solutions following the method given in Refs. 22 and 23.

The paper is organized as follows. In Sec. 2, we will present the Lax pair of (1.2) and recover its potentials in matrix forms by IST. In Sec. 3, we will prove that the potentials recovered in Sec. 2 solve (1.2). Several kinds of explicit solutions will be given in Sec. 4. We conclude the paper in Sec. 5.

2. The Lax Pair of DNLSE and Its Inverse Scattering Transform

DNLSE (1.2) has the Lax pair[27,28]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \begin{pmatrix}-i\zeta^2 & q\zeta\\ -r\zeta & i\zeta^2+\frac{i}{2}qr\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},$$
$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t = \begin{pmatrix}-2i\zeta^4-i\zeta^2qr & 2\zeta^3q+i\zeta q_x+\frac{1}{2}\zeta q^2r\\ 2\zeta^3r-i\zeta r_x+\frac{1}{2}\zeta qr^2 & 2i\zeta^4+i\zeta^2qr-\frac{1}{2}(q_xr-qr_x)+\frac{i}{4}q^2r^2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{2.1}$$

where ζ is the spectral parameter. From Ref. 27, we find that the Marchenko equation (1.2) is

$$\begin{cases} K(x,y)-\bar{F}(y)-\frac{i}{2}\int_x^\infty dz\int_x^\infty dsK(x,z)F(z+s-x)\frac{\partial\bar{F}(s+y-x)}{\partial s}=0\,,\\ \bar{K}(x,y)-F(y)+\frac{i}{2}\int_x^\infty dz\int_x^\infty ds\frac{\partial\bar{K}(x,z)}{\partial z}\bar{F}(z+s-x)F(s+y-x)=0\end{cases} \tag{2.2}$$

and the potentials can be recovered by

$$\begin{cases} q(x)=K(x,x)\,,\\ r(x)=\bar{K}(x,x)\,.\end{cases} \tag{2.3}$$

Lemma 1. *Let A and $\bar{A}$ be $l\times l$ constant matrices, B and $\bar{B}$ be $l\times 1$ constant column vectors and C and $\bar{C}$ be $1\times l$ constant row vectors. Assume that A and $\bar{A}$ satisfy $\lim_{x\to\infty}e^{-Ax}=0$ and $\lim_{x\to\infty}e^{-\bar{A}x}=0$, respectively, where 0 is a zero matrix of size $l\times l$. Then upon taking*

$$M=\int_0^\infty e^{-\bar{A}z}\bar{B}Ce^{-Az}dz,\quad \bar{M}=\int_0^\infty e^{-Az}B\bar{C}e^{-\bar{A}z}dz\,,$$

we have the following coupled Sylvester relations:

$$\bar{A}M+MA=\bar{B}C,\quad A\bar{M}+\bar{M}\bar{A}=B\bar{C}\,. \tag{2.4}$$

Proof. Here, we only prove the first relation, while the second relation can be proved similarly,

$$\bar{A}M+MA=\int_0^\infty(\bar{A}e^{-\bar{A}s}\bar{B}Ce^{-As}+e^{-\bar{A}s}\bar{B}Ce^{-As}A)ds=-e^{\bar{A}s}\bar{B}Ce^{As}|_{s=0}^\infty=\bar{B}C\,.$$

□

For convenience, from now on we set

$$\Omega(x) = e^{-\bar{A}x} M e^{-Ax}, \quad \bar{\Omega}(x) = e^{-Ax} \bar{M} e^{-\bar{A}x}, \tag{2.5a}$$

$$\Xi(t) = e^{i\bar{T}t} M e^{-iTt}, \quad \bar{\Xi}(t) = e^{-iTt} \bar{M} e^{i\bar{T}t}, \tag{2.5b}$$

where T and $\bar{T}$ are $l \times l$ matrices, $M, \bar{M}, A, \bar{A}, B, \bar{B}, C$ and $\bar{C}$ are matrices defined in Lemma 1, and they enjoy the coupled Sylvester relations (2.4).

Setting

$$\begin{cases} \Gamma = I + \dfrac{i}{2} \Omega(x) \bar{\Xi}(t) \bar{A}, \\ \bar{\Gamma} = I - \dfrac{i}{2} A \bar{\Omega}(x) \Xi(t), \end{cases} \tag{2.6}$$

where I is a $l \times l$ unit matrix, if Γ and $\bar{\Gamma}$ are nondegenerate matrices, we have the following theorem.

