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A scalar fourth-order integrable equation associated with so(3, $\mathbb{R}$)

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Abstract We construct a scalar fourth-order integrable equation reduced from an integrable system associated with the special orthogonal Lie algebra so(3, $\mathbb{R}$). The resulting reduced fourth-order integrable equation inherits a bi-Hamiltonian structure, which leads to infinitely many commuting symmetries and conservation laws.

1 Introduction

Zero curvature equations associated with matrix Lie algebras are used to generate integrable equations [1,2], whose Hamiltonian structures could be furnished by the trace identity [3,4]. Among the well-known integrable equations associated with simple Lie algbras are the Korteweg-de Vries equation, the nonlinear Schrödinger equation and the derivative nonlinear Schrödinger equation [5–7].

More generally, zero curvature equations associated with non-simple Lie algebras yield so-called integrable couplings [8], and the variational identity [9] provides a basic tool for determining their Hamiltonian structures, which often lead to abundant hereditary recursion operators in block matrix form [10].

We will use the special orthogonal Lie algebra $\mathfrak{g} = \mathrm{so}(3, \mathbb{R})$, and it could be presented by all 3×3 trace-free, skew-symmetric real matrices. A basis can be chosen as follows:

$$e_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.1}$$

and the corresponding structure equations read

$$[e_1, e_2] = e_3, \; [e_2, e_3] = e_1, \; [e_3, e_1] = e_2. \tag{1.2}$$

This is one of the only two three-dimensional real Lie algebras, whose derived algebra is equal to itself. The other such Lie algebra is the special linear algebra sl(2, $\mathbb{R}$), which has been frequently used in studying integrable equations [2].

The following matrix loop algebra

$$\tilde{\mathfrak{g}} = \widetilde{\mathrm{so}}(3, \mathbb{R}) = \{M \in \mathrm{so}(3, \mathbb{R}) \,|\, \text{entries of } M \; - \; \text{Laurent series in } \lambda\}, \tag{1.3}$$

where λ is a spectral parameter and will be used in our construction. This loop algebra includes matrices of the form $\lambda^{m_1} e_1 + \lambda^{m_2} e_2 + \lambda^{m_3} e_3$ with arbitrary integers $m_i, \; 1 \le i \le 3$. This matrix loop algebra has been used in constructing integrable equations [11,12]. Based on the perturbation-type loop algebras of $\widetilde{\mathrm{so}}(3, \mathbb{R})$, we can also construct integrable couplings [13].

In this paper, starting from zero curvature equations, we would like to recall an application of so(3, $\mathbb{R}$) to integrable equations [11], with a slightly modified spectral matrix. We will then make an integrable reduction for an associated integrable system to present a fourth-order scalar integrable equation, which possesses the Liouville integrability, i.e., possesses infinitely many commuting symmetries and conservation laws. The presented scalar integrable equation reads

$$\begin{aligned} i p_t = & -p^*_{xxxx} + \frac{5}{2}(p^*)^2 p^*_{xx} + \frac{5}{2} p^* (p^*_x)^2 + \frac{3}{2} p^2 p^*_{xx} \\ & + 3 p p_x p^*_x + p p_{xx} p^* - \frac{1}{2} p_x^2 p^* - \frac{3}{8} p^* \left[p^2 + (p^*)^2\right]^2, \end{aligned}$$

where p^* denotes the complex conjugate of p.

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2 Lax pair and Hamiltonian structure

2.1 Lax pair

Let us consider a Lax pair of matrix spectral problems:

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ -i\phi_t = V\phi = V(u,\lambda)\phi, \tag{2.1}$$

with

$$U = U(u,\lambda) = \begin{bmatrix} 0 & -q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix}, \tag{2.2}$$

and

$$V = V(u,\lambda) = \sum_{l=0}^{4} \begin{bmatrix} 0 & -c_l & -a_l \\ c_l & 0 & -b_l \\ a_l & b_l & 0 \end{bmatrix} \lambda^l. \tag{2.3}$$

In the above spectral problems, i is the unit imaginary number, λ is a spectral parameter, $u = (p,q)^T$ is a potential, $\phi = (\phi_1,\phi_2,\phi_3)^T$ is a column eigenfunction, and a_l, b_l, c_l are defined by

$$\begin{aligned}
&a_0 = -1,\ b_0 = c_0 = 0;\\
&b_1 = -p,\ c_1 = -q,\ a_1 = 0;\\
&b_2 = iq_x,\ c_2 = -ip_x,\ a_2 = \frac{1}{2}\left(p^2+q^2\right);\\
&b_3 = -p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2,\\
&c_3 = -q_{xx} + \frac{1}{2}p^2q + \frac{1}{2}q^3,\\
&a_3 = i(p_xq - pq_x);\\
&b_4 = i\left(q_{xxx} - \frac{3}{2}p^2q_x - \frac{3}{2}q^2q_x\right),\\
&c_4 = i\left(-p_{xxx} + \frac{3}{2}p^2p_x + \frac{3}{2}p_xq^2\right),\\
&a_4 = pp_{xx} + qq_{xx} - \frac{1}{2}p_x^2 - \frac{1}{2}q_x^2 - \frac{3}{8}(p^2+q^2)^2;\\
&b_5 = -p_{xxxx} + \frac{5}{2}p^2p_{xx} + \frac{5}{2}pp_x^2 + \frac{3}{2}p_{xx}q^2 + 3p_xqq_x\\
&\qquad + pqq_{xx} - \frac{1}{2}pq_x^2 - \frac{3}{8}p\left(p^2+q^2\right)^2,\\
&c_5 = -q_{xxxx} + \frac{5}{2}q^2q_{xx} + \frac{5}{2}qq_x^2 + \frac{3}{2}p^2q_{xx} + 3pp_xq_x\\
&\qquad + pp_{xx}q - \frac{1}{2}p_x^2q - \frac{3}{8}q\left(p^2+q^2\right)^2,\\
&a_5 = i\Big(p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_xq_{xx}\\
&\qquad - \frac{3}{2}p^2p_xq + \frac{3}{2}pq^2q_x + \frac{3}{2}p^3q_x - \frac{3}{2}p_xq^3\Big).
\end{aligned}$$

