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An so $(3, \mathbb{R})$ Counterpart of the Dirac Soliton Hierarchy and its Bi-Integrable Couplings

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Abstract We derive a counterpart hierarchy of the Dirac soliton hierarchy from zero curvature equations associated with a matrix spectral problem from so$(3, \mathbb{R})$. Inspired by a special class of non-semisimple loop algebras, we construct a hierarchy of bi-integrable couplings for the counterpart soliton hierarchy. By applying the variational identities which cope with the enlarged Lax pairs, we generate the corresponding Hamiltonian structure for the hierarchy of the resulting bi-integrable couplings. To show Liouville integrability, infinitely many commuting symmetries and conserved densities are presented for the counterpart soliton hierarchy and its hierarchy of bi-integrable couplings.

Keywords Hamiltonian structure · Bi-integrable couplings · Symmetry · Conservation law · Matrix loop algebra

1 Introduction

The theory of integrable couplings [1] presents various perspectives for future investigations on integrable systems, providing fresh insights into existing areas of soliton theory and building up many new soliton hierarchies based on enlarged Lax pairs or zero curvature equations.

Integrable couplings generalize the symmetry problem [2] as well as provide clues towards complete classification of integrable systems [3, 4]. The perturbation equations are specific examples of integrable couplings generalizing the symmetry problem, exhibiting diverse integrable structures that the multiplicity of integrable systems possess [1, 5]. The

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KdV case was extensively studied by perturbation in [6], indeed, which shows the existence of local bi-Hamiltonian structures in $2+1$ dimensions. One of other realizations of integrable couplings is the theory of related Virasoro symmetry algebras or τ-symmetry algebras of integrable systems in soliton theory [3, 4].

Recalling that an arbitrary Lie algebra has a semidirect sum structure of a solvable Lie algebra and a semisimple Lie algebra [7], integrable coupling are associated with semi-direct sums of loop algebras, and therefore, they represent general integrable systems, providing algebraic approaches for classifying multi-component integrable systems. A series of research works have shown different connections between semi-direct sums of Lie algebras and integrable couplings, both discrete and continuous [8–10]. Hamiltonian structures of integrable couplings are normally furnished by the variational identity [4, 11]. The main idea is to search for non-degenerate, symmetric and ad-invariant bilinear forms on the underlying non-semisimple loop algebras.

An integrable coupling of a given system $u_t = K(u)$ is a triangular integrable system of the following form [1]:

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v), \end{cases} \tag{1.1}$$

where v is a new column vector of dependent variables. An integrable coupling contains the given system as a sub-system, and the v_t part corresponds a solvable Lie algebra. Integrable couplings are triangular systems, which are associated with non-semisimple Lie algebras and keep various integrable properties [12] of the original systems. An integrable system of the form

$$\begin{cases} u_t = K(u), \\ v_{1,t} = S_1(u, v_1), \\ v_{2,t} = S_2(u, v_1, v_2), \end{cases} \tag{1.2}$$

is called a bi-integrable coupling of the given system [13].

We shall use the simisimple real Lie algebra of the special orthogonal group, so$(3, \mathbb{R})$, the Lie algebra of 3×3 trace-free, skew-symmetric real matrices. It has the basis

$$e_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.3}$$

with which, the structure equations of so$(3, \mathbb{R})$ read

$$[e_1, e_2] = e_3, \ [e_2, e_3] = e_1, \ [e_3, e_1] = e_2.$$

Based on so$(3, \mathbb{R})$, we shall introduce a new matrix spectral problem and generate a counterpart of the Dirac soliton hierarchy. Moreover, we shall use a class of non-semisimple loop algebras, presented in [14], to generate bi-integrable couplings for the counterpart soliton hierarchy.

This paper proceeds as follows. In Section 2, we would like to present an so$(3, \mathbb{R})$ counterpart of the Dirac soliton hierarchy. In Section 3, we shall construct Hamiltonian bi-integrable couplings through semi-direct sums of Lie algebras. Hamiltonian structures of the resulting bi-integrable couplings will be furnished by applying the variational identity. The last section is devoted to a few concluding remarks, where we pose a few open questions as well.