Theorem 1. *If $AT = TA$ and $\bar{A}\bar{T} = \bar{T}\bar{A}$, the potentials of the DNLS spectral problem* (2.1) *can be recovered as*

$$\begin{cases} q(x,t) = K(x,x,t) = \bar{C} e^{i\bar{T}t} \Gamma^{-1} e^{-\bar{A}x} \bar{B}, \\ r(x,t) = \bar{K}(x,x,t) = C e^{-iTt} \bar{\Gamma}^{-1} e^{-Ax} B. \end{cases} \tag{2.7}$$

Proof. Let F and $\bar{F}$ be the matrix exponential forms as follows:

$$F(x) = C e^{-Ax} B, \quad \bar{F}(x) = \bar{C} e^{-\bar{A}x} \bar{B}.$$

Suppose that the time evolutions of $F(x)$ and $\bar{F}(x)$ are

$$F(x,t) = C e^{-iTt - Ax} B, \quad \bar{F}(x,t) = \bar{C} e^{i\bar{T}t - \bar{A}x} \bar{B} \tag{2.8}$$

and we may take

$$K(x,y,t) = H(x,t) e^{-\bar{A}y} \bar{B}, \quad \bar{K}(x,y,t) = \bar{H}(x,t) e^{-Ay} B, \tag{2.9}$$

accordingly, where $H(x,t), \bar{H}(x,t)$ are $1 \times l$ row vectors. Substituting (2.8) and (2.9) into the first equation of (2.2), we have

$$\begin{aligned} \bar{C} e^{i\bar{T}t} - H(x,t) = \frac{i}{2} H(x,t) \left(\int_x^\infty e^{-\bar{A}z} \bar{B} C e^{-Az} dz \right) e^{-iTt + Ax} \\ \times \left(\int_x^\infty e^{-As} B \bar{C} e^{-\bar{A}s} ds \right) e^{i\bar{T}t + \bar{A}x} \bar{A}. \end{aligned}$$

In the light of

$$\int_x^\infty e^{-\bar{A}z} \bar{B} C e^{-Az} dz = e^{-\bar{A}x} M e^{-Ax}, \quad \int_x^\infty e^{-Az} B \bar{C} e^{-\bar{A}z} dz = e^{-Ax} \bar{M} e^{-\bar{A}x},$$

we obtain

$$H(x,t) = \bar{C} e^{i\bar{T}t} \Gamma^{-1}.$$

Similarly, we can arrive at

$$\bar{H}(x,t) = Ce^{-iTt}\bar{\Gamma}^{-1}\,.$$

Finally, we recover the potentials of the DNLSE spectral problem (2.1) as

$$\begin{cases} q(x,t) = K(x,x,t) = \bar{C}e^{i\bar{T}t}\Gamma^{-1}e^{-\bar{A}x}\bar{B}\,, \\ r(x,t) = \bar{K}(x,x,t) = Ce^{-iTt}\bar{\Gamma}^{-1}e^{-Ax}B\,, \end{cases}$$

by taking advantage of (2.3). □

3. Generalized Matrix Exponential Solutions to the Coupled DNLSE II

In this section, we will derive generalized matrix exponential solutions to the DNLSE (1.2).

Lemma 2. *Suppose that $AT = TA$ and $\bar{A}\bar{T} = \bar{T}\bar{A}$. Then we have*

$$\bar{\Gamma}e^{-Ax}\bar{M}e^{i\bar{A}^2t}\bar{A} + Ae^{-Ax}\bar{M}e^{i\bar{A}^2t}\Gamma = e^{-Ax}B\bar{C}e^{i\bar{A}^2t}\,, \tag{3.1}$$

where Γ and $\bar{\Gamma}$ are defined by (2.6).

