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Lie Algebraic Approach to Nonlinear Integrable Couplings of Evolution Type

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Abstract

Based on two higher-dimensional extensions of Lie algebras, three kinds of specific Lie algebras are introduced. Upon constructing proper loop algebras, six isospectral matrix spectral problems are presented and they yield nonlinear integrable couplings of the Ablowitz-Kaup-Newell-Segur hierarchy, the Broer-Kaup hierarchy and the Kaup-Newell hierarchy. Especially, the reduced cases of the resulting integrable couplings give nonlinear integrable couplings of the nonlinear Schrödinger equation and the classical Boussinesq equation. Two linear functionals are introduced on two loop algebras of dimension 6 and Hamiltonian structures of the obtained nonlinear integrable couplings are worked out by employing the associated variational identity. The proposed approach can also be used to generate nonlinear integrable couplings for other integrable hierarchies.

1 Introduction

Integrable couplings are coupled systems of integrable equations, which contain given integrable equations as their sub-systems [1–3]. The general definition on integrable couplings is as follows: for a given integrable system of evolution type $u_t = K(u)$, an integrable coupling is a new bigger triangular integrable system

$$\begin{cases} u_t = K(u), \\ v_t = S(u,v), \end{cases} \tag{1}$$

where the vector-valued function S should satisfy the non-triviality condition $\partial S/\partial[u] \neq 0$, $[u]$ denoting a vector consisting of all derivatives of u with respect to the space variable. In the paper [4], a kind of 3×3 matrix Lie algebras was introduced and the associated integrable coupling of the TD hierarchy was obtained. A general procedure for generating integrable couplings was proposed in [3], based on semi-direct sums of Lie algebras. The basic idea is as follows.

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Let G be a matrix Lie algebra. Assume that a continuous integrable system of evolution type

$$u_t = K(u) = K(u, u_x, u_{xx}, \ldots) \tag{2}$$

is linked with G. That is, there exists a pair of Lax matrices U and V in G so that the matrix spectral problem

$$\phi_x = U\phi = U(u, \lambda)\phi \tag{3}$$

and the associated matrix spectral problem

$$\phi_t = V\phi = V(u, u_x, \ldots, \frac{\partial^{n_0} u}{\partial x^{n_0}}; \lambda) \tag{4}$$

generate the integrable system (2) through the isospectral ($\lambda_t = 0$) compatibility condition

$$U_t - V_x + [U, V] = 0. \tag{5}$$

To construct an integrable coupling of Eq.(2), we enlarge the Lie algebra G by using semi-direct sums of Lie algebras as follows

$$\bar{G} = G \in\!\!\!\!+ G_c, \tag{6}$$

where $[G, G_c] = \{[A, B] | A \in G, B \in G_c\}$, and G and G_c satisfy that

$$[G, G_c] \subset G_c. \tag{7}$$

Therefore, G_c is an ideal Lie sub-algebra of $\bar{G}$. Take a pair of new Lax matrices in the semi-direct sum $\bar{G}$:

$$\bar{U} = U + U_c, \bar{V} = V + V_c, U_c, V_c \in G_c. \tag{8}$$

Then the compatibility condition of the Lax pair $\bar{U}$ and $\bar{V}$, i.e., the enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0$$

is equivalent to

$$\begin{cases} U_t - V_x + [U, V] = 0, \\ U_{c,t} - V_{c,x} + [U, V_c] + [U_c, V] + [U_c, V_c] = 0. \end{cases} \tag{9}$$

The first equation (9) presents Eq. (2). The whole system (9) provides an integrable coupling of Eq. (2). In particular, we can choose the enlarged spectral matrices $\bar{U}$ and $\bar{V}$ as follows [3]:

$$\bar{U} = \begin{pmatrix} U & U_{a_1} & \ldots & U_{a_\nu} \\ 0 & U & \ddots & \vdots \\ \vdots & \ddots & \ddots & U_{a_1} \\ 0 & \ldots & 0 & U \end{pmatrix}, \bar{V} = \begin{pmatrix} V & V_{a_1} & \ldots & V_{a_\nu} \\ 0 & V & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{a_1} \\ 0 & \ldots & 0 & V \end{pmatrix}, \tag{10}$$

Thus, the coupling system (9) becomes that

$$\begin{cases} u_t - V_x + [U, V] = 0, \\ U_{a_i,t} - V_{a_i,t} + \sum_{l+k=i, k,l \geq 0} [U_{a_k}, V_{a_l}] = 0, \end{cases} \tag{11}$$

where $U_{a_0} = U, V_{a_0} = 0$. Besides the form of Lax pairs in (10), another enlarged form of the spectral matrices [5]

$$\bar{U} = \begin{pmatrix} U & U_a \\ 0 & 0 \end{pmatrix} \tag{12}$$

corresponds to the semi-direct sum of Lie algebras

$$G \Subset G_c, G = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\},\ G_c = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}.$$

More generally, the enlargement of Lax spectral matrices [3]:

$$\bar{U} = \begin{pmatrix} U & U_{a_1} \\ 0 & U_{a_2} \end{pmatrix} \tag{13}$$

can be taken from the following semi-direct sum of Lie algebras

$$G \Subset G_c, G = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\},\ G_c = \left\{ \begin{pmatrix} 0 & B_1 \\ 0 & B_2 \end{pmatrix} \right\},$$

where A, B_1 and B_2 have the same sizes as U, U_{a_1} and U_{a_2}, respectively. Even more general matrix forms than (12) and (13) can be taken as follows [3]

$$\bar{U} = \begin{pmatrix} U & U_{a_1} & U_{a_2} \\ 0 & U & U_{a_3} \\ 0 & 0 & 0 \end{pmatrix}, \bar{U} = \begin{pmatrix} U & U_{a_1} & U_{a_2} \\ 0 & U & U_{a_3} \\ 0 & 0 & U_{a_4} \end{pmatrix}. \tag{14}$$

All this presents a few simple categories of Lax pairs which yield integrable couplings under the framework of semi-direct sums of Lie algebras. Based on this idea, some integrable couplings of integrable systems were obtained [6–9], from which we see that there are much richer mathematical structures behind integrable couplings than scalar integrable equations.

Tu's trace identities have been broadly generalized to semi-direct sums of Lie algebras recently [10,11]. The generalizations form a general identity called the variational identity, which provides a powerful tool for generating Hamiltonian structures of integrable couplings [11]. More recently, the variational identity has been extensively studied and extended to the case of super Lie algebras, and the supertrace identity has been established to deduce super-Hamiltonian structures of a super-AKNS soliton hierarchy and a super-Dirac soliton hierarchy [12].

Usually, integrable equations can have different integrable couplings. For example, a known equation or system by Eq. (1) may have two integrable couplings [13]:

$$\bar{u}_{1,t} = \bar{K}_1(\bar{u}) = \begin{pmatrix} K(u) \\ S(u,v) \end{pmatrix},\ \bar{u}_1 = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{15}$$

$$\bar{u}_{2,t} = \bar{K}_2(\bar{u}_2) = \begin{pmatrix} K(u) \\ T(u,v) \end{pmatrix},\ \bar{u}_2 = \begin{pmatrix} u \\ w \end{pmatrix}. \tag{16}$$

Putting (15) and (16) together forms a bigger system

$$\bar{u}_{3,t} = \bar{K}_3(\hat{u}) = \begin{pmatrix} K(u) \\ S(u,v) \\ T(u,w) \end{pmatrix},\ \hat{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{17}$$

If (17) is still integrable, then we call it a bi-integrable coupling of Eq. (1) [13]. Two kinds of matrix Lie algebras consisting of square matrices of block forms were constructed to obtain different bi-integrable couplings of the standard nonlinear schrödinger equation. Following this idea, two kinds of higher-dimensional Lie algebras were presented in [14,15] to produce bi-integrable couplings of the KdV hierarchy and the BPT hierarchy. However, the integrable couplings obtained this way are all linear with respect to the second variable v in the system $v_t = S(u,v)$, that is, the whole system is linear with respect to the variable v and its derivatives with respect to the space variable. One of the reasons may be the fact that the Lie algebra G_c is nilpotent.

Recently based on a kind of new special non-semisimple Lie algebras, two feasible schemes for constructing nonlinear continuous and discrete integrable couplings were proposed in [16,17]. Variational identities [9,18] over the corresponding loop algebras were used to furnish Hamiltonian structures for the resulting nonlinear integrable couplings.

In the paper, enlightened by the idea adopted in [16,17], we want to establish three kinds of specific Lie algebras and make use of them to generate nonlinear continuous integrable couplings of evolution type for given integrable evolution equations. We take the AKNS hierarchy, the BK hierarchy and the KN hierarchy as examples to illustrate the suggested approach. Nonlinear integrable couplings of the nonlinear Schrödiner equation and the classical Boussinesq equation are presented as reductions of the computed examples. Moreover, two linear functionals are introduced on the corresponding loop algebras and Hamiltonian structures of the resulting nonlinear integrable couplings are furnished by employing the associated variational identity.

2 Three Lie Algebras: G, H and Q

Let us consider a vector space [14,15]:

$$L_3 = \{a = (a_1, a_2, a_3)^T, a_i \in R\}.$$

Denote by

$$K(L_3) = \{A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, A_1, A_2, A_3 \in L_3\}. \tag{18}$$

For $\forall A = (A_1, A_2, A_3)^T, B = (B_1, B_2, B_3)^T \in K(L_3)$, define an operation

$$[A,B] = \begin{pmatrix} [A_1,B_1] \\ [A_1,B_2]+[A_2,B_1] \\ [A_1,B_3]+[A_3,B_1]+[A_3,B_3] \end{pmatrix}. \tag{19}$$

It can be verified that $K(L_3)$ is a Lie algebra equipped with (19). In the vector space L_3, two different commutative operations are given by

$$[a,b] = \begin{pmatrix} a_2b_3 - a_3b_2 \\ 2a_1b_2 - 2a_2b_1 \\ 2a_3b_1 - 2a_1b_3 \end{pmatrix} \tag{20}$$

and

$$[a,b] = \begin{pmatrix} a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 \\ a_1b_2 - a_2b_1 \end{pmatrix}. \tag{21}$$

It can be verified that L_3 is a Lie algebra if equipped with (20) or (21), where

$$a = (a_1, a_2, a_3)^T, b = (b_1, b_2, b_3)^T \in L_3.$$

Now we extend the vector space L_3 into a higher-dimensional one as follows:

$$L_9 = \{a = (a_1, \ldots, a_9)^T, a_i \in R\}. \tag{22}$$

For $\forall a = (a_1, \ldots, a_9)^T, b = (b_1, \ldots, b_9)^T \in L_9$, note $a = (A_1, A_2, A_3)^T, b = (B_1, B_2, B_3)^T$, where

$$A_1 = (a_1, a_2, a_3)^T, A_2 = (a_4, a_5, a_6)^T, A_3 = (a_7, a_8, a_9)^T,$$
$$B_1 = (b_1, b_2, b_3)^T, B_2 = (b_4, b_5, b_6)^T, B_3 = (b_7, b_8, b_9)^T.$$