2 An so(3, $\mathbb{R}$) Counterpart of the Dirac Soliton Hierarchy

2.1 The Dirac Soliton Hierarchy

The sl(2, $\mathbb{R}$) matrix spectral problem of the Dirac soliton hierarchy reads

$$\phi_x = U\phi, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad U = U(u,\lambda) = \begin{bmatrix} p & \lambda + q \\ -\lambda + q & -p \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{2.1}$$

where λ is the spectral parameter. Suppose that a solution to the stationary zero curvature equation $W_x = [U, W]$ can be written as

$$W = \begin{bmatrix} c & a+b \\ a-b & -c \end{bmatrix} = \sum_{i=0}^{\infty} W_{0,i}\lambda^{-i} = \sum_{i=0}^{\infty} \begin{bmatrix} c_i & a_i + b_i \\ a_i - b_i & -c_i \end{bmatrix} \lambda^{-i}, \tag{2.2}$$

where the initial values are taken as

$$a_0 = 0, \quad b_0 = -1, \quad c_0 = 0, \tag{2.3}$$

and

$$\begin{cases} a_{i+1} = qb_i + c_{i,x}, \\ c_{i+1} = pb_i - a_{i,x}, \\ b_{i+1} = 2pa_{i+1} - 2qc_{i+1}, \end{cases} \quad i \geq 0. \tag{2.4}$$

Introduce a sequence of Lax operators

$$V^{[m]} = (\lambda^m W)_+ = \sum_{i=0}^{m} W_{0,i}\lambda^{m-i}, \quad m \geq 0, \tag{2.5}$$

and then, the zero curvature equations (see also [15]):

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \tag{2.6}$$

leads to the isospectral ($\lambda_{t_m} = 0$) Dirac soliton hierarchy:

$$u_{t_m} = K_m = \begin{bmatrix} 2a_{m+1} \\ -2c_{m+1} \end{bmatrix} = \Phi^m K_0 = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad K_0 = \begin{bmatrix} 2q \\ -2p \end{bmatrix}, \quad m \geq 0, \tag{2.7}$$

where the Hamiltonian operator J, the hereditary recursion operator Φ and the Hamiltonian functionals $\mathcal{H}_m$ are defined by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \Phi = \begin{bmatrix} 2q\partial^{-1}p & -\frac{1}{2}\partial + 2q\partial^{-1}q \\ \frac{1}{2}\partial - 2p\partial^{-1}p & -2p\partial^{-1}q \end{bmatrix}, \ \mathcal{H}_m = \int \left(-\frac{2b_{m+2}}{m+1} \right) dx. \tag{2.8}$$

The inverse scattering problem, the binary nonlinearization of Lax pairs and the τ-symmetry algebra were studied for the Dirac soliton hierarchy (2.7) in [16–18], respectively.

2.2 A Counterpart Associated with so(3, $\mathbb{R}$)

To drive a counterpart of the Dirac soliton hierarchy (2.1), we introduce a spatial spectral problem associated with so(3, $\mathbb{R}$):

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{2.9a}$$

where the spectral matrix U is chosen as the same linear combination of basis vectors as in the sl(2, $\mathbb{R}$) case:

$$U=U(u,\lambda)=pe_1+(\lambda+q)e_2+(-\lambda+q)e_3=\begin{bmatrix}0 & \lambda-q & -p\\ -\lambda+q & 0 & -\lambda-q\\ p & \lambda+q & 0\end{bmatrix}. \quad (2.10)$$

As normal, we can take a solution

$$W=W(u,\lambda)=\sum_{i=0}^{\infty}W_{0,i}\lambda^{-i},\quad W_{0,i}\in \mathrm{so}(3,\mathbb{R}),\quad i\geq 0, \quad (2.11)$$

to the stationary zero curvature equation

$$W_x=[U,W]. \quad (2.12)$$

If we choose a special form of W as

$$W=ce_1+(a+b)e_2+(a-b)e_3=\begin{bmatrix}0 & -a+b & -c\\ a-b & 0 & -a-b\\ c & a+b & 0\end{bmatrix}, \quad (2.13)$$

direct calculations show that (2.12) is equivalent to

$$\begin{cases} a_x=pb-\lambda c,\\ b_x=-pa+qc,\\ c_x=2\lambda a-2qb.\end{cases} \quad (2.14)$$