Proof. Since

$$\begin{cases} \Gamma = I + \dfrac{i}{2}\Omega(x)\bar{\Xi}(t)\bar{A}\,, \\ \bar{\Gamma} = I - \dfrac{i}{2}A\bar{\Omega}(x)\Xi(t)\,, \end{cases}$$

we can obtain

$$(\bar{\Gamma} - I)e^{-Ax}\bar{M}e^{i\bar{T}t}\bar{A} = -Ae^{-Ax}\bar{M}e^{i\bar{T}t}(\Gamma - I)\,,$$

i.e.

$$\bar{\Gamma}e^{-Ax}\bar{M}e^{i\bar{T}t}\bar{A} + Ae^{-Ax}\bar{M}e^{i\bar{T}t}\Gamma = e^{-Ax}(A\bar{M} + \bar{M}\bar{A})e^{i\bar{T}t}\,.$$

By Lemma 1, we get (3.1). □

Let

$$\Upsilon(x,t) = e^{-\bar{A}x}\bar{B}Ce^{-iA^2t}, \quad \bar{\Upsilon}(x,t) = e^{-Ax}B\bar{C}e^{i\bar{A}t}\,, \tag{3.2}$$

$$\omega(x,t) = e^{-\bar{A}x}Me^{-iA^2t}, \quad \bar{\omega}(x,t) = e^{-Ax}\bar{M}e^{i\bar{A}^2t} \tag{3.3}$$

and

$$w_1 = \omega(x,t)\bar{\omega}(x,t)\bar{A}^3, \quad w_2 = \omega(x,t)A^2\bar{\omega}(x,t)\bar{A}, \quad w_3 = \Upsilon(x,t)A\bar{\omega}(x,t)\bar{A}\,,$$

$$w_4 = \bar{A}\Upsilon(x,t)\bar{\omega}(x,t)\bar{A}, \quad w_5 = \Upsilon(x,t)\bar{\Gamma}^{-1}A\bar{\omega}(x,t)\bar{A}, \quad w_6 = \Upsilon(x,t)\bar{\omega}(x,t)\bar{A}\Gamma^{-1}\bar{A}\Gamma\,,$$

$$w_7 = \Upsilon(x,t)\bar{\Gamma}^{-1}\bar{\Upsilon}(x,t)\Gamma^{-1}\bar{A}\Gamma, \quad w_8 = \Upsilon(x,t)\bar{\Gamma}^{-1}A\bar{\omega}(x,t)\Upsilon(x,t)\bar{\omega}(x,t)\bar{A}\,,$$
$$w_9 = \Upsilon(x,t)\bar{\omega}(x,t)\bar{A}\Gamma^{-1}\Upsilon(x,t)\bar{\omega}(x,t)\bar{A},$$
$$w_{10} = \Upsilon(x,t)\bar{\Gamma}^{-1}\bar{\Upsilon}(x,t)\Gamma^{-1}\Upsilon(x,t)\bar{\omega}(x,t)\bar{A}$$

and then we have following relations.

Lemma 3. *Let $T = A^2$ and $\bar{T} = \bar{A}^2$ in Γ and $\bar{\Gamma}$, respectively. Then we have*

$$w_7 = w_5 + w_6, \quad w_{10} = w_8 + w_9, \quad w_8 + 2iw_5 = 2iw_3\,,$$
$$\bar{A}^2\Gamma - \Gamma\bar{A}^2 + \tfrac{i}{2}(w_1 - w_2 + w_3 - w_4) = 0\,.$$

Proof. Directly computing and using Lemma 2 give the proof. □

Theorem 2. *If $T = A^2$ and $\bar{T} = \bar{A}^2$ in $\Xi(t)$ and $\bar{\Xi}(t)$, respectively, the recovered potentials* (2.7) *are solutions to Eq.* (1.2).

Proof. We only prove the first equation of (1.2), and the second equation can be proved by a similar way.

$$\begin{aligned}
&iq_t + q_{xx} - iqrq_x \\
&\quad = \bar{C}e^{i\bar{A}^2t}\Gamma^{-1}[\bar{A}^2\Gamma - \Gamma\bar{A}^2 + \frac{i}{2}(w_1 - w_2 - w_3 - w_4 - 2w_6 + 2w_7) \\
&\qquad - \frac{1}{2}(w_9 - w_{10})]\Gamma^{-1}e^{-\bar{A}x}\bar{B} \\
&\quad = \bar{C}e^{i\bar{A}^2t}\Gamma^{-1}\left[\bar{A}^2\Gamma - \Gamma\bar{A}^2 + \frac{i}{2}(w_1 - w_2 - w_3 - w_4) + iw_5 + \frac{1}{2}w_8\right]\Gamma^{-1}e^{-\bar{A}x}\bar{B} \\
&\quad = \bar{C}e^{i\bar{A}^2t}\Gamma^{-1}\left[\bar{A}^2\Gamma - \Gamma\bar{A}^2 + \frac{i}{2}(w_1 - w_2 + w_3 - w_4)\right]\Gamma^{-1}e^{-\bar{A}x}\bar{B} \\
&\quad = 0\,,
\end{aligned}$$