According to (20) and (21) combining with (19), we define the following two operations in L_9 :

$$[a,b] = \begin{pmatrix} [A_1, B_1]_1 \\ [A_1, B_2]_1 + [A_2, B_1]_1 \\ [A_1, B_3]_1 + [A_3, B_1]_1 + [A_3, B_3]_1 \end{pmatrix}, \tag{23}$$

where

$$[A_1, B_1]_1 = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ 2a_1 b_2 - 2a_2 b_1 \\ 2a_3 b_1 - 2a_1 b_3 \end{pmatrix},$$
$$[A_1, B_2]_1 + [A_2, B_1]_1 = \begin{pmatrix} a_2 b_6 - a_6 b_2 + a_5 b_3 - a_3 b_5 \\ 2a_1 b_5 - 2a_5 b_1 + 2a_4 b_2 - 2a_2 b_4 \\ 2a_6 b_1 - 2a_1 b_6 + 2a_3 b_4 - 2a_4 b_3 \end{pmatrix},$$
$$[A_1, B_3]_1 + [A_3, B_1]_1 + [A_3, B_3]_1 = \begin{pmatrix} a_2 b_9 - a_9 b_2 + a_8 b_3 - a_3 b_8 + a_8 b_9 - a_9 b_8 \\ 2a_1 b_8 - 2a_8 b_1 + 2a_7 b_2 - 2a_2 b_7 + 2a_7 b_8 - 2a_8 b_7 \\ 2a_9 b_1 - 2a_1 b_9 + 2a_3 b_7 - 2a_7 b_3 + 2a_9 b_7 - 2a_7 b_9 \end{pmatrix},$$

$$[a,b] = \begin{pmatrix} [A_1, B_1]_2 \\ [A_1, B_2]_2 + [A_2, B_1]_2 \\ [A_1, B_3]_2 + [A_3, B_1]_2 + [A_3, B_3]_2 \end{pmatrix}, \tag{24}$$

where

$$[A_1, B_1]_2 = \begin{pmatrix} a_3 b_2 - a_2 b_3 \\ a_1 b_3 - a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{pmatrix},$$
$$[A_1, B_2]_2 + [A_2, B_1]_2 = \begin{pmatrix} a_3 b_5 - a_5 b_3 + a_6 b_2 - a_2 b_6 \\ a_1 b_6 - a_6 b_1 + a_4 b_3 - a_3 b_4 \\ a_1 b_5 - a_5 b_1 + a_4 b_2 - a_2 b_4 \end{pmatrix},$$
$$[A_1, B_3]_2 + [A_3, B_1]_2 + [A_3, B_3]_2 = \begin{pmatrix} a_3 b_8 - a_8 b_3 + a_9 b_2 - a_2 b_9 + a_9 b_8 - a_8 b_9 \\ a_1 b_9 - a_9 b_1 + a_7 b_3 - a_3 b_7 + a_7 b_9 - a_9 b_7 \\ a_1 b_8 - a_8 b_1 + a_7 b_2 - a_2 b_7 + a_7 b_8 - a_8 b_7 \end{pmatrix},$$

Noting $e_i = (e_{i1}, \ldots, e_{i9})^T, e_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$, in terms of (23), we have a commutative relation as follows:

$$\begin{aligned}
&[e_1,e_2] = 2e_2, [e_1,e_3] = -2e_3, [e_2,e_3] = e_1, [e_1,e_4] = 0, [e_1,e_5] = 2e_5, [e_1,e_6] = -2e_6,\\
&[e_1,e_7] = 0, [e_1,e_8] = 2e_8, [e_1,e_9] = -2e_9, [e_2,e_4] = -2e_5, [e_2,e_5] = 0, [e_2,e_6] = e_4,\\
&[e_2,e_7] = -2e_8, [e_2,e_8] = 0, [e_2,e_9] = e_7, [e_3,e_4] = 2e_6, [e_3,e_5] = -e_4, [e_3,e_6] = 0,\\
&[e_3,e_7] = 2e_9, [e_3,e_8] = -e_7, [e_3,e_9] = [e_4,e_5] = [e_4,e_6] = [e_4,e_7] = [e_4,e_8] = [e_4,e_9] =\\
&[e_5,e_6] = [e_5,e_7] = [e_5,e_8] = [e_5,e_9] = 0, [e_6,e_7] = [e_6,e_8] = [e_6,e_9] = 0,\\
&[e_7,e_8] = 2e_8, [e_7,e_9] = -2e_9, [e_8,e_9] = e_7.
\end{aligned}$$

Let us make a linear transformation

$$f_1 = \frac{1}{2}e_7,\, f_2 = \frac{1}{2}(e_8 + e_9),\, f_3 = \frac{1}{2}(e_8 - e_9),$$

then we have a new Lie algebra

$$G = \mathrm{span}\{e_1, e_2, e_3, f_1, f_2, f_3\}, \tag{25}$$

where the commutative operations are the following:

$$\begin{aligned}
&[e_1,e_2] = 2e_2, [e_1,e_3] = -2e_3, [e_2,e_3] = e_1, [e_1,f_1] = 0, [e_1,f_2] = 2f_3, [e_1,f_3] = 2f_2,\\
&[e_2,f_1] = -f_2 - f_3, [e_2,f_2] = f_1, [e_2,f_3] = -f_1, [e_3,f_1] = f_2 - f_3, [e_3,f_2] = [e_3,f_3] = -f_1,\\
&[f_1,f_2] = f_3, [f_1,f_3] = f_2, [f_2,f_3] = -f_1.
\end{aligned}$$

Denote

$$G_1 = \mathrm{span}\{e_1, e_2, e_3\}, G_2 = \mathrm{span}\{f_1, f_2, f_3\},$$

and then we have

$$G = G_1 \Subset G_2, [G_1, G_2] \subset G_2,$$

which satisfy the sufficient condition for generating integrable couplings. We remark that here G_1 and G_2 are all simple Lie algebras.

If we make another linear transformation

$$h_1 = \sqrt{3}i f_1, h_2 = -\frac{\sqrt{6}i}{2} f_2 + \frac{3}{\sqrt{2}} f_3, h_3 = -\frac{\sqrt{6}i}{2} f_2 - \frac{3}{\sqrt{2}} f_3,$$

where

$$[f_1, f_2] = f_3, [f_1, f_3] = f_2, [f_2, f_3] = -f_1, i^2 = -1,$$

then we obtain that

$$\begin{aligned}
&[h_1,h_2] = -h_2 - 2h_3, [h_1,h_3] = 2h_2 + h_3, [h_2,h_3] = -3h_1, [e_1,h_1] = 0,\\
&[e_1,h_2] = \frac{2i}{\sqrt{3}}(h_2 + 2h_3), [e_1,h_3] = -\frac{2i}{\sqrt{3}}(2h_2 + h_3), [e_2,h_1] = \frac{3\sqrt{2} - \sqrt{6}i}{6}h_2 + \frac{3\sqrt{2} + \sqrt{6}i}{6}h_3,\\
&[e_2,h_2] = \frac{\sqrt{3}i - 1}{\sqrt{2}}h_1, [e_2,h_3] = -\frac{1 + \sqrt{3}i}{\sqrt{2}}h_1, [e_3,h_1] = -\frac{i + \sqrt{3}}{\sqrt{6}}h_2 + \frac{i - \sqrt{3}}{\sqrt{6}}h_3, [e_3,h_2] = \frac{1 + \sqrt{3}i}{\sqrt{2}}h_1,\\
&[e_3,h_3] = \frac{1 - \sqrt{3}i}{\sqrt{2}}h_1.
\end{aligned}$$

Denoting by

$$H = \mathrm{span}\{e_1, e_2, e_3, h_1, h_2, h_3\}, \tag{26}$$

then H is also a Lie algebra equipped with the above commutator. Similarly, according to (24), a basis of the Lie algebra R^9,

$$e_i = (e_{i1}, \ldots, e_{i9})^T, e_{ij} = \begin{cases} 1, \ i = j, \\ 0, \ i \neq j, \end{cases} 1 \le i, j \le 9,$$

possesses the following commutative relations

$$\begin{aligned}
&[e_1,e_2] = e_3, [e_1,e_3] = e_2, [e_2,e_3] = -e_1, [e_1,e_4] = 0, [e_1,e_5] = e_6, [e_1,e_6] = e_5, [e_1,e_7] = 0, \\
&[e_1,e_8] = e_9, [e_1,e_9] = e_8, [e_2,e_4] = -e_6, [e_2,e_5] = 0, [e_2,e_6] = -e_4, [e_2,e_7] = -e_9, \\
&[e_2,e_8] = 0, [e_2,e_9] = -e_7, [e_3,e_4] = -e_5, [e_3,e_5] = e_4, [e_3,e_6] = 0, [e_3,e_7] = -e_8, \\
&[e_3,e_8] = e_7, [e_3,e_9] = [e_4,e_5] = [e_4,e_6] = [e_4,e_7] = [e_4,e_8] = [e_4,e_9] = [e_5,e_6] = [e_5,e_7], \\
&[e_5,e_8] = [e_5,e_9] = [e_6,e_7] = [e_6,e_8] = [e_6,e_9] = 0, \\
&[e_7,e_8] = e_9, [e_7,e_9] = e_8, [e_8,e_9] = -e_7.
\end{aligned}$$

If make a linear transformation

$$p_1 = 2e_7, p_2 = 2(e_8 + e_9), p_3 = 2(e_8 - e_9),$$

then we have

$$\begin{aligned}
&[p_1,p_2] = 2p_2, [p_1,p_3] = -2p_3, [p_2,p_3] = 4p_1, [e_1,p_1] = 0, [e_1,p_2] = p_2, [e_1,p_3] = -p_3, \\
&[e_2,p_1] = \tfrac{1}{2}(p_3 - p_2), [e_2,p_2] = -p_1, [e_2,p_3] = p_1, [e_3,p_1] = -\tfrac{1}{2}(p_2 + p_3), [e_3,p_2] = p_1, \\
&[e_3,p_3] = p_1.
\end{aligned}$$

Setting

$$Q = \mathrm{span}\{e_1, e_2, e_3, p_1, p_2, p_3\}, \tag{27}$$

then Q forms a Lie algebra equipped with the above commutative operations.

3 Nonlinear Integrable Couplings

In the section, we shall use the loop algebras of the Lie algebras G, H and Q to construct nonlinear integrable couplings for the AKNS hierarchy, the BK hierarchy and the KN hierarchy, respectively. Specially, we shall obtain nonlinear integrable couplings of the standard nonlinear Schrödinger equation and the classical Boussinesq equation.

3.1 Two Nonlinear Integrable Couplings of the AKNS Hierarchy

Introduce a loop algebra of the Lie algebra G:

$$\tilde{G} = \mathrm{span}\{e_1(n), e_2(n), e_3(n), f_1(n), f_2(n), f_3(n)\},$$

where

$$e_i(n) = e_i\lambda^n, \ f_i(n) = f_i\lambda^n, \ i = 1, 2, 3, \ n \in Z.$$

The corresponding commutative operations read that

$$[e_i(m),e_j(n)]=[e_i,e_j]\lambda^{m+n},\ [f_i(m),f_j(n)]=[f_i,f_j]\lambda^{m+n},\ 1\le i,j\le 3,\ m,n\in Z.$$

By using the loop algebra $\tilde{G}$, we introduce a Lax pair of zero curvature equations

$$\begin{cases} U=e_1(1)+qe_2(0)+re_3(0)+u_1f_2(0)+u_2f_3(0),\\ V=\sum_{m\ge 0}\sum_{i=1}^{3}V_{im}e_i(-m)+\sum_{m\ge 0}\sum_{j=4}^{6}V_{jm}f_{j-3}(-m).\end{cases} \tag{28}$$

A compatibility condition of the Lax pair (28) gives rise to a recursion relation

$$\begin{cases} V_{1,mx}=qV_{3m}-rV_{2m},\\ V_{2,mx}=2V_{2,m+1}-2qV_{1m},\\ V_{3,mx}=-2V_{3,m+1}+2rV_{1m},\\ V_{4,mx}=(q-r+u_2)V_{5m}-(q+r+u_1)V_{6m}+(u_1+u_2)V_{3m}+(u_2-u_1)V_{2m},\\ V_{5,mx}=2V_{6,m+1}+(r-q)V_{4m}-u_2(2V_{1m}+V_{4m}),\\ V_{6,mx}=2V_{5,m+1}-(q+r)V_{4m}-u_1(2V_{1m}+V_{4m}).\end{cases} \tag{29}$$

Note that

$$V_+^{(n)}=(\lambda^nV)_+=\sum_{m=0}^{n}\sum_{i=1}^{3}V_{im}e_i(n-m)+\sum_{m=0}^{n}\sum_{j=4}^{6}V_{jm}f_{j-3}(n-m)=\lambda^nV-V_-^{(n)}.$$