Further letting

$$a=\sum_{i=0}^{\infty}a_i\lambda^{-i},\quad b=\sum_{i=0}^{\infty}b_i\lambda^{-i},\quad c=\sum_{i=0}^{\infty}c_i\lambda^{-i}, \quad (2.15)$$

we find that the system (2.14) leads equivalently to

$$\begin{cases} a_{i+1}=\frac{1}{2}c_{i,x}+qb_i,\\ c_{i+1}=-a_{i,x}+pb_i,\\ b_{i+1,x}=qc_{i+1}-pa_{i+1},\end{cases}\quad i\geq 0, \quad (2.16)$$

upon taking the initial values

$$a_0=0,\quad b_0=-1,\quad c_0=0. \quad (2.17)$$

Therefore, with the constants of integration being chosen as zero, a series of differential polynomial functions in u with respect to x can be uniquely computed as follows:

$$\begin{aligned}
&a_1=-q,\quad c_1=-p,\quad b_1=0;\\
&a_2=-\tfrac{1}{2}p_x,\quad c_2=q_x,\quad b_2=\tfrac{1}{2}q^2+\tfrac{1}{4}p^2;\\
&a_3=\tfrac{1}{2}q_{xx}+\tfrac{1}{2}q^3+\tfrac{1}{4}p^2q,\quad c_3=\tfrac{1}{2}p_{xx}+\tfrac{1}{4}p^3+\tfrac{1}{2}pq^2,\quad b_3=\tfrac{1}{2}p_xq-\tfrac{1}{2}pq_x;\\
&a_4=\tfrac{1}{4}p_{xxx}+\tfrac{3}{8}p^2p_x+\tfrac{3}{4}q^2p_x,\quad c_4=-\tfrac{1}{2}q_{xxx}-\tfrac{3}{4}p^2q_x-\tfrac{3}{2}q^2q_x,\\
&b_4=-\tfrac{1}{4}p_{xx}p+\tfrac{1}{8}p_x^2-\tfrac{3}{32}p^4-\tfrac{3}{8}p^2q^2-\tfrac{1}{2}q_{xx}q+\tfrac{1}{4}q_x^2-\tfrac{3}{8}q^4.
\end{aligned}$$

Similarly, consider a sequence of zero curvature equations:

$$U_{t_m}-V_x^{[m]}+[U,V^{[m]}]=0,\quad V^{[m]}=(\lambda^mW)_+=\sum_{i=0}^{m}W_{0,i}\lambda^{m-i},\quad m\geq 0. \quad (2.18)$$

By virtue of (2.16), we see that all zero curvature equations in (2.18) engender a hierarchy of evolution equations:

$$u_{t_m} = K_m = \begin{bmatrix} 2a_{m+1} \\ -c_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{2.19}$$

Through

$$K_{m+1} = \Phi K_m, \quad m \geq 0, \tag{2.20}$$

the recursion relation (2.16) determines that the recursion operator Φ (see, e.g., [19] for details on recursion operators):

$$\Phi = \begin{bmatrix} -q\partial^{-1}p & -\partial - 2q\partial^{-1}q \\ \frac{1}{2}\partial + \frac{1}{2}p\partial^{-1}p & p\partial^{-1}q \end{bmatrix}. \tag{2.21}$$

In what follows, we shall construct Hamiltonian structures for the counterpart solion hierarchy (2.19) by the trace identity:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(\frac{\partial U}{\partial \lambda}W\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\left(\frac{\partial U}{\partial u}W\right), \quad \gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln\left|\operatorname{tr}\left(W^2\right)\right|. \tag{2.22}$$

In this so(3, $\mathbb{R}$) case, we can easily work out

$$\operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = -4b, \quad \operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) = -2c, \quad \operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) = -4a.$$

Balancing coefficients of all powers of λ in the corresponding trace identity presents

$$\frac{\delta}{\delta u}\int 2b_{m+1}dx = (\gamma - m)\begin{bmatrix} c_m \\ 2a_m \end{bmatrix}, \quad m \geq 0. \tag{2.23}$$

Checking the case with $m = 1$ yields $\gamma = 0$, and thus, we obtain the Hamiltonian structures for the counterpart soliton hierarchy (2.19):

$$u_{t_m} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = K_m = \begin{bmatrix} 2a_{m+1} \\ -c_{m+1} \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0, \tag{2.24}$$

with the Hamilton operator

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.25}$$

and the Hamiltonian functionals

$$\mathcal{H}_m = \int \left(-\frac{2b_{m+2}}{m+1}\right) dx, \quad m \geq 0. \tag{2.26}$$