where we have used Lemmas 2 and 3. □

Corollary 1. *If $\bar{A} = A^*$, $T = A^2$ and $\bar{T} = \bar{A}^2$ in $\Xi(t)$ and $\bar{\Xi}(t)$, then the recovered potentials* (2.7) *are solutions to Eq.* (1.1).

Proof. It is easy to prove that

$$r = q^*,$$

if $\bar{A} = A^*$, $T = A^2$ and $\bar{T} = \bar{A}^2$ in $\Xi(t)$ and $\bar{\Xi}(t)$. The result in the corollary is a consequence of that in the above theorem. □

4. Explicit Solutions to the Coupled DNLSE II

In this section, we will construct different kinds of explicit solutions to Eq. (1.2) by taking different kinds of matrices A, B, C and $\bar{A}, \bar{B}, \bar{C}$ through Mathematica symbolic computations.

- One-soliton solutions: Taking $A = k_1, B = b_1, C = c_1$ and $\bar{A} = k_2, \bar{B} = b_2, \bar{C} = c_2$, we get

$$\begin{cases} q = \dfrac{2(k_1+k_2)^2 b_2 c_2 e^{ik_2^2 t - k_2 x}}{2(k_1+k_2)^2 + ib_1 b_2 c_1 c_2 k_2 e^{-i(k_1^2-k_2^2)t-(k_1+k_2)x}}, \\ r = \dfrac{2(k_1+k_2)^2 b_1 c_1 e^{ik_1^2 t - k_1 x}}{2(k_1+k_2)^2 - ib_1 b_2 c_1 c_2 k_1 e^{-i(k_1^2-k_2^2)t-(k_1+k_2)x}}, \end{cases} \tag{4.1}$$

  where $k_j, b_j, c_j \in \mathbb{C}$ and $\operatorname{Re} k_j < 0, j = 1, 2$.
- Two-soliton solutions: Taking

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \bar{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C = \bar{C} = \begin{pmatrix} 1, & 1 \end{pmatrix},$$

  we have

$$\begin{cases} q = \dfrac{18 e^{4it+x}}{4ie^{3it} + 9e^{3x}}, \\ r = \dfrac{18}{9e^{it+x} - 2ie^{4it-2x}}. \end{cases} \tag{4.2}$$

- The Matveev solutions: Taking

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

  and

$$B = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T, \quad \bar{C} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}^T,$$

  we have

$$\begin{cases} q = \dfrac{18e^{7it+x}(5-6it-3x) + 18e^{4it+x}(-i+12t+3ix)}{4ie^{6it} - ie^{6x} + e^{3it+x}[8-27x+18(8t^2+it(5-2x)+x^2)])}, \\ r = \dfrac{3e^{3x}(-2+2it-x) - 3e^{3it}(i+8t+2ix)}{e^{7it-2x} - e^{it+4x} + e^{4it+x}[72it^2+18t(x-5)+i(-5-18x+9x^2)]}. \end{cases} \tag{4.3}$$

- Complexiton solutions: Taking

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$$

and

$$B=\begin{pmatrix}3\\3\end{pmatrix},\quad \bar{B}=\begin{pmatrix}3\\9\end{pmatrix},\quad C=\begin{pmatrix}1\\1\end{pmatrix}^T,\quad \bar{C}=\begin{pmatrix}3\\1\end{pmatrix}^T,$$

we have

$$\begin{cases} q=\dfrac{6e^{x}\varpi_1}{e^{6x}-8+e^{3x}\varpi_2}\,,\\[2ex] r=\dfrac{12e^{2x}\varpi_3}{2e^{6x}-4+e^{3x}\varpi_4}\,,\end{cases}\tag{4.4}$$

where

$$\varpi_1=14i\cosh(2t-ix)+2\sinh(2t-ix)+e^{3x}[3\cosh(8+2ix)+4i\sinh(8t+2ix)]\,,$$
$$\varpi_2=9i\cosh(10t+ix)-i\cosh(6t+3ix)-9\sinh(10t+ix)-7\sinh(6t+3ix)\,,$$
$$\varpi_3=[e^{3x}\cosh(2t-ix)+i\cosh(8t+2ix)+\sinh(8t+2ix)]\,,$$
$$\varpi_4=[9i\cosh(10t+ix)+i\cosh(6t+3ix)+9\sinh(10t+ix)+7\sinh(6t+3ix)]\,.$$