According to the Tu scheme [19,20], we need to compute the left-hand side of the following equation

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}].$$

A direct calculation gives that

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=-2V_{2,n+1}e_2(0)+2V_{3,n+1}e_3(0)-2V_{5,n+1}f_3(0)-2V_{6,n+1}f_2(0).$$

Let $V^{(n)}=V_+^{(n)}$. Then the zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0$$

admits a Lax integrable hierarchy

$$u_{t_n}=\begin{pmatrix} q\\ r\\ u_1\\ u_2\end{pmatrix}_{t_n} F=\begin{pmatrix} 2V_{2,n+1}\\ -2V_{3,n+1}\\ 2V_{6,n+1}\\ 2V_{5,n+1}\end{pmatrix}=L\begin{pmatrix} 2V_{2n}\\ -2V_{3n}\\ 2V_{6n}\\ 2V_{5n}\end{pmatrix}, \tag{30}$$

where L is a recurrence operator, i.e.,

$$L=\begin{pmatrix} \frac{\partial}{2}-q\partial^{-1}r & q\partial^{-1}q & 0 & 0\\ r\partial^{-1}r & -\frac{\partial}{2}+r\partial^{-1}q & 0 & 0\\ \frac{1}{2}(q-r+u_2)\partial^{-1}(u_1-u_2) & -\frac{1}{2}(q-r+u_2)\partial^{-1}(u_1+u_2) & l_1 & l_2s\\ l_3 & l_4 & l_5 & l_6\end{pmatrix},$$

with

$$l_1 = -\frac{1}{2}(q-r+u_2)\partial^{-1}(q+r+u_1), l_2 = \frac{\partial}{2} + \frac{1}{2}(q-r+u_2)\partial^{-1}(q-r+u_2),$$
$$l_3 = \frac{1}{2}(q+r-u_1)\partial^{-1}(u_2-u_1) - u_1\partial^{-1}r, l_4 = u_1\partial^{-1}q + \frac{1}{2}(q+r-u_1)\partial^{-1}(u_1+u_2),$$
$$l_5 = \frac{\partial}{2} - \frac{1}{2}(q+r-u_1)\partial^{-1}(q+r+u_1), l_6 = \frac{1}{2}(q+r-u_1)\partial^{-1}(q-r+u_2).$$

It is easy to check that

$$\begin{pmatrix} 2V_{2,n+1} \\ -2V_{3,n+1} \\ 2V_{6,n+1} \\ 2V_{5,n+1} \end{pmatrix} = L \begin{pmatrix} 2V_{2n} \\ -2V_{3n} \\ 2V_{6n} \\ 2V_{5n} \end{pmatrix}.$$

When set $u_1 = u_2 = 0$, the hierarchy (30) reduces to the well-known AKNS hierarchy.
Setting $V_{1,0} = \alpha, V_{2,0} = V_{3,0} = V_{5,0} = V_{6,0} = 0$, then from (29) we have

$$V_{4,0} = 0, V_{2,1} = \alpha q, V_{3,1} = \alpha r, V_{6,1} = \alpha u_2, V_{5,1} = \alpha u_1, V_{1,1} = V_{4,1} = 0, V_{2,2} = \frac{\alpha}{2}q_x,$$
$$V_{3,2} = -\frac{\alpha}{2}r_x, V_{1,2} = -\frac{\alpha}{2}qr, V_{2,3} = \frac{\alpha}{4}q_{xx} - \frac{\alpha}{2}q^2r, V_{3,3} = \frac{\alpha}{4}r_{xx} - \frac{\alpha}{2}qr^2, V_{1,3} = \frac{\alpha}{4}(qr_x - q_xr),$$
$$V_{6,2} = \frac{\alpha}{2}u_{1x}, V_{5,2} = \frac{\alpha}{2}u_{2x}, V_{4,2} = \frac{\alpha}{2}(q-r)u_2 - \frac{\alpha}{2}(q+r)u_1 - \frac{\alpha}{4}(u_1^2 - u_2^2),$$
$$V_{5,3} = \frac{\alpha}{4}u_{1xx} + \frac{\alpha}{4}(q+r+u_1)[(q-r)u_2 - (q+r)u_1] - \frac{\alpha}{8}(q+r+u_1)(u_1^2-u_2^2) - \frac{\alpha}{2}qru_1,$$
$$V_{6,3} = \frac{\alpha}{4}u_{2xx} + \frac{\alpha}{4}(q-r+u_2)[(q-r)u_2 - (q+r)u_1] - \frac{\alpha}{8}(q-r+u_2)(u_1^2-u_2^2) - \frac{\alpha}{2}qru_2, \ldots,$$

When $n = 2$, the hierarchy (30) reduces to a nonlinear integrable coupling of the standard Schrödinger equation

$$\begin{cases} q_{t_2} = \frac{\alpha}{2}q_{xx} - \alpha q^2 r, \\ r_{t_2} = -\frac{\alpha}{2}r_{xx} + \alpha qr^2, \\ u_{1,t_2} = \frac{\alpha}{2}u_{2xx} + \frac{\alpha}{2}(q-r+u_2)[(q-r)u_2 - (q+r)u_1] - \frac{\alpha}{4}(q-r+u_2)(u_1^2-u_2^2) - \alpha qru_2, \\ u_{2,t_2} = \frac{\alpha}{2}u_{1xx} + \frac{\alpha}{2}(q+r+u_1)[(q-r)u_2 - (q+r)u_1] - \frac{\alpha}{4}(q+r+u_1)(u_1^2-u_2^2) - \alpha qru_1. \end{cases} \tag{31}$$

When taking $u_1 = u_2 = 0$, (30) reduces to the nonlinear Schrödinger equation. When taking $n = 2, 3, \ldots$, the hierarchy (30) all reduces to nonlinear integrable coupling equations. Hence, (30) presents a hierarchy of nonlinear integrable coupling for the AKNS hierarchy.

In what follows, we employ the Lie algebra H to construct a second hierarchy of nonlinear integrable couplings for the AKNS hierarchy. A loop algebra of the Lie algebra H is chosen as

$$\tilde{H} = \text{span}\{e_1(n), e_2(n), e_3(n), h_1(n), h_2(n), h_3(n)\},$$

where

$$e_i(n) = e_i\lambda^n, h_i(n) = h_i\lambda^n,\ i = 1,2,3,\ n \in Z.$$

The corresponding commutative operations read

$$[e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n},\ [h_i(m), h_j(n)] = [h_i, h_j]\lambda^{m+n},\ 1 \le i, j \le 3,\ m,n \in Z.$$

By utilizing $\tilde{H}$, a Lax pair is introduced as follows

$$\begin{cases} U = e_1(1) + qe_2(0) + re_3(0) + s_1h_2(0) + s_2h_3(0), \\ V = \sum\limits_{m\ge 0}(\sum\limits_{i=1}^{3} V_{im}e_i(-m) + \sum\limits_{j=4}^{6} V_{jm}h_{j-3}(-m)). \end{cases} \tag{32}$$

The stationary zero curvature equation

$$[U,V] = V_x$$

is equivalent to the following

$$\begin{cases} V_{1,mx} = qV_{3m} - rV_{2m}, \\ V_{2,mx} = 2V_{2,m+1} - 2qV_{1m}, \\ V_{3,mx} = -2V_{3,m+1} + 2rV_{1m}, \\ V_{4,mx} = \left(\frac{1+\sqrt{3}i}{\sqrt{2}}s_2 + \frac{1-\sqrt{3}i}{\sqrt{2}}s_1\right)V_{2m} + \left(\frac{\sqrt{3}i-1}{\sqrt{2}}s_2 - \frac{1+\sqrt{3}}{\sqrt{2}}s_1\right)V_{3m} \\ \qquad + \left(\frac{\sqrt{3}i-1}{\sqrt{2}}q + \frac{1+\sqrt{3}i}{\sqrt{2}}r + 3s_2\right)V_{5m} + \left(\frac{1-\sqrt{3}i}{\sqrt{2}}r - \frac{1+\sqrt{3}i}{\sqrt{2}}q - 3s_1\right)V_{6m}, \\ V_{5,mx} = \frac{2i}{\sqrt{3}}V_{5,m+1} - \frac{4i}{\sqrt{3}}V_{6,m+1} + \left(\frac{3\sqrt{2}-\sqrt{3}i}{6}q - \frac{i+\sqrt{3}}{\sqrt{6}}r + s_1 - 2s_2\right)V_{4m} \\ \qquad + \left(\frac{4i}{\sqrt{3}}s_2 - \frac{2is_1}{\sqrt{3}}\right)V_{1m}, \\ V_{6,mx} = \frac{4i}{\sqrt{3}}V_{5,m+1} - \frac{2i}{\sqrt{3}}V_{6,m+1} + \left(\frac{3\sqrt{2}-\sqrt{6}i}{6}q + \frac{i-\sqrt{3}}{\sqrt{6}}r + s_1 - s_2\right)V_{4m} \\ \qquad + \left(\frac{2i}{\sqrt{3}}s_2 - \frac{4i}{\sqrt{3}}s_1\right)V_{1m}. \end{cases} \tag{33}$$

Setting $V_{1,0} = \alpha, V_{2,0} = V_{3,0} = V_{5,0} = V_{6,0} = 0$, then we obtain from (33)

$$V_{4,0}=0, V_{5,1}=\alpha s_1, V_{6,1}=\alpha s_2, V_{4,1}=0, V_{2,1}=\alpha q, V_{3,1}=\alpha r, V_{2,2}=\frac{\alpha}{2}q_x, V_{3,2}=-\tfrac{\alpha}{2}r_x,$$

$$V_{1,2}=-\frac{\alpha}{2}qr, V_{5,2}=\frac{\alpha}{2\sqrt{3}i}(2s_{2x}-s_{1x}), V_{6,2}=\frac{\alpha}{2\sqrt{3}i}(s_{2x}-2s_{1x}),$$

$$V_{4,2}=\frac{\sqrt{3}i+1}{2\sqrt{2}}\alpha q s_2+\frac{1-\sqrt{3}i}{2\sqrt{2}}\alpha q s_1+\frac{1-\sqrt{3}i}{2\sqrt{2}}\alpha r s_2+\frac{1+\sqrt{3}i}{2\sqrt{2}}\alpha r s_1+\frac{3\alpha}{2\sqrt{3}i}(s_1^2+s_2^2-s_1s_2),$$

$$\begin{aligned}V_{5,3}=&\frac{\alpha}{4}s_{1xx}+\frac{i\alpha}{3}\left(\frac{3\sqrt{2}-\sqrt{6}i}{12\sqrt{3}}q+\frac{3i-\sqrt{3}}{6\sqrt{2}}r+\frac{3s_1}{2\sqrt{3}}\right)\left[\frac{\sqrt{3}i+1}{2\sqrt{2}}qs_2+\frac{1-\sqrt{3}i}{2\sqrt{2}}qs_1+\left(\frac{1-\sqrt{3}i}{2\sqrt{2}}s_2\right.\right.\\&\left.\left.+\frac{1+\sqrt{3}i}{2\sqrt{2}}s_1\right)r-\frac{\sqrt{3}i}{2}(s_1^2+s_2^2-s_1s_2)\right]-\frac{\alpha}{2}qrs_1,\end{aligned}$$

$$\begin{aligned}V_{6,3}=&\frac{\alpha}{4}s_{2xx}-\frac{\alpha i}{6}\left(\frac{3\sqrt{2}-\sqrt{6}i}{6\sqrt{3}}q-\frac{3i+\sqrt{3}}{2\sqrt{2}}r-\sqrt{3}s_2\right)\left[\left(\frac{\sqrt{3}i+1}{2\sqrt{2}}s_2+\frac{1-\sqrt{3}i}{2\sqrt{2}}s_1\right)q\right.\\&\left.+\left(\frac{1-\sqrt{3}i}{2\sqrt{2}}s_2+\frac{1+\sqrt{3}i}{2\sqrt{2}}s_1\right)r-\frac{\sqrt{3}i}{2}(s_1^2+s_2^2-s_1s_2)\right]-\frac{\alpha}{2}qrs_2.\end{aligned}$$