3 Hamiltonian Bi-Integrable Couplings

3.1 Bi-Integrable Couplings

We would here like to generate a class of bi-integrable couplings (see, e.g., [3, 10] for details) for the counterpart hierarchy (2.19) of the Dirac soliton hierarchy. We shall use a class of non-semisimple Lie algebras presented in [14], which can be written as semi-direct sums of two Lie subalgebras:

$$\mathfrak{g}(\lambda) = \mathfrak{g} \Subset \mathfrak{g}_c. \tag{3.1}$$

That class of matrix Lie algebras, consisting of 3×3 block matrices, reads

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha A_2 \\ 0 & 0 & A_1 \end{bmatrix}, \tag{3.2}$$

where A_1, A_2 and A_3 are square matrices of the same size and α is a fixed real constant, and it can be written as

$$\mathfrak{g}(\lambda) = \mathfrak{g} \Subset \mathfrak{g}_c, \quad \mathfrak{g} = \{M(A_1, 0, 0)\}, \quad \mathfrak{g}_c = \{M(0, A_2, A_3)\}. \tag{3.3}$$

Let us particularly introduce a new spectral matrix from $\bar{\mathfrak{g}}(\lambda)$:

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2), \quad \bar{u} = (p, q, r, s, v, w)^T, \tag{3.4}$$

where U is defined by (2.10) and U_1 and U_2 are determined by

$$U_1 = U_1(u_1) = \begin{bmatrix} 0 & -s & -r \\ s & 0 & -s \\ r & s & 0 \end{bmatrix} \in \mathrm{so}(3, \mathbb{R}), \quad u_1 = \begin{bmatrix} r \\ s \end{bmatrix}, \tag{3.5}$$

and

$$U_2 = U_2(u_2) = \begin{bmatrix} 0 & -w & -v \\ w & 0 & -w \\ v & w & 0 \end{bmatrix} \in \mathrm{so}(3, \mathbb{R}), \quad u_2 = \begin{bmatrix} v \\ w \end{bmatrix}. \tag{3.6}$$

As usual, to solve the enlarged stationary zero curvature equation

$$\bar{W}_x = [\bar{U}, \bar{W}], \tag{3.7}$$

we try a solution of the following type

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2) = \sum_{i=0}^{\infty} \bar{W}_i \lambda^{-i}, \tag{3.8}$$

where W is defined by (2.13) and W_1 and W_2 are assumed to be

$$\begin{aligned} W_1 = W_1(u, u_1, \lambda) &= \begin{bmatrix} 0 & e-f & -g \\ -e+f & 0 & -e-f \\ g & e+f & 0 \end{bmatrix} \in \mathrm{so}(3, \mathbb{R}), \\ W_2 = W_2(u, u_1, u_2, \lambda) &= \begin{bmatrix} 0 & e'-f' & -g' \\ -e'+f' & 0 & -e'-f' \\ g' & e'+f' & 0 \end{bmatrix} \in \mathrm{so}(3, \mathbb{R}). \end{aligned} \tag{3.9}$$

The enlarged stationary zero curvature equation (3.8) requires the original zero curvature equation (2.12) and

$$\begin{aligned} W_{1,x} &= [U, W_1] + [U_1, W], \\ W_{2,x} &= [U, W_2] + [U_2, W] + \alpha[U_1, W_1]. \end{aligned} \tag{3.10}$$

The above equations generate

$$\begin{cases} e_x = pf - g\lambda + rb, \\ f_x = -pe + gq - ra + sc, \\ g_x = 2\lambda e - 2qf - 2sb, \end{cases} \tag{3.11}$$

and

$$\begin{cases} e'_x = pf' - g'\lambda + vb + \alpha rf, \\ f'_x = -pe' + g'q - va + wc - \alpha er + \alpha sg, \\ g'_x = 2\lambda e' - 2qf' - 2wb - 2\alpha sf. \end{cases} \tag{3.12}$$