Actually, here A and $\bar{A}$ both have two conjugate complex eigenvalues. This kind solution can also be obtained by the Wronskian technique.[29–33]

- Three soliton solutions: Taking

$$A=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix},\quad \bar{A}=\begin{pmatrix}3&0&0\\0&2&0\\0&0&1\end{pmatrix},$$

$$B=\begin{pmatrix}2\\0\\1\end{pmatrix},\quad \bar{B}=\begin{pmatrix}0\\-1\\2\end{pmatrix},\quad C^T=\begin{pmatrix}-1\\0\\1\end{pmatrix},\quad \bar{C}^T=\begin{pmatrix}0\\-1\\1\end{pmatrix},$$

we have

$$\begin{cases} q=\dfrac{ie^{4it+x}(200ie^{8it+2x}+3600e^{8it+4x}+7200e^{5it+5x}-27i)}{2ie^{3it}-225e^{3x}-144e^{3it+2x}+800e^{11it+4x}+1800e^{8it+5x}(1+2ie^{2x})}\,,\\[2ex] r=\dfrac{-2(64ie^{3it}+225ie^{x}-1800e^{3x}+3600e^{8it+5x})}{3e^{4it-x}-216ie^{4it+x}+400e^{3(4it+x)}-675ie^{it+2x}+1800e^{9it+4x}(i+2e^{2x})}\,.\end{cases}\tag{4.5}$$

- The Matveev solutions: Taking

$$A=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix},\quad \bar{A}=\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix},$$

$$B = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad C^T = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \quad \bar{C}^T = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix},$$

we have

$$\begin{cases} q = \dfrac{2e^{it-x}[e^{2x}(2t+2i-ix-x)]}{1+18(t-i)+2x(x-2)-i\sinh 2x}, \\ r = \dfrac{4e^{-it-x}[1+2it+e^{2x}(2t+i-ix-x)]}{3+8t^2+2x(x-2)+i\sinh 2x}. \end{cases} \tag{4.6}$$

5. Conclusions

To sum up, we have solved the DNLSE (1.2) through the IST on half-line. In general, in IST the reflection coefficient $R(k)$ are defined only for real k, but for potentials satisfying additional decay conditions, the reflection coefficient has analytic or meromorphic extensions off the real axis in the complex k-plane. In this case, the complex eigenvalues of A correspond to the poles of the refection coefficient $R(k)$, and $K(x, y, t)$ may provide solutions to the soliton equations in a certain quadrant of the (x, t)-plane.[22] The triplet $(A; B; C)$ provides a solution $K(x, y, t)$ to the Marchenko equation. If $T = A^2$ and $\bar{T} = \bar{A}^2$, it can be proved that the potentials (2.7) satisfy the DNLS equations (1.2) for arbitrary A and $\bar{A}$, which enlarges the scope of solutions.

By the suggested method, we can solve soliton equations more directly, and obtain different kinds of exact solutions more systematically than by the traditional IST. Some classic solutions of physical significance have been obtained this way. Those solutions can also be computed by the Wronskian technique (see e.g. Refs. 33 and 34). If we take other special matrices for the involved matrices (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, we can obtain many other kinds of exact solutions to soliton equations.

Acknowledgments

The authors declare that they have no conflict of interest. The first author is very grateful to Professor Tuncay Aktosun for his enthusiastic guidance and help. The work was supported in part by the National Natural Science Foundation of China under the Grant Nos. 11671177, 11771186, 11571079, 11371086 and 1371361, National Science Foundation under the Grant No. DMS-1664561, and the Distinguished Professorships by Shanghai University of Electric Power, China, and North-West University, South Africa.

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