Setting

$$V^{(n)}=\sum_{m=0}^{n}\left(\sum_{i=1}^{3}V_{im}e_i(n-m)+\sum_{j=4}^{6}V_{jm}h_{j-3}(n-m)\right),$$

a direct calculation gives rise to

$$\begin{aligned}-V_x^{(n)}+[U,V^{(n)}]=&-2V_{2,n+1}e_2(0)+2V_{3,n+1}e_3(0)+\frac{2i}{\sqrt{3}}(2V_{6,n+1}-V_{5,n+1})h_2(0)\\&+\frac{2i}{\sqrt{3}}(V_{6,n+1}-2V_{5,n+1})h_3(0).\end{aligned}$$

The equation

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0$$

leads to

$$u_{t_n}=\begin{pmatrix}q\\r\\s_1\\s_2\end{pmatrix}_{t_n}=\begin{pmatrix}2V_{2,n+1}\\-2V_{3,n+1}\\\frac{2i}{\sqrt{3}}(V_{5,n+1}-2V_{6,n+1})\\\frac{2i}{\sqrt{3}}(2V_{5,n+1}-V_{6,n+1})\end{pmatrix}. \tag{34}$$

When setting $s_1=s_2=0$, (34) reduces to the AKNS hierarchy, and so it gives a hierarchy of nonlinear integrable couplings for the AKNS hierarchy. Specially, if we take $n=2$, the hierarchy (34) reduces to another nonlinear integrable coupling of the nonlinear Schrödinger equation:

$$\begin{cases} q_{t_2} = \frac{\alpha}{2} q_{xx} - \alpha q^2 r, \\ r_{t_2} = -\frac{\alpha}{2} r_{xx} + \alpha q r^2, \\ s_{1,t_2} = -\frac{i\alpha}{2\sqrt{3}}(2s_{2xx} - s_{1xx}) + \frac{\imath\alpha}{\sqrt{3}}(2s_2 - s_1)qr + \frac{\alpha}{\sqrt{3}}\left(\frac{\sqrt{6}i - 3\sqrt{2}}{6\sqrt{3}} q + \frac{i+\sqrt{3}}{3\sqrt{2}} r - \frac{s_1 - 2s_2}{\sqrt{3}}\right) \\ \qquad \left[\left(\frac{\sqrt{3}i+1}{2\sqrt{2}} s_2 + \frac{1-\sqrt{3}i}{2\sqrt{2}} s_1\right) q + \left(\frac{1-\sqrt{3}i}{2\sqrt{2}} s_2 + \frac{1+\sqrt{3}i}{2\sqrt{2}}\right) r - \frac{\sqrt{3}i}{2}(s_1^2 + s_2^2 - s_1 s_2)\right], \\ s_{2,t_2} = -\frac{i\alpha}{2\sqrt{3}}(s_{2xx} - 2s_{1xx}) + \frac{i\alpha}{\sqrt{3}}(s_2 - 2s_1)qr - \frac{\alpha}{\sqrt{3}}\left(\frac{3\sqrt{2} - \sqrt{6}i}{6\sqrt{3}} q + \frac{i-\sqrt{3}}{3\sqrt{2}} r + \frac{2s_1 - s_2}{\sqrt{3}}\right) \\ \qquad \left[\left(\frac{\sqrt{3}i+1}{2\sqrt{2}} s_2 + \frac{1-\sqrt{3}i}{2\sqrt{2}} s_1\right) q + \left(\frac{1\sqrt{3}i}{2\sqrt{2}} s_2 + \frac{1+\sqrt{3}i}{2\sqrt{2}} s_1\right) r - \frac{\sqrt{3}i}{2}(s_1^2 + s_2^2 - s_1 s_2)\right]. \end{cases} \tag{35}$$

3.2 Three Nonlinear Integrable Couplings of the BK Hierarchy

In this sub-section, we first make use of the Lie algebra G to establish two different Lax pairs to generate the corresponding different nonlinear integrable couplings of the BK hierarchy. As reduced cases, various nonlinear integrable couplings of the classical Boussinesq equation are obtained. Employing the Lie algebra H, we introduce a Lax pair whose compatibility condition gives rise to the third nonlinear integrable couplings of the BK hierarchy.

Applying the loop algebra $\tilde{G}$ of the Lie algebra G introduces a Lax pair as follows:

$$\begin{cases} U = -e_1(1) + \frac{v}{2} e_1(0) + e_2(0) - w e_3(0) + u_1 f_2(0) + u_2 f_3(0), \\ V = \sum_{m \geq 0} \sum_{j=1}^{3} V_{jm} e_j(-m) + \sum_{m \geq 0} \sum_{k=4}^{6} V_{km} f_{k-3}(-m). \end{cases} \tag{36}$$

Equation $V_x = [U, V]$ gives that

$$\begin{cases} V_{1,mx} = V_{3m} + w V_{2m}, \\ V_{2,mx} = -2V_{2,m+1} + v V_{2m} - 2V_{1m}, \\ V_{3,mx} = 2V_{3,m+1} - v V_{2m} - 2w V_{1m}, \\ V_{4,mx} = (1 + w + u_2)V_{5m} + (-1 + w - u_1)V_{6m} + (u_1 - u_1)V_{2m} + (u_1 + u_2)V_{3m}, \\ V_{5,mx} = -2V_{6,m+1} + v V_{6m} + (-1 + w - u_2)V_{4m} - 2u_2 V_{1m}, \\ V_{6,mx} = -2V_{5,m+1} + v V_{5m} + (-1 + w - u_1)V_{4m} - 2u_1 V_{1m}. \end{cases} \tag{37}$$

Noting

$$V_+^{(n)} = (lambda^n V)_+ = \sum_{m=0}^{n} \left(\sum_{j=1}^{3} V_{jm} e_j(n-m) + \sum_{k=4}^{6} V_{km} f_{k-3}(n-m) \right) = \lambda^n V - V_-^{(n)},$$

then Eq.$V_x = [U,V]$ can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}].$$

A direct computation reads

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = 2V_{2,n+1}e_2(0) - 2V_{3,n+1}e_3(0) + 2V_{5,n+1}f_3(0) + 2V_{6,n+1}f_2(0).$$

Setting $V^{(n)} = V_+^{(n)} + \Delta_1, \Delta_1 = V_{2,n+1}e_1(0)$, we have

$$\begin{aligned}-V_x^{(n)} + [U, V^{(n)}] = & -V_{2,n+1x}e_1(0) + (-2wV_{2,n+1} - 2V_{3,n+1})e_3(0) \\ & +(-2u_2V_{2,n+1} + 2V_{6,n+1})f_2(0) + (-2u_1V_{2,n+1} + 2V_{5,n+1})f_3(0).\end{aligned}$$

Thus, the compatibility of the Lax pair U and $V^{(n)}$ gives rise to

$$u_{t_n} = \begin{pmatrix} v \\ w \\ u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} 2V_{2,n+1x} \\ -2wV_{2,n+1} - 2V_{3,n+1} \\ 2u_2V_{2,n+1} - 2V_{6,n+1} \\ 2u_1V_{2,n+1} - 2V_{5,n+1} \end{pmatrix} = \begin{pmatrix} 2V_{2,n+1x} \\ -2V_{1,n+1x} \\ 2u_2V_{2,n+1} - 2V_{6,n+1} \\ 2u_1V_{2,n+1} - 2V_{5,n+1} \end{pmatrix}. \tag{38}$$

Letting $u_1 = u_2 = 0$, (38) reduces to the BK hierarchy. Setting $V_{1,0} = \alpha, V_{2,0} = V_{3,0} = V_{5,0} = V_{6,0} = 0$, then one infers from (37) that

$$\begin{aligned}
&V_{1,1} = 0, V_{2,1} = -\alpha, V_{3,1} = \alpha w, V_{1,2} = \frac{\alpha}{2}w, V_{2,2} = -\frac{\alpha}{2}v, V_{3,1} = \alpha w, V_{1,3} = \frac{\alpha}{4}w_{xx} + \frac{\alpha}{2}wv, \\
&V_{3,2} = \frac{\alpha}{2}(w_x + wv), V_{2,3} = \frac{\alpha}{4}v_x - \frac{\alpha}{4}v^2 - \frac{\alpha}{2}w, V_{3,3} = \frac{\alpha}{4}(w_x + wv)_x + \frac{\alpha}{4}v(w_x + wv) + \frac{\alpha}{2}w^2, \\
&V_{6,1} = 0, V_{6,2} = \frac{\alpha}{2}(u_{1x} - vu_2), V_{5,2} = \frac{\alpha}{2}(u_{2x} - vu_1), \\
&V_{4,2} = \frac{\alpha}{2}\left(u_2 - u_1 + wu_2 + \frac{1}{2}u_2^2 - \frac{1}{2}u_1^2 + wu_1\right), \\
&V_{6,3} = -\frac{\alpha}{4}(u_{2xx} - (vu_1)_x) + \frac{\alpha}{4}v(u_{1x} - vu_2) - \frac{\alpha}{4}(1 + w + u_2)(u_2 - u_1 + wu_1 + wu_2 - \frac{1}{2}u_1^2 - \frac{1}{2}u_2^2) - \frac{\alpha}{2}u_2w, \\
&V_{5,3} = -\frac{\alpha}{4}(u_{1xx} - (vu_2)_x) + \frac{\alpha}{4}v(u_{2x} - vu_1) + \frac{\alpha}{4}(-1 + w - u_1)\left(u_2 - u_1 + wu_1 + wu_2 + \frac{1}{2}u_2^2 - \frac{1}{2}u_1^2\right) \\
&\qquad -\frac{\alpha}{2}u_1w, \ldots
\end{aligned}$$

Letting $n = 2$, then (38) reduces to the following nonlinear integrable equations

$$\begin{aligned}
v_{t_2} &= \frac{\alpha}{2}v_{xx} - \alpha vv_x - \alpha w_x, \\
w_{t_2} &= -\frac{\alpha}{2}w_{xx} - \alpha(wv)_x, \\
u_{1,t_2} &= \frac{\alpha}{2}u_2v_x - \frac{\alpha}{2}u_2v^2 - \alpha u_2w + \frac{\alpha}{2}(u_{2xx} - (vu_1)_x) - \frac{\alpha}{2}v(u_{1x} - vu_2) + \frac{\alpha}{2}(1 + w + u_2) \\
&\quad \left[u_2 - u_1 + w(u_1 + u_2) - \frac{1}{2}(u_1^2 + u_2^2)\right] + \alpha u_2w, \\
u_{2,t_2} &= \frac{\alpha}{2}u_1v_x - \frac{\alpha}{2}u_1v^2 - \alpha u_1w - \frac{\alpha}{2}(u_{1xx} - (vu_2)_x) - \frac{\alpha}{2}(-1 + w - u_1)[u_2 - u_1 + w(u_1 + u_2) \\
&\quad + \frac{1}{2}(u_2^2 - u_1^2)] + \alpha u_1w.
\end{aligned} \tag{39}$$

Setting $u_1 = u_2 = 0, \alpha = -1, t_2 = t$, (39) reduces to the BK system

$$\begin{cases} v_t = -\frac{1}{2}v_{xx} + vv_x + w_x, \\ w_t = \left(vw + \frac{1}{2}w_x\right)_x, \end{cases} \tag{40}$$

which was obtained in [21]. Two basic Darboux transformations and some new solutions were obtained in [22]. Li and Zhang [23] obtained the third Darboux transformation of the BK system (40) and produced some multi-soliton solutions. (39) presents a nonlinear integrable coupling for the BK system (40). The Lax integrable hierarchy (38) gives a hierarchy of nonlinear integrable couplings for the BK hierarchy.