Upon setting

$$\begin{cases} \mathrm{e} = \sum_{i=0}^{\infty} e_i\lambda^{-i}, \quad f = \sum_{i=0}^{\infty} f_i\lambda^{-i}, \quad g = \sum_{i=0}^{\infty} g_i\lambda^{-i}, \\ e' = \sum_{i=0}^{\infty} e'_i\lambda^{-i}, \quad f' = \sum_{i=0}^{\infty} f'_i\lambda^{-i}, \quad g' = \sum_{i=0}^{\infty} g'_i\lambda^{-i}, \end{cases} \tag{3.13}$$

we have the recursion relations to define W_1 and W_2:

$$\begin{cases} e_{i+1}=\frac{1}{2}g_{i,x} + qf_i + sb_i, \\ g_{i+1} = -e_{i,x} + pf_i + rb_i, \\ f_{i+1,x} = -pe_{i+1} + qg_{i+1} - ra_{i+1} + sc_{i+1}, \end{cases} \quad i \geq 0, \tag{3.14}$$

and

$$\begin{cases} e'_{i+1}=\frac{1}{2}g'_{i,x} + qf_i{}' + wb_i + \alpha sf_i, \\ g'_{i+1} = -e'_{i,x} + pf_i{}' + vb_i + \alpha rf_i, \\ f'_{i+1,x} = -pe'_{i+1} + qg'_{i+1} - va_{i+1} + wc_{i+1} - \alpha re_{i+1} + \alpha sg_{i+1}, \end{cases} \quad i \geq 0, \tag{3.15}$$

if we take the initial data as follows:

$$f_0 = -1, \quad e_0 = g_0 = 0; \quad f'_0 = -1, \quad e'_0 = g'_0 = 0. \tag{3.16}$$

From the above recursion relations and taking the constants of integration as zero, one therefore can compute

$$e_1 = -q - s, \quad g_1 = -p - r, \quad f_1 = 0; \tag{3.17}$$

$$e_2 = -\frac{1}{2}(r_x + p_x), \quad g_2 = q_x + s_x, \quad f_2 = \frac{1}{4}(p^2 + 2q^2) + \frac{1}{2}(pr + 2sq); \tag{3.18}$$

$$\begin{cases} e_3 = \frac{1}{2}(q_{xx} + s_{xx}) + \frac{1}{4}(p^2 + 2q^2)(q + s) + \frac{1}{2}(pr + sq)q, \\ g_3 = \frac{1}{2}(p_{xx} + r_{xx}) + \frac{1}{4}(p^2 + 2q^2)(p + r) + \frac{1}{2}(pr + sq)p, \\ f_3 = -\frac{1}{2}(pq_x + ps_x - p_xq - p_xs - qr_x + q_xr); \end{cases} \tag{3.19}$$

and

$$e'_1 = -q - w - \alpha s, \quad g'_1 = -p - v - \alpha r, \quad f'_1 = 0; \tag{3.20}$$

$$\begin{cases} e'_2 = -\frac{1}{2}(v_x + p_x) - \frac{1}{2}\alpha r_x, \\ g'_2 = q_x + w_x + \alpha s_x, \\ f'_2 = \frac{1}{4}(p^2 + 2q^2) + \frac{\alpha}{4}(r^2 + 2s^2) + \frac{1}{2}(\alpha r + v)p + (\alpha s + w)q; \end{cases} \tag{3.21}$$

$$\begin{cases} e_3' = \frac{1}{2}(q_{xx} + \alpha s_{xx} + w_{xx}) + \frac{1}{4}(q + w + \alpha s)(p^2 + q^2) \\ \qquad + \frac{1}{4}\alpha(r^2 + s^2)q + \frac{1}{2}(qw + pv)q + \frac{1}{2}\alpha(pr + qs)(q + s), \\ g_3' = \frac{1}{2}(p_{xx} + \alpha v_{xx} + r_{xx}) + \frac{1}{4}(p + v + \alpha r)(p^2 + q^2) \\ \qquad + \frac{1}{4}\alpha(r^2 + s^2)p + \frac{1}{2}(qw + pv)p + \frac{1}{2}\alpha(pr + qs)(p + r), \\ f_3' = \frac{1}{2}[(q + \alpha s + w)p_x - (p + \alpha r + v)q_x \\ \qquad + \alpha(q + s)r_x - \alpha(p + r)s_x + qv_x - pw_x]. \end{cases} \tag{3.22}$$

Now the enlarged zero curvature equations

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0, \quad \bar{V}^{[m]} = (\lambda^m \bar{W})_+ = \sum_{i=0}^{m} \bar{W}_i \lambda^{m-i}, \quad m \geq 0, \tag{3.23}$$

generate a hierarchy of bi-integrable couplings for the counterpart soliton hierarchy (2.19):