In what follows, we deduce a nonlinear integrable coupling for the classical Boussinesq equation. By making a linear transformation [22,23]:

$$v = -u, w = \xi + 1 + \frac{v_x}{2}, \tag{41}$$

the BK system (40) is transformed to the following classical Boussinesq equation

$$\begin{cases} u_t + uu_x + \xi_x = 0, \\ \xi_t + ((1+\xi)u)_x + \frac{1}{4}u_{xxx} = 0, \end{cases} \tag{42}$$

where ξ is the elevation of the water wave, u is the surface velocity of water along x-direction. Substituting (41) into (39) yields a nonlinear integrable coupling of (42):

$$\begin{cases} u_t + uu_x + \xi_x = 0, \\ \xi_t + ((1+\xi)u)_x + \frac{1}{4}u_{xxx} = 0, \\ u_{1t} = \frac{1}{2}u_2u_x - \frac{1}{2}u_2u^2 + u_2\left(1+\xi+\frac{v_x}{2}\right) - \frac{1}{2}(u_{2xx} + (uu_x)_x) - \frac{1}{2}(u_{1x} + uu_2) \\ \qquad -\frac{1}{2}(2+\xi+\frac{v_x}{2}+u_2)\left[u_2 - u_1 + \left(1+\xi+\frac{v_x}{2}\right)(u_1+u_2) - \frac{1}{2}(u_1^2+u_2^2)\right] - u_2(1+\xi+\frac{v_x}{2}), \\ u_{2t} = \frac{1}{2}u_1u_x + \frac{1}{2}u_1u^2 + u_1\left(1+\xi+\frac{v_x}{2}\right) + \frac{1}{2}(u_{1xx} + (uu_2)_x) \\ \qquad +\frac{1}{2}\left(\xi+\frac{v_x}{2}-u_1\right)\left[u_2 - u_1 + (u_1+u_2)\left(1+\xi+\frac{v_x}{2}\right) + \frac{1}{2}(u_2^2-u_1^2)\right] - u_1\left(1+\xi+\frac{v_x}{2}\right), \end{cases} \tag{43}$$

In the following, we deduce the second nonlinear coupling of the BK hierarchy. We still use the loop algebra $\tilde{G}$ to introduce a Lax pair:

$$\begin{cases} U = -e_1(1) + \frac{v}{2}e_1(0) + e_2(0) - we_3(0) + s_1 f_1(0) + s_2 f_3(0), \\ V = \sum_{m\geq 0}\left(\sum_{i=1}^{3} V_{im}e_i(-m) + \sum_{j=4}^{6} V_{jm} f_{j-3}(-m)\right). \end{cases} \tag{44}$$

Similar to the above discussion, a recurrence relation reads as follows

$$\begin{cases} V_{1,mx} = V_{3m} + wV_{2m}, \\ V_{2,mx} = -2V_{2,m+1} + vV_{2m} - 2V_{1m}, \\ V_{3,mx} = 2V_{3,m+1} - vV_{3m} - 2wV_{1m}, \\ V_{4,mx} = (w-1)V_{6m} + (1+w+s_2)V_{5m} + s_2(V_{2m} + V_{3m}), \\ V_{5,mx} = -2V_{6,m+1} + (v+s_1)V_{6m} - (1+w+s_2)V_{4m} - 2s_2V_{1m} + s_1(V_{2m} - V_{3m}), \\ V_{6,mx} = -2V_{5,m+1} + (v+s_1)V_{5m} + (-1+w)V_{4m} + s_1(V_{2m} + V_{3m}). \end{cases} \tag{45}$$

Taking some initial values such as $V_{1,0} = \alpha, V_{2,0} = V_{3,0} = V_{5,0} = V_{6,0} = 0$, then from (45) one infers that

$$\begin{aligned} & V_{1,1} = V_{4,0} = 0, V_{2,1} = -\alpha, V_{3,1} = \alpha w, V_{1,2} = \frac{\alpha}{2}w, V_{2,2} = -\frac{\alpha}{2}v, V_{3,1} = \alpha w, \\ & V_{1,3} = \frac{\alpha}{4}w_{xx} + \frac{\alpha}{2}wv, V_{3,2} = \frac{\alpha}{2}(w_x + wv), V_{2,3} = \frac{\alpha}{4}v_x - \frac{\alpha}{4}v^2 - \frac{\alpha}{2}w, \\ & V_{3,3} = \frac{\alpha}{4}(w_x + wv)_x + \frac{\alpha}{4}v(w_x + wv) + \frac{\alpha}{2}w^2, V_{6,1} = -\alpha s_2, V_{5,0} = 0, V_{4,0} = 0, \\ & V_{6,2} = -\frac{\alpha}{2}(s_2v + s_1s_2 + s_1 + s_1w), V_{5,2} = \frac{\alpha}{2}(s_{2x} - s_1 + s_1w), \\ & V_{4,2} = \frac{\alpha}{2}\left(s_2 + s_2w + \frac{1}{2}s_2^2\right), \\ & V_{6,3} = -\frac{\alpha}{4}(s_{2xx} - s_{1x} + (s_1w)_x) - \frac{\alpha}{4}(v+s_1)(s_2v + s_1s_2 + s_1 + s_1w) - \frac{\alpha}{4}(1+w+s_2)\left(s_2 + s_2w + \frac{1}{2}s_2^2\right) \\ & \qquad -\frac{\alpha}{2}s_2w - \frac{\alpha}{2}(s_1v + s_1w_x + s_1wv), \\ & V_{5,3} = \frac{\alpha}{4}(s_2v + s_1s_2 + s_1 + s_1w)_x + \frac{\alpha}{4}(v+s_1)(s_{2x} - s_1 + s_1w) + \frac{\alpha}{4}(w-1)\left(s_2 + s_2w + \frac{1}{2}s_2^2\right) \\ & \qquad -\frac{\alpha}{4}(s_1v - s_1w_x - s_1wv), \ldots \end{aligned}$$

Setting

$$V_+^{(n)} = \sum_{m=0}^{n}\left(\sum_{i=1}^{3} V_{im}e_i(n-m) + \sum_{j=4}^{6} V_{jm}f_{j-3}(n-m)\right),$$

we obtain that

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = 2V_{2,n+1}e_2(0) - 2V_{3,n+1}e_3(0) + 2V_{5,n+1}f_3(0) + 2V_{6,n+1}f_2(0).$$

Noting

$$V^{(n)} = V_+^{(n)} + \Delta_2, \Delta_2 = V_{2,n+1}e_1(0) + \frac{2V_{6,n+1} - 2s_2V_{2,n+1}}{1+w+s_2}f_1(0),$$

then we have

$$\begin{aligned} -V_x^{(n)} + [U, V^{(n)}] = & -V_{2,n+1x}e_1(0) - 2V_{1,n+1x}e_3(0) - \left(\frac{2V_{6,n+1} - 2s_2V_{2,n+1}}{1+w+s_2}\right)_x f_1(0) + (2V_{5,n+1} \\ & + \frac{w-1}{1+w+s_2}(2V_{6,n+1} - 2s_2V_{2,n+1}))f_3(0). \end{aligned}$$

The compatibility condition of the Lax pair U and $V^{(n)}$ leads to

$$u_{t_n}=\begin{pmatrix} v\\ w\\ s_1\\ s_2\end{pmatrix}_{t_n}=\begin{pmatrix} 2V_{2,n+1x}\\ -2V_{1,n+1x}\\ \left(\dfrac{2V_{6,n+1}-2s_2V_{2,n+1}}{1+w+s_2}\right)_x\\ -2V_{5,n+1}+\dfrac{1-w}{1+w+s_2}(2V_{6,n+1}-2s_2V_{2,n+1})\end{pmatrix}. \tag{46}$$

When $s_1=s_2=0$, (46) reduces to the BK hierarchy. When taking $n=2,\alpha=-1,t_2=t$, (46) reduces to a nonlinear integrable coupling of the BK system (40):

$$\begin{cases} v_t=-\dfrac{1}{2}v_{xx}+vv_x+w_x,\\ w_t=\left(vw+\dfrac{1}{2}w_x\right)_x,\\ s_{1t}=\left(\dfrac{1}{1+w+s_2}\left[\dfrac{1}{2}(s_{2xx}-s_{1x}+(s_1w)_x)+\dfrac{1}{2}(v+s_1)(s_2v+s_1s_2+s_1+s_1w)\right.\right.\\ \qquad\left.\left.+\dfrac{1}{2}(1+w+s_2)\left(s_2+s_2w+\dfrac{1}{2}s_2^2\right)+s_1v+s_1w_x+s_1wv+\dfrac{1}{2}(s_2v_x-s_2v^2)\right]\right)_x,\\ s_{2t}=\dfrac{1}{2}(s_2v+s_1s_2+s_1+s_1w)_x+\dfrac{1}{2}(v+s_1)(s_{2x}-s_1+s_1w)+\dfrac{1}{2}(w-1)\left(s_2+s_2w+\dfrac{1}{2}s_2^2\right)\\ \qquad-\dfrac{1}{2}(s_1v-s_1w_x-s_1wv)+\dfrac{1-w}{2(1+w+s_2)}\Big[s_{2xx}-s_{1x}+(s_1w)_x+(v+s_1)(s_2v+s_1s_2+s_1+s_1w)\\ \qquad+(1+w+s_2)\left(s_2+s_2w+\dfrac{1}{2}s_2^2\right)+2s_1v+2s_1w_x+2s_1wv+s_2v_x-s_2v^2)\Big]. \end{cases} \tag{47}$$

Obviously, (47) is different from (39). They are all nonlinear integrable couplings of the BK system. When making the linear transformation, it is easy to transform the hierarchy (47) into a hierarchy of nonlinear integrable couplings for the classical Boussinesq equation. Here we omit it due to complicatedness.

Next, we construct the third nonlinear integrable coupling of the BK hierarchy by using a loop algebra $\tilde{H}$ of the Lie algebra H.
Setting

$$\begin{cases} U=-e_1(1)+\frac{v}{2}e_1(0)+e_2(0)-we_3(0)+w_1h_2(0)+w_2h_3(0),\\ V=\sum\limits_{m\geq 0}\left(\sum\limits_{i=1}^{3}V_{im}e_i(-m)+\sum\limits_{j=4}^{6}V_{jm}h_{j-3}(-m)\right), \end{cases} \tag{48}$$

Eq. $[U,V]=V_x$ leads to

$$\begin{cases} V_{1,mx} = V_{3m} + wV_{2m}, \\ V_{2,mx} = -2V_{2,m+1} + vV_{2m} - 2V_{1m}, \\ V_{3,mx} = 2V_{3,m+1} - vV_{3m} - wV_{1m}, \\ V_{4,mx} = \left(\frac{\sqrt{3}i-1}{\sqrt{2}} - \frac{1+\sqrt{3}i}{\sqrt{2}}w + 3w_2\right)V_{5m} - \left(\frac{1+\sqrt{3}i}{\sqrt{2}}\right. \\ \qquad\left. + \frac{1-\sqrt{3}i}{\sqrt{2}}w + 3w_1\right)V_{6m} + \left(\frac{1-\sqrt{3}i}{\sqrt{2}}w_1 + \frac{1+\sqrt{3}i}{\sqrt{2}}w_2\right)V_{2m}, \\ V_{5,mx} = -\frac{2i}{\sqrt{3}}V_{5,m+1} + \frac{4i}{\sqrt{3}}V_{6,m+1} + \frac{i}{\sqrt{3}}vV_{5m} - \frac{2i}{\sqrt{3}}vV_{6m} + \left(\frac{3\sqrt{2}-\sqrt{6}i}{6} + \frac{i+\sqrt{3}}{\sqrt{6}}w + w_1 + 2w_2\right)V_{4m} \\ \qquad + \left(\frac{2i}{\sqrt{3}}w_1 + \frac{4i}{\sqrt{3}}\right)V_{1m}, \\ V_{6,mx} = -\frac{4i}{\sqrt{3}}V_{5,m+1} + \frac{2i}{\sqrt{3}}V_{6,m+1} + \frac{2i}{\sqrt{3}}vV_{5m} - \frac{i}{\sqrt{3}}vV_{6m} + \left(\frac{3\sqrt{2}+\sqrt{6}i}{6} + \frac{i-\sqrt{3}}{\sqrt{6}}w + 2w_1 + w_2\right)V_{4m} \\ \qquad + \left(-\frac{4i}{\sqrt{3}}w_1 + \frac{2i}{\sqrt{3}}w_2\right)V_{1m}. \end{cases} \tag{49}$$

Setting $V_{1,0} = \alpha, V_{2,0} = V_{3,0} = V_{5,0} = V_{6,0} = 0$, then from (49) we have

$V_{4,0} = 0, V_{5,1} = -\alpha w_1, V_{6,1} = -\alpha w_2, V_{4,1} = 0, V_{6,2} = \frac{i\alpha}{\sqrt{3}}w_{1x} - \frac{i\alpha}{2\sqrt{3}}w_{2x} + \frac{\alpha}{3}\left(2vw_1 - \frac{5}{2}vw_2\right),$

$V_{5,2} = \frac{i\alpha}{2\sqrt{3}}w_{1x} - \frac{i\alpha}{\sqrt{3}}w_{2x} + \frac{\alpha}{3}\left(\frac{5}{2}vw_1 - 2vw_2\right),$

$V_{4,2} = \frac{\sqrt{3}-3i}{2\sqrt{6}}\alpha w_1 + \frac{3i+\sqrt{3}}{2\sqrt{6}}\alpha w_2 - \frac{1+\sqrt{3}i}{2\sqrt{2}}\alpha ww_1 + \frac{\sqrt{3}i-1}{2\sqrt{2}}\alpha ww_2 - \frac{\sqrt{3}i}{2}\alpha w_1^2 - \frac{\sqrt{3}i}{2}w_2^2 + \frac{\sqrt{3}i}{2}\alpha w_1 w_2,$

$V_{in}(1 \le i, j \le 3)$ are the same with the previous those in (45).
Defining

$$V_+^{(n)} = \sum_{m=0}^{n}\left(\sum_{i=1}^{3} V_{im}e_i(n-m) + \sum_{j=4}^{6} V_{jm}h_{j-3}(n-m)\right) = \lambda^n V - V_-^{(n)},$$

we obtain that

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = 2V_{2,n+1}e_2(0) - 2V_{3,n+1}e_3(0) + \frac{2i}{\sqrt{3}}(-V_{5,n+1} + 2V_{6,n+1})h_2(0) + \frac{2i}{\sqrt{3}}(-2V_{5,n+1} + V_{6,n+1})h_3(0).$$

If setting $V^{(n)} = V_+^{(n)} + V_{2,n+1}e_1(0)$, we have

$$-V_x^{(n)} + [U, V^{(n)}] = -V_{2,n+1x}e_1(0) - (2wV_{2,n+1} + 2V_{3,n+1})e_3(0) + \frac{2i}{\sqrt{3}}[-V_{5,n+1} + 2V_{6,n+1} + (2w_2 - w_1)V_{2,n+1}]h_2(0) + \frac{2i}{\sqrt{3}}[-2V_{5,n+1} + V_{6,n+1} + (w_2 - 2w_1)V_{2,n+1}]h_3(0).$$

Hence, we get a nonlinear integrable coupling of the BK hierarchy from the zero curvature equation:

$$u_{t_n}=\begin{pmatrix}v\\w\\w_1\\w_2\end{pmatrix}_{t_n}=\begin{pmatrix}2V_{2,n+1x}\\-2V_{1,n+1x}\\-\frac{2i}{\sqrt{3}}[-V_{5,n+1}+2V_{6,n+1}+(2w_2-w_1)V_{2,n+1}]\\-\frac{2i}{\sqrt{3}}[-2V_{5,n+1}+V_{6,n+1}+(w_2-2w_1)V_{2,n+1}]\end{pmatrix}. \tag{50}$$

When setting $w_1=w_2=0$, (50) reduces to the BK hierarchy. When taking $n=2,\alpha=-1,t_2=t$, we get a nonlinear integrable coupling of the BK system (40):

$$\begin{cases}v_t=-\frac{1}{2}v_{xx}+vv_x+w_x,\\ w_t=\left(vw+\frac{1}{2}w_x\right)_x,\\ w_{1t}=-\frac{2i}{\sqrt{3}}\Big[-\frac{1}{4}w_{1xx}+\frac{1}{2}w_{2xx}+\frac{i}{2\sqrt{3}}\left(\frac{5}{2}vw_1-2vw_2\right)_x-\frac{1}{4\sqrt{3}}w_{1x}+\frac{1}{2\sqrt{3}}w_{2x}-\frac{i}{3}vw_2\\ \quad+\left(\frac{5i}{12}-\frac{i}{\sqrt{3}}\right)vw_1+\frac{i}{2\sqrt{3}}vw_{2x}-\frac{1}{3}\left(2vw_1-\frac{5}{2}vw_2\right)\\ \quad+\frac{\sqrt{3}i}{2}\left(\frac{3\sqrt{2}-\sqrt{6}i}{6}+\frac{i+\sqrt{3}}{\sqrt{6}}w+w_1+2w_2\right)V_{4,2}|_{\alpha=-1}-\frac{1}{2}w((w_1-2w_2)]\\ \quad+(2w_2-w_1)\left(-\frac{1}{4}v_x+\frac{1}{4}v^2+\frac{1}{2}w\right),\\ w_{2t}=-\frac{2i}{\sqrt{3}}\Big[-\frac{1}{2}w_{1xx}+\frac{1}{4}w_{2xx}+\frac{i}{2\sqrt{3}}\left(2vw_1-\frac{5}{2}vw_x\right)_x+\frac{i}{2\sqrt{3}}vw_{1x}-\frac{3i}{4\sqrt{3}}vw_{2x}+\frac{1}{2}v^2w_1\\ \quad-\frac{1}{4}v^2w_2-\frac{\sqrt{3}i}{2}\left(\frac{3\sqrt{2}+\sqrt{6}i}{6}+\frac{i-\sqrt{3}}{\sqrt{6}}w+2w_1+w_2\right)V_{4,2}|_{\alpha=-1}+\frac{1}{2}w(w_2-2w_1)\Big]\\ \quad+(w_2-2w_1)\left(-\frac{1}{4}v_x+\frac{1}{4}v^2+\frac{1}{2}w\right).\end{cases} \tag{51}$$

3.3 A Nonlinear Integrable Coupling of the KN Hierarchy

In the section, we shall use a loop algebra of the Lie algebra Q to introduce a Lax pair, from which a nonlinear integrable coupling of the KN hierarchy is obtained. Set

$$\tilde{Q}=\text{span}\{e_1(n),e_2(n),e_3(n),p_1(n),p_2(n),p_3(n)\},$$

where

$$e_1(n)=e_1\lambda^{2n},e_2(n)=e_2\lambda^{2n+1},e_3(n)=e_3\lambda^{2n+1},$$
$$p_1(n)=p_1\lambda^{2n},p_2(n)=p_2\lambda^{2n+1},p_3(n)=p_3\lambda^{2n+1},n\in Z.$$

It is easy to verify that the following commutative relations hold:

$$\begin{aligned}
&[e_1(m),e_2(n)]=e_3(m+n),[e_1(m),e_3(n)]=e_2(m+n),[e_2(m),e_3(n)]=-e_1(m+n+1),\\
&[p_1(m),p_2(n)]=2p_2(m+n),[p_1(m),p_3(n)]=-2p_3(m+n),[p_2(m),p_3(n)]=4p_1(m+n),\\
&[e_1(m),p_1(n)]=0,[e_1(m),p_2(n)]=p_2(m+n),[e_1(m),p_3(n)]=-p_3(m+n),\\
&[e_2(m),p_1(n)]=\frac{1}{2}(p_3(m+n)-p_2(m+n)),[e_2(m),p_2(n)]=-p_1(m+n+1),\\
&[e_2(m),p_3(n)]=p_1(m+n+1),[e_3(m),p_1(n)]=-\frac{1}{2}(p_2(m+n)+p_3(m+n)),\\
&[e_3(m),p_2(n)]=p_1(m+n+1),[e_3(m),p_3(n)]=p_1(m+n+1),m,n\in Z.
\end{aligned}$$

By employing the loop algebra $\tilde{Q}$, we consider a Lax pair

$$\begin{cases} U=e_1(1)+qe_2(0)+re_3(0)+u_1p_2(0)+u_2p_3(0),\\ V=\sum_{m\geq 0}\left(\sum_{i=1}^{3}V_{im}e_i(-m)+\sum_{j=4}^{6}V_{jm}p_{j-3}(-m)\right).\end{cases} \tag{52}$$

The stationary zero curvature equation of the compatibility condition of (52) leads to the following

$$\begin{cases} V_{1,mx}=-qV_{3,m+1}+rV_{2,m+1}=-qV_{2,mx}+rV_{3,mx},\\
V_{2,mx}=V_{3,m+1}-rV_{1m},\\
V_{3,mx}=V_{2,m+1}-qV_{1m},\\
V_{4,mx}=(q+r+4u_1)V_{6,m+1}+(-q+r-4u_2)V_{5,m+1}+(u_1-u_2)V_{2,m+1}-(u_1+u_2)V_{3,m+1}\\
\qquad\quad =-(q+r+4u_1)V_{6,mx}+(-q+r-4u_2)V_{5,mx}+(u_1-u_2)V_{3,mx}-(u_1+u_2)V_{2,mx},\\
V_{5,mx}=V_{5,m+1}-\left(\frac{1}{2}q+\frac{1}{2}r+2u_1\right)V_{4m}-u_1V_{1m},\\
V_{6,mx}=-V_{6,m+1}+\left(\frac{1}{2}q-\frac{1}{2}r+2u_2\right)V_{4m}+u_2V_{1m}.\end{cases} \tag{53}$$

Setting

$$V_+^{(n)}=\sum_{m=0}^{n}(\sum_{i=1}^{3}V_{im}e_i(-m)+\sum_{j=4}^{6}V_{jm}p_{j-3}(-m))\lambda^{2n}=\lambda^{2n}V-V_-^{(n)},$$

then we obtain that

$$\begin{aligned}
-V_{+x}^{(n)}+[U,V_+^{(n)}]=&(-rV_{2,n+1}+qV_{3,n+1})e_1(0)-V_{3,n+1}e_2(0)-V_{2,n+1}e_3(0)-V_{5,n+1}p_2(0)+V_{6,n+1}p_3(0)\\
&+[(q-r+4u_2)V_{5,n+1}-(q+r+4u_1)V_{6,n+1}+(u_2-u_1)V_{2,n+1}+(u_1+u_2)V_{3,n+1}]p_1(0)\\
=&-V_{1,nx}e_1(0)-V_{3,n+1}e_2(0)-V_{2,n+1}e_3(0)-V_{5,n+1}p_2(0)+V_{6,n+1}p_3(0)-V_{4,nx}p_1(0).
\end{aligned}$$

Choosing a modified term of $V_+^{(n)}$ as that $V^{(n)}=V_+^{(n)}-V_{1,n}e_1(0)-V_{4,n}p_1(0)$, a direct calculation yields that

$$\begin{aligned}
-V_x^{(n)}+[U,V^{(n)}]=&(-V_{3,n+1}+rV_{1n})e_2(0)+(-V_{2,n+1}+qV_{1n})e_3(0)+\Big(-V_{5,n+1}+\frac{1}{2}qV_{4n}+\frac{1}{2}rV_{4n}+u_1V_{1n}\\
&+2u_1V_{4n}\Big)p_2(0)+\left(V_{6,n+1}+\left(-\frac{1}{2}q+\frac{1}{2}r-2u_2\right)V_{4n}-u_2V_{1n}\right)p_3(0)\\
=&-V_{2,nx}e_2(0)-V_{3,nx}e_3(0)-V_{5,nx}p_2(0)-V_{6,nx}p_3(0).
\end{aligned}$$