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}_{t_m} = \bar{K}_m(u) = \begin{bmatrix} 2a_{m+1} \\ -c_{m+1} \\ 2e_{m+1} \\ -g_{m+1} \\ 2{e'}_{m+1} \\ -{g'}_{m+1} \end{bmatrix}, \quad m \geq 0. \tag{3.24}$$

3.2 Hamiltonian Structures

In order to generate Hamiltonian structures for the bi-integrable couplings (3.24), we apply the variational identity (see [3, 11] for details):

$$\frac{\delta}{\delta u} \int \langle W, U_\lambda \rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \langle W, U_u \rangle, \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\langle W, W \rangle|, \tag{3.25}$$

where $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric and ad-invariant bilinear form on the underlying loop algebra $\bar{\mathfrak{g}}(\lambda)$. For the brevity, we define a mapping as follows:

$$\sigma : \bar{\mathfrak{g}}(\lambda) \to \mathbb{R}^9, \quad A \mapsto a = (a_1, a_2, \cdots, a_9)^T, \tag{3.26}$$

where

$$A = M(A_1, A_2, A_3), \; A_i = \begin{bmatrix} 0 & a_{3i-1} - a_{3i-2} & -a_{3i} \\ a_{3i-2} - a_{3i-1} & 0 & -a_{3i-2} - a_{3i-1} \\ a_{3i} & a_{3i-2} + a_{3i-1} & 0 \end{bmatrix}, \tag{3.27}$$

where $1 \leq i \leq 3$. As usual (see, e.g., [4, 14]), we compute the corresponding Lie bracket $[\,\cdot\,, \,\cdot\,]$ on $\mathbb{R}^9$.

From the defining equation

$$[a, b]^T = a^T R(b), \tag{3.28}$$

we can get

$$R(b) = M(R_1, R_2, R_3), \quad R_i = \begin{bmatrix} 0 & b_{3i} & -2b_{3i-1} \\ -b_{3i} & 0 & 2b_{3i-2} \\ b_{3i-1} & -b_{3i-2} & 0 \end{bmatrix}, \quad 1 \leq i \leq 3. \tag{3.29}$$

A bilinear form defined by

$$\langle a, b\rangle = a^T F b \tag{3.30}$$

which requires in (3.25), needs to satisfy

$$F(R(b))^T = -R(b)F, \tag{3.31}$$

together with

$$F^T = F, \quad \det(F) \neq 0. \tag{3.32}$$

We can therefore obtain

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.33}$$

where $\eta_i,\ 1 \leq i \leq 3$, are arbitrary constants. The non-degenerate condition requires

$$\det(F) = -64\eta_3^9\alpha^3 \neq 0. \tag{3.34}$$

So, the bilinear form on the semi-direct sum $\bar{\mathfrak{g}}(\lambda)$ is given by

$$\begin{aligned} \langle A, B\rangle &= (a_1, \cdots, a_9)\, F(b_1, \cdots, b_9)^T \\ &= (2a_1b_1 + 2a_2b_2 + a_3b_3)\,\eta_1 + (2a_4b_1 + 2a_5b_2 + a_6b_3 + 2a_1b_4 + 2a_2b_5 + a_3b_6)\,\eta_2 \\ &+ (2a_7b_1 + 2a_8b_2 + a_9b_3 + 2a_1b_7 + 2a_2b_8 + a_3b_9 + 2\alpha a_4b_4 + 2\alpha a_5b_5 + \alpha a_6b_6)\,\eta_3. \end{aligned} \tag{3.35}$$

Presently, as mentioned above, we can compute

$$\left\langle \bar{W}, \bar{U}_\lambda \right\rangle = 2b\eta_1 + 2f\eta_2 + 2f'\eta_3, \tag{3.36}$$

and

$$\left\langle \bar{W}, \bar{U}_{\bar{u}} \right\rangle = \begin{bmatrix} c\eta_1 + g\eta_2 + g'\eta_3 \\ 2a\eta_1 + 2e\eta_2 + 2e'\eta_3 \\ \alpha g\eta_3 + c\eta_2 \\ 2\alpha e\eta_3 + 2a\eta_2 \\ c\eta_3 \\ 2a\eta_3 \end{bmatrix}. \tag{3.37}$$