Therefore, the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0$$

admits a hierarchy of equations of evolution type

$$u_{t_n} = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} V_{2,nx} \\ V_{3,nx} \\ V_{5,nx} \\ V_{6,nx} \end{pmatrix}. \tag{54}$$

A recurrence operator of (54) is

$$L = \begin{pmatrix} -q\partial^{-1}q\partial & \partial + q\partial^{-1}r\partial & 0 & 0 \\ \partial - r\partial^{-1}q\partial & r\partial^{-1}r\partial & 0 & 0 \\ A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \end{pmatrix},$$

which satisfies that

$$\begin{pmatrix} V_{2,n+1} \\ V_{3,n+1} \\ V_{5,n+1} \\ V_{6,n+1} \end{pmatrix} = L \begin{pmatrix} V_{2,n} \\ V_{3,n} \\ V_{5,n} \\ V_{6,n} \end{pmatrix},$$

where

$$\begin{aligned}
A_1 &= -u\partial^{-1}q\partial - \frac{1}{2}(q + r + 4u_1)\partial^{-1}(u_1 + u_2)\partial, \\
A_2 &= u_1\partial^{-1}r\partial + \frac{1}{2}(q + r + 4u_1)\partial^{-1}(u_1 - u_2)\partial, \\
A_3 &= \partial - \frac{1}{2}(q + r + 4u_1)\partial^{-1}(q + r - 4u_2)\partial, \\
A_4 &= -\frac{1}{2}(q + r + 4u_1)\partial^{-1}(q + r + 4u_1)\partial, \\
B_1 &= -u_2\partial^{-1}q\partial - \frac{1}{2}(q - r + 4u_2)\partial^{-1}(u_1 + u_2)\partial, \\
B_2 &= u_2\partial^{-1}r\partial + \frac{1}{2}(q - r + 4u_2)\partial^{-1}(u_1 - u_2)\partial, \\
B_3 &= -\frac{1}{2}(q - r + 4u_2)\partial^{-1}(q + r - 4u_2)\partial, \\
B_4 &= -\partial - \frac{1}{2}(q - r + 4u_2)\partial^{-1}(q + r + 4u_1)\partial..
\end{aligned}$$

Letting $V_{1,0} = \alpha, V_{2,0} = V_{3,0} = V_{5,0} = V_{6,0} = 0$, then we have from (53)

$$V_{3,1} = \alpha r, V_{2,1} = \alpha q, V_{1,0} = 0, V_{5,1} = \alpha u_1, V_{6,1} = \alpha u_2, V_{4,0} = 0, V_{1,1} = \frac{\alpha}{2}(r^2 - q^2),$$
$$V_{4,1} = \alpha(-qu_2 - qu_1 + ru_1 - ru_2 - 4u_1u_2), V_{2,2} = \alpha r_x + \frac{\alpha}{2}q(r^2 - q^2),$$
$$V_{3,2} = \alpha q_x + \frac{\alpha}{2}r(r^2 - q^2),$$
$$V_{5,2} = \alpha u_{1x} + \frac{\alpha}{2}(q + r + 4u_1)(-qu_2 - qu_1 + ru_1 - ru_2 - 4u_1u_2) + \frac{1}{2}u_1(r^2 - q^2),$$
$$V_{6,2} = -\alpha u_{2x} + \frac{\alpha}{2}(q - r + 4u_2)(-qu_2 - qu_1 + ru_1 - ru_2 - 4u_1u_2) + \frac{\alpha}{2}u_2(r^2 - q^2), \ldots$$

When taking $n = 2$, the hierarchy (54) reduces to a nonlinear evolution equation

$$\begin{cases} q_{t_2} = \alpha r_{xx} + \frac{\alpha}{2}(qr^2 - q^3)_x, \\ r_{t_2} = \alpha q_{xx} + \frac{1}{2}(r^3 - rq^2)_x, \\ u_{1,t_2} = \alpha u_{1xx} + \frac{\alpha}{2}[(q + r + 4u_1)(r(u_1 - u_2) - q(u_1 + u_2) - 4u_1u_2)]_x + \frac{\alpha}{2}(u_1(r^2 - q^2))_x, \\ u_{2,t_2} = -\alpha u_{2xx} + \frac{\alpha}{2}((q - r + 4u_2)(-q(u_1 + u_2) + r(u_1 - u_2) - 4u_1u_2))_x + \frac{\alpha}{2}(u_2(r^2 - q^2))_x. \end{cases} \tag{55}$$

Setting $u_1 = u_2 = 0$, (55) reduces to a nonlinear coupled KN equation

$$\begin{cases} q_{t_2} = \alpha r_{xx} + \frac{\alpha}{2}(qr^2 - q^3)_x, \\ r_{t_2} = \alpha q_{xx} + \frac{1}{2}(r^3 - rq^2)_x. \end{cases} \tag{56}$$

Obviously, (55) is a nonlinear integrable coupling of (56). Thus, (54) is a hierarchy of nonlinear integrable couplings of the KN hierarchy.

4 Vector representations of the Lie algebras G and Q as well as some Hamiltonian structures of nonlinear integrable couplings

We find that it is difficult to express the Lie algebras G and Q as square matric representations. Therefore, we consider their vector representations so that we can deduce Hamiltonian structures of the obtained nonlinear integrable couplings by using the variational identities [11,18]. For $\forall a, b \in G$, we can express them as

$$a = \sum_{i=1}^{3} a_i e_i + \sum_{j=4}^{6} a_j f_{j-3}, b = \sum_{i=1}^{3} b_i e_i + \sum_{j=4}^{6} b_j f_{j-3} \in G,$$

and we have

$$\begin{aligned} [a,b] = &(a_2b_3 - a_3b_2)e_1 + (2a_1b_2 - 2a_2b_1)e_2 + (2a_3b_1 - 2a_1b_3)e_3 \\ &+(a_2b_5 - a_5b_2 + a_5b_3 - a_3b_5 + a_6b_2 - a_2b_6 + a_6b_3 - a_3b_6 + a_6b_5 - a_5b_6) f_1 \\ &+(2a_1b_6 - 2a_6b_1 + a_4b_6 - a_6b_4 + a_3b_4 - a_4b_3 + a_4b_2 - a_2b_4) f_2 \\ &+(2a_1b_5 - 2a_5b_1 + a_4b_5 - a_5b_4 + a_4b_2 - a_2b_4 + a_4b_3 - a_3b_4) f_3. \end{aligned} \tag{57}$$

If noting $a=(a_1,\dots,a_6)^T, b=(b_1,\dots,b_6)^T$, then we can write (57) as

$$[a,b]=\begin{pmatrix}[A_1,B_1]_1\\ [a,b]_1\end{pmatrix}, \tag{58}$$

where

$$[a,b]_1=\begin{pmatrix}a_2b_5-a_5b_2+a_5b_3-a_3b_5+a_6b_2-a_2b_6+a_6b_3-a_3b_6+a_6b_5-a_5b_6\\ 2a_1b_6-2a_6b_1+a_4b_6-a_6b_4+a_3b_4-a_4b_3+a_4b_2-a_2b_4\\ 2a_1b_5-2a_5b_1+a_4b_5-a_5b_4+a_4b_2-a_2b_4+a_4b_3-a_3b_4\end{pmatrix}.$$

It can be verified that the vector space $R^6=\{a=(a_1,\dots,a_6)^T\}$ is a Lie algebra if equipped with (58). Hence, the Lie algebra G is isomorphic to the Lie algebra R^6. In order to apply the variational identity to deduce Hamiltonian structures of nonlinear integrable couplings, we need to get a constant symmetric matrix F which satisfies the matrix equation

$$R(b)F=-(R(b)F)^T, F^T=F, \tag{59}$$

where $R(b)$ comes from rewriting (58) as the following form

$$[a,b]=a^TR_1(b). \tag{60}$$

It is easy to see that

$$R_1(b)=\begin{pmatrix}0&2b_2&-2b_3&0&2b_6&2b_5\\ b_3&-2b_1&0&b_5-b_6&-b_4&-b_4\\ -b_2&0&2b_1&-b_5-b_6&b_4&-b_4\\ 0&0&0&0&b_6-b_3+b_2&b_5+b_2+b_3\\ 0&0&0&-b_2+b_3-b_6&0&-2b_1-b_4\\ 0&0&0&b_2+b_3+b_5&-2b_1-b_4&0\end{pmatrix}.$$

Solving (59) yields

$$F_1=\begin{pmatrix}2&0&0&2&0&0\\ 0&0&1&0&1&-1\\ 0&1&0&0&1&1\\ 2&0&0&1&0&0\\ 0&1&1&0&1&0\\ 0&-1&1&0&0&-1\end{pmatrix}. \tag{61}$$

Similarly, for

$$a=\sum_{i=1}^{3}a_ie_i+\sum_{j=4}^{6}a_jp_{j-3}, b=\sum_{i=1}^{3}b_ie_i+\sum_{j=4}^{6}b_jp_{j-3}\in Q,$$

define that

$$[a,b]=\begin{pmatrix}[A_1,B_1]_2\\ [a,b]_2\end{pmatrix}=a^TR_2(b), \tag{62}$$

where

$$[a,b]_2 = \begin{pmatrix} a_2b_6 - a_6b_2 + a_3b_6 - a_6b_3 + a_5b_2 - a_2b_5 + 4a_5b_6 - 4a_6b_5 + a_3b_5 - a_5b_3 \\ a_1b_5 - a_5b_1 + \frac{1}{2}a_4b_3 - \frac{1}{2}a_3b_4 + \frac{1}{2}a_4b_2 - \frac{1}{2}a_2b_4 + 2a_4b_5 - 2a_5b_4 \\ a_6b_1 - a_1b_6 + \frac{1}{2}a_2b_4 - \frac{1}{2}a_4b_2 + \frac{1}{2}a_4b_3 - \frac{1}{2}a_3b_4 + 2a_6b_4 - 2a_4b_6 \end{pmatrix}.$$

Solving Eq.(59) for F gives

$$F_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 4 & 0 \end{pmatrix}. \tag{63}$$

In what follows, we construct Hamiltonian structures of the nonlinear integrable couplings of (30) and (54) by using the variational identity [11,18]. At first, we make use of F_1 and F_2 to define two linear functionals.
For $\forall a = (a_1,\dots,a_6)^T, b = (b_1,\dots,b_6)^T \in R^6$, using F_1 we define

$$\begin{aligned}\{a,b\} = {} & 2a_1b_1 + 2a_4b_1 + 2a_1b_4 + a_3b_2 + a_2b_3 + a_5b_2 \\ & + a_2b_5 + a_5b_3 + a_3b_5 + a_6b_3 + a_3b_6 - a_2b_6 - a_6b_2 \\ & + a_4b_4 + a_5b_5 - a_6b_6.\end{aligned} \tag{64}$$

Using F_2, we define that

$$\begin{aligned}\{a,b\} = {} & a_1b_1 + a_4b_1 + a_1b_4 + a_2b_2 + a_5b_2 + a_2b_5 + a_6b_2 + a_2b_6 \\ & - a_3b_3 - a_5b_3 - a_3b_5 + a_6b_3 + a_3b_6 + 2a_4b_4 + 4a_6b_5 + 4a_5b_6.\end{aligned} \tag{65}$$

Rewrite the Lax pair (28) as follows

$$U = (\lambda, q, r, 0, u_1, u_2)^T, V = (V_1,\dots,V_6)^T,$$

where $V_1 = \sum_{m\geq 0} V_{1m}\lambda^{-m},\dots$ Using (64), we obtain

$$\begin{aligned}&\left\{V, \frac{\partial U}{\partial q}\right\} = V_3 + V_5 - V_6, \left\{V, \frac{\partial U}{\partial r}\right\} = V_2 + V_5 + V_6, \left\{V, \frac{\partial U}{\partial u_1}\right\} = V_2 + V_3 + V_5, \\ &\left\{V, \frac{\partial U}{\partial u_2}\right\} = -V_2 + V_3 - V_6, \left\{V, \frac{\partial U}{\partial \lambda}\right\} = 2V_1 + 2V_4.\end{aligned}$$