Moreover, we can easily see that $\gamma = 0$, and so, the corresponding variational identity reads

$$\frac{\delta}{\delta u}\int\left(-\frac{2b_{m+1}\eta_1 + 2f_{m+1}\eta_2 + 2f'_{m+1}\eta_3}{m}\right)dx = \begin{bmatrix} c_m\eta_1 + g_m\eta_2 + g'_m\eta_3 \\ 2a_m\eta_1 + 2e_m\eta_2 + 2e'_m\eta_3 \\ \alpha g_m\eta_3 + c_m\eta_2 \\ 2\alpha e_m\eta_3 + 2a_m\eta_2 \\ c_m\eta_3 \\ 2a_m\eta_3 \end{bmatrix}, m \geq 1. \tag{3.38}$$

Consequently, we have the following Hamiltonian structures for the hierarchy (3.24) of bi-integrable couplings:

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \quad m \geq 0, \tag{3.39}$$

with the Hamiltonian functions

$$\bar{\mathcal{H}}_m = \int\left(-\frac{2b_{m+2}\eta_1 + 2f_{m+2}\eta_2 + 2f'_{m+2}\eta_3}{m+1}\right)dx \tag{3.40}$$

with the Hamiltonian operator

$$\bar{J} = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix}^{-1} \otimes J, \tag{3.41}$$

where J is defined by (2.25).

3.3 Symmetries and Conserved Functionals

Let us now check the recursion relation

$$\bar{K}_{m+1} = \bar{\Phi}\bar{K}_m, \quad m \geq 0. \tag{3.42}$$

This determines the recursion operator $\bar{\Phi}$:

$$\bar{\Phi} = (M(\Phi, \Phi_1, \Phi_2))^T, \tag{3.43}$$

where Φ is given by (2.21) and

$$\Phi_1 = \begin{bmatrix} -q\partial^{-1}r - s\partial^{-1}p & 2q\partial^{-1}s + 2s\partial^{-1}q \\ \frac{1}{2}p\partial^{-1}r + \frac{1}{2}r\partial^{-1}p & p\partial^{-1}s + r\partial^{-1}q \end{bmatrix}, \tag{3.44}$$

$$\Phi_2 = \begin{bmatrix} -q\partial^{-1}v - w\partial^{-1}p - s\partial^{-1}r & -2q\partial^{-1}w - w\partial^{-1}q - 2\alpha s\partial^{-1}s \\ \frac{1}{2}p\partial^{-1}v + \frac{1}{2}v\partial^{-1}p + \frac{1}{2}\alpha r\partial^{-1}r & p\partial^{-1}w + v\partial^{-1}q + \alpha r\partial^{-1}s \end{bmatrix}. \tag{3.45}$$

It is direct to check that

$$\bar{\Phi}'(\bar{u})\left[\bar{\Phi}\bar{T}_1\right]\bar{T}_2 - \bar{\Phi}\bar{\Phi}'(\bar{u})\left[\bar{T}_1\right]\bar{T}_2 \tag{3.46}$$

is symmetric with respect to $\bar{T}_1$ and $\bar{T}_2$, which satisfies

$$\bar{\Phi}'(\bar{u})\left[\bar{\Phi}\bar{T}_1\right]\bar{T}_2 - \bar{\Phi}\bar{\Phi}'(\bar{u})\left[\bar{T}_1\right]\bar{T}_2 = \Phi'(u)[\Phi\bar{T}_2]\bar{T}_1 - \Phi\Phi'(u)[\bar{T}_2]\bar{T}_1, \tag{3.47}$$

and the two operators, $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$, constitute a Hamiltonian pair. i.e. any linear combination N of J and M satisfies

$$\int K^T N'(u)[NS]T\,dx + \int S^T N'(u)[NT]K\,dx + \int T^T N'(u)[NK]S\,dx = 0, \tag{3.48}$$

for all vector fields K, S and T. The condition (3.48) is equivalent to

$$L_{\bar{\Phi}K}\bar{\Phi} = \bar{\Phi}L_K\bar{\Phi}, \tag{3.49}$$

where K is an arbitrary vector field. The Lie derivative $L_K\bar{\Phi}$ is defined by