Substituting the above into the variational identity gives rise to

$$\frac{\delta}{\delta}\int^x (2V_1 + 2V_4)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} V_3 + V_5 - V_6 \\ V_2 + V_5 + V_6 \\ V_2 + V_3 + V_5 \\ -V_2 + V_3 - V_6 \end{pmatrix}. \tag{66}$$

Comparing the coefficients of λ^{-n-2} on both sides in (66) yields

$$\frac{\delta}{\delta u}\int^{x}(2V_{1,n+2}+2V_{4,n+2})dx=(-n-1+\gamma)\begin{pmatrix} V_{3,n+1}+V_{5,n+1}-V_{6,n+1}\\ V_{2,n+1}+V_{5,n+1}+V_{6,n+1}\\ V_{2,n+1}+V_{3,n+1}+V_{5,n+1}\\ -V_{2,n+1}+V_{3,n+1}-V_{6,n+1}\end{pmatrix}.$$

It is easy to verify that $\gamma=0$. Thus, we have

$$\begin{pmatrix} V_{3,n+1}+V_{5,n+1}-V_{6,n+1}\\ V_{2,n+1}+V_{5,n+1}+V_{6,n+1}\\ V_{2,n+1}+V_{3,n+1}+V_{5,n+1}\\ -V_{2,n+1}+V_{3,n+1}-V_{6,n+1}\end{pmatrix}=\frac{\delta}{\delta u}\left(-\frac{\int^{x}(2V_{1,n+2}+2V_{4,n+2})dx}{n+1}\right)=:\frac{\delta H_{n+1}}{\delta u},$$

where

$$H_{n+1}=-\frac{\displaystyle\int^{x}(2V_{1,n+2}+2V_{4,n+2})dx}{n+1}.$$

Therefore, (30) can be written as a Hamiltonian form

$$u_{t_n}=\begin{pmatrix} q\\ r\\ u_1\\ u_2\end{pmatrix}_{t_n}=\begin{pmatrix} 0 & -2 & 2 & -2\\ 2 & 0 & -2 & -2\\ -2 & 2 & 0 & 2\\ 2 & 2 & -2 & 0\end{pmatrix}\begin{pmatrix} V_{3,n+1}+V_{5,n+1}-V_{6,n+1}\\ V_{2,n+1}+V_{5,n+1}+V_{6,n+1}\\ V_{2,n+1}+V_{3,n+1}+V_{5,n+1}\\ -V_{2,n+1}+V_{3,n+1}-V_{6,n+1}\end{pmatrix}=J\frac{\delta H_{n+1}}{\delta u}, \tag{67}$$

where J is obviously Hamiltonian. From (29), we can obtain $V_{1,4}$ and $V_{4,4}$. Thus, we can obtain the Hamiltonian structure of the nonlinear integrable coupling (31) of the nonlinear Schrödinger equation if substituting $V_{1,4}, V_{4,4}$ into (67).

In the following, we deduce the Hamiltonian structure of (54). Rewrite the Lax pair (52) as

$$\begin{cases} U=(\lambda^2,q\lambda,r\lambda,0,u_1\lambda,u_2\lambda)^T,\\ V=(V_1,V_2\lambda,V_3\lambda,V_4,V_5\lambda,V_6\lambda)^T,\end{cases} \tag{68}$$

where $V_1=\sum\limits_{m\geq 0}V_{1m}\lambda^{-2m},\ldots$ Using the linear functional (65) along with (68), we have

$$\begin{aligned}
&\left\{V,\frac{\partial U}{\partial q}\right\}=(V_2+V_5+V_6)\lambda^2,\{V,\frac{\partial U}{\partial r}\}=(-V_3-V_5+V_6)\lambda^2,\left\{V,\frac{\partial U}{\partial u_1}\right\}=(V_2-V_3+4V_6)\lambda^2,\\
&\left\{V,\frac{\partial U}{\partial u_2}\right\}=(V_2+V_3+4V_5)\lambda^2,\\
&\left\{V,\frac{\partial U}{\partial \lambda}\right\}=[2V_1+(q+u_1+u_2)V_2+(-r-u_1+u_2)V_3+2V_4+(-r+4u_2+q)V_5+(q+r+4u_1)V_6]\lambda.
\end{aligned}$$

Substituting them into the variational identity gives

$$\begin{aligned}&\frac{\delta}{\delta u}\int^x\{(2V_1+(q+u_1+u_2)V_2+(-r-u_1+u_2)V_3+2V_4\\ &+(-r+4u_2+q)V_5+(q+r+4u_1)V_6)\}\lambda\,dx\\ &=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}(V_2+V_5+V_6)\lambda^2\\ (-V_3-V_5+V_6)\lambda^2\\ (V_2-V_3+4V_6)\lambda^2\\ (V_2+v_3+4V_5)\lambda^2\end{pmatrix}.\end{aligned}\tag{69}$$

Comparing the coefficients of λ^{-2n+1} on both sides of (69) yields

$$\begin{aligned}&\frac{\delta}{\delta u}\int^x(2V_{1n}+(q+u_1+u_2)V_{2n})dx=(-r-u_1+u_2)V_{3n}+2V_{4n}\\ &+(-r+4u_2+q)V_{5n}+(q+r+4u_1)V_{6n}dx=(-2n+2+\gamma)\begin{pmatrix}V_{2n}+V_{5n}+V_{6n}\\ -V_{3n}-V_{5n}+V_{6n}\\ V_{2n}-V_{3n}+4V_{6n}\\ V_{2n}+V_{3n}+4V_{5n}\end{pmatrix}.\end{aligned}$$

Utilizing the initial values in (53), we have $\gamma=2$. Hence, (54) can be written as the Hamiltonian form

$$u_{t_n}=\begin{pmatrix}q\\ r\\ u_1\\ u_2\end{pmatrix}_{t_n}=\begin{pmatrix}2\partial & 0 & -\frac{\partial}{2} & -\frac{\partial}{2}\\ 0 & -2\partial & \frac{\partial}{2} & -\frac{\partial}{2}\\ -\frac{\partial}{2} & \frac{\partial}{2} & 0 & \frac{\partial}{2}\\ -\frac{\partial}{2} & -\frac{\partial}{2} & \frac{\partial}{2} & 0\end{pmatrix}\begin{pmatrix}V_{2n}+V_{5n}+V_{6n}\\ -V_{3n}-V_{5n}+V_{6n}\\ V_{2n}-V_{3n}+4V_{6n}\\ V_{2n}+V_{3n}+4V_{5n}\end{pmatrix}=J\frac{\delta H_n}{\delta u},\tag{70}$$

where

$$H_n=\frac{1}{4-2n}\int^x(2V_{1n}+(q+u_1u_2)V_{2n}+(-r-u_1+u_2)V_{3n}+2V_{4n}+(q-r+4u_2)V_{5n}+(q+r+4u_1)V_{6n})dx,$$

and J is a Hamiltonian operator. As for Hamiltonian structures of the other obtained nonlinear integrable couplings, we can make a similar analysis but we omit here.

5 Discussions

We have constructed three Lie algebras of semi-direct sum form: $\overline{G}=G\Subset G_c$, where G and G_c are all simple Lie sub-algebras and satisfy that

$$[G,G]\subset G,[G_c,G_c]\subset G_c,[G,G_c]\subset G_c.\tag{71}$$

We require that the commutator of G is different from that of G_c to generate non-trivial nonlinear integrable couplings. Various nonlinear integrable couplings of the AKNS hierarchy and the BK hierarchy were obtained from different loop algebras. We also remark that the problem of how to apply the approach in [13-15] to nonlinear bi-integrable couplings of the AKNS hierarchy and the BK hierarchy deserves future investigation.

In order to generate more interesting nonlinear integrable couplings, it should be good to study different isospectral matrix spectral problems. Ablowitz et al. [24] proposed a simple expression of the self-dual Yang-Mills equation by the following isospectral problem

$$\begin{cases} \partial_\sigma + \xi \partial_{\tilde{\tau}} \psi = (A_\sigma + \xi A_{\tilde{\tau}})\psi, \\ (\partial_\tau - \xi \partial_{\tilde{\sigma}})\psi = (A_\tau - \xi A_{\tilde{\sigma}})\psi, \end{cases} \tag{72}$$

where $\sigma, \tilde{\sigma}, \tau, \tilde{\tau}$ are all null coordinates, and $A = A_\mu d\mu = A_\sigma d\sigma + A_{\tilde{\sigma}} d\tilde{\sigma} + A_\tau d\tau + A_{\tilde{\tau}} d\tilde{\tau}$. The compatibility condition of (72) leads to

$$\begin{cases} A_{\sigma\tau} - A_{\tau\sigma} + A_\sigma A_\tau - A_\tau A_\sigma = 0, \\ A_{\tau\tilde{\tau}} - A_{\tilde{\sigma}\sigma} + A_{\sigma\tilde{\sigma}} - A_{\tilde{\tau}\tau} + [A_\tau, A_{\tilde{\tau}}] + [A_\sigma, A_{\tilde{\sigma}}] = 0, \\ A_{\tilde{\sigma}\tilde{\tau}} - A_{\tilde{\tau}\tilde{\sigma}} + [A_{\tilde{\sigma}}, A_{\tilde{\tau}}] = 0. \end{cases} \tag{73}$$

From this, some reduced cases can be presented as follows:

(1) Assume that A_X, $X = \sigma, \tau, \tilde{\sigma}, \tilde{\tau}$, are functions of $x = \tilde{\sigma}$ and $t = \tilde{\tau}$ only. Set $A_\sigma = 0, A_\tau = P, A_{\tilde{\sigma}} = Q, A_{\tilde{\tau}} = R$. Then we have

$$\begin{cases} P_t + [P, R] = 0, \\ Q_t - R_x + [Q, R] = 0. \end{cases} \tag{74}$$

(2) Let $x = \sigma + \tilde{\sigma}, y = \tau, t = \tilde{\tau}, A_{\tilde{\sigma}} = 0, A_\sigma = P, A_\tau = R, A_{\tilde{\tau}} = Q$. Then we obtain

$$\begin{cases} P_y - R_x + [P, R] = 0, \\ R_t + P_x - P_y + [R, Q] = 0, \\ Q_x = 0. \end{cases} \tag{75}$$

(3) If A_σ is a function of x and t only, $x = \tau + \tilde{\tau}, t = \tilde{\sigma}, A_\tau = 0, P = A_\sigma, Q = A_{\tilde{\tau}}, R = A_{\tilde{\sigma}}$, then we get [24]

$$\begin{cases} P_x = 0, \\ P_t - Q_x + [P, R] = 0, \\ R_x - Q_t + R, Q] = 0. \end{cases} \tag{76}$$

(4) If A_X, $X = \sigma, \tau, \tilde{\sigma}, \tilde{\tau}$, are functions of $x = \sigma$ and $t = \tau$ only, taking $A_\sigma = U + B\partial_y, A_\tau = V + C\partial_y, \psi = Ge^{-\xi y}$, leads to [24]

$$\begin{cases} \partial_x G = (U + B\partial_y)G, \\ \partial_t G = (V + C\partial_y)G. \end{cases} \tag{77}$$

This gives rise to

$$\begin{cases} \partial_t U - \partial_x V + [U, V] - C\partial_y U + B\partial_y V = 0, \\ [B, V] = [C, U]. \end{cases} \tag{78}$$

We can generate new nonlinear integrable couplings of both (1+1)-dimension and (2+1)-dimension, by using our general construction procedure and combining the above generalized zero curvature equations. It is also interesting for us to see if the resulting integrable couplings possess the linear superposition principle for exponential waves, or more generally, to see if there are linear subspaces of their solution spaces (see [25] for details). These will be one of our future research topics.

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