$$(L_K\bar{\Phi})S = \bar{\Phi}[K, S] - [K, \bar{\Phi}S],$$

where $[\cdot,\cdot]$ is the Lie bracket of vector fields. Note that an autonomous operator $\bar{\Phi} = \bar{\Phi}(\bar{u}, \bar{u}_x, \cdots)$ is a recursion operator of a given evolution equation $\bar{u}_t = K = K(\bar{u})$ if and only if $\bar{\Phi}$ satisfies

$$L_K\bar{\Phi} = 0. \tag{3.50}$$

The operator $\bar{\Phi}$ satisfies

$$L_{K_0}\bar{\Phi} = 0,\ K_0 = \begin{bmatrix} -2q \\ p \\ -2q-2s \\ p+s \\ -2q-2w-2\alpha s \\ p+r+\alpha r \end{bmatrix} \tag{3.51}$$

and thus

$$L_{K_m}\bar{\Phi} = L_{\bar{\Phi}K_{m-1}}\bar{\Phi} = \bar{\Phi}L_{K_{m-1}}\bar{\Phi} = \ldots = \bar{\Phi}L_{K_0}\bar{\Phi} = 0,\ m \geq 1. \tag{3.52}$$

Therefore, all bi-integrable couplings (3.24) of the counterpart hierarchy possess a bi-Hamiltonian structure, and so, they are Liouville integrable.

The bi-Hamiltonian theory also tells the two commuting algebras of infinitely many symmetries and conserved functionals:

$$[K_k, K_l] = K_k'(u)[K_l] - K_l'(u)[K_k] = 0, k, l \geq 0, \tag{3.53}$$

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J = \int \left(\frac{\delta \mathcal{H}_k}{\delta \bar{u}}\right)^T J \frac{\delta \mathcal{H}_l}{\delta \bar{u}} dx = 0, k, l \geq 0, \tag{3.54}$$

and

$$\{\mathcal{H}_k, \mathcal{H}_l\}_M = \int \left(\frac{\delta \mathcal{H}_k}{\delta \bar{u}}\right)^T M \frac{\delta \mathcal{H}_l}{\delta \bar{u}} dx = 0, k, l \geq 0. \tag{3.55}$$

4 Concluding Remarks

By introducing a matrix spectral problem associated with so$(3, \mathbb{R})$, we generates a counterpart soliton hierarchy of the Dirac soliton hierarchy associated with sl$(2, \mathbb{R})$. From a special class of non-semisimple loop algebras, Hamiltonian bi-integrable couplings of the counterpart soliton hierarchy were worked out. The Hamiltonian structures of the resulting bi-integrable couplings were established by applying the variational identity over the underlying the non-semisimple loop algebra.

The hierarchy of bi-integrable couplings shows us the diversity of commuting flows defined by evolution equations. It is helpful to see more soliton hierarchies of integrable couplings while classifying multi-component integrable systems. It is normally difficult to construct soliton hierarchies from higher order matrix spectral problems, and we hope that more different examples of integrable couplings can be exploited to know the structures of integrable systems.

We can also study different types of so$(3, \mathbb{R})$ matrix spectral problems which generalize our spectral problem (2.10). Two examples are the Kaup-Newell type and the WKI type of so$(3, \mathbb{R})$ matrix spectral problems:

$$U = \begin{bmatrix} 0 & \lambda^2 - \lambda q & -\lambda p \\ -\lambda^2 + \lambda q & 0 & -\lambda^2 - \lambda q \\ \lambda p & \lambda^2 + \lambda q & 0 \end{bmatrix},$$

and

$$U = \begin{bmatrix} 0 & \lambda - \lambda q & -\lambda p \\ -\lambda + \lambda q & 0 & -\lambda - \lambda q \\ \lambda p & \lambda + \lambda q & 0 \end{bmatrix}.$$

Another generalization of the WKI type reads

$$U = \begin{bmatrix} 0 & \lambda^2 - \lambda^2 q & -\lambda^2 p \\ -\lambda^2 + \lambda^2 q & 0 & -\lambda^2 - \lambda^2 q \\ \lambda^2 p & \lambda^2 + \lambda^2 q & 0 \end{bmatrix}.$$

Also there remain some open questions. Are there any integrable couplings of the Dirac systems in (2.7) with five dependent variables? How can one construct Hamiltonian integrable couplings for general cases? For example, is there any Hamiltonian structure for

$$\begin{cases} u_t = K(u), \\ v_t = K'[u](v), \\ w_t = K'[u](w)? \end{cases}$$

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