The original spectral matrix in [11] is just U, but here the spectral matrix is iU with a factor difference i. This brings us convenience in determining integrable reductions. The coefficients a_l, b_l, c_l are determined by

$$\begin{cases} b_{l+1} = -ic_{l,x} + pa_l, \\ c_{l+1} = ib_{l,x} + qa_l, \qquad l \geq 0. \\ a_{l+1,x} = i(pc_{l+1} - qb_{l+1}), \end{cases} \tag{2.4}$$

under the integration conditions

$$a_l|_{u=0} = 0, \ l \geq 1,$$

i.e., take the constant of integration as zero. Such a matrix

$$W = ae_1 + be_2 + ce_3 = \sum_{l=0}^{\infty} \begin{bmatrix} 0 & -c_l & -a_l \\ c_l & 0 & -b_l \\ a_l & b_l & 0 \end{bmatrix} \lambda^{-l} \tag{2.5}$$

solves the stationary zero curvature equation

$$W_x = i[U, W]. \tag{2.6}$$

Now, the zero curvature equation

$$U_t - V_x + i[U, V] = 0, \tag{2.7}$$

generates a fourth-order integrable system:

$$\begin{cases} p_t = -i\Big[-q_{xxxx} + \frac{5}{2}q^2 q_{xx} + \frac{5}{2}q q_x^2 + \frac{3}{2}p^2 q_{xx} + 3pp_x q_x \\ \qquad +pp_{xx}q - \frac{1}{2}p_x^2 q - \frac{3}{8}q(p^2+q^2)^2\Big], \\ q_t = i\Big[-p_{xxxx} + \frac{5}{2}p^2 p_{xx} + \frac{5}{2}pp_x^2 + \frac{3}{2}p_{xx}q^2 + 3p_x q q_x \\ \qquad +pqq_{xx} - \frac{1}{2}pq_x^2 - \frac{3}{8}p(p^2+q^2)^2\Big]. \end{cases} \tag{2.8}$$

2.2 Hamiltonian structure

We apply the trace identity [3] with our spectral matrix iU:

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right)dx = \lambda^{-\gamma}\frac{\lambda}{\partial \lambda}\lambda^{\gamma}\operatorname{tr}\left(W\frac{\partial U}{\partial u}\right), \tag{2.9}$$

where the constant γ is determined by [10]

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle W, W\rangle|, \tag{2.10}$$

and then, we can obtain the following bi-Hamiltonian structure [14] for the integrable system (2.8):

$$u_t = J\frac{\delta \mathcal{H}_2}{\delta u} = M\frac{\delta \mathcal{H}_1}{\delta u}, \tag{2.11}$$

where the Hamiltonian pair, J and M, is given by

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{2.12}$$

and

$$M = i\begin{bmatrix} -\partial + q\partial^{-1}q & -q\partial^{-1}p \\ -p\partial^{-1}q & -\partial + p\partial^{-1}p \end{bmatrix}, \tag{2.13}$$

and the Hamiltonian functionals, $\mathcal{H}_1$ and $\mathcal{H}_2$, are determined by

$$\mathcal{H}_1 = \frac{1}{4}\int \Big(p_{xxx}q - pq_{xxx} - p_{xx}q_x + p_x q_{xx} \\ -\frac{3}{2}p^2 p_x q + \frac{3}{2}pq^2 q_x + \frac{3}{2}p^3 q_x - \frac{3}{2}p_x q^3\Big)\,dx, \tag{2.14}$$

and

$$\mathcal{H}_2 = -\frac{i}{5}\int \Big[pp_{xxxx} + qq_{xxxx} - p_x p_{xxx} - q_x q_{xxx} + \frac{1}{2}p_{xx}^2 + \frac{1}{2}q_{xx}^2 \\ -\frac{5}{2}p(p^2+q^2)p_{xx} - \frac{5}{2}q(p^2+q^2)q_{xx} - 5pqp_xq_x \\ -\frac{5}{4}(p^2-q^2)p_x^2 + \frac{5}{4}(p^2-q^2)q_x^2 + \frac{5}{16}(p^2+q^2)^3 \Big] dx. \tag{2.15}$$

This leads to infinitely many symmetries and conservation laws for the integrable system (2.8), which can often be generated through symbolic computation by computer algebra systems (see, e.g., [15,16]). The operator

$$\Phi = MJ^{-1} = i\begin{bmatrix} q\partial^{-1}p & -\partial + q\partial^{-1}q \\ \partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix},\ \partial = \frac{\partial}{\partial x}, \tag{2.16}$$

is a hereditary [17] recursion operator [18] for the integrable system (2.8).

3 A scalar integrable equation by reduction

Let us consider a specific reduction for the spectral matrix

$$(U(\lambda^*))^\dagger = CU(\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{3.1}$$

where † denotes the Hermitian transpose. This leads to the potential reduction

$$p = q^*. \tag{3.2}$$

Under this potential reduction, one has

$$a_l^* = a_l,\ b_l^* = c_l,\ l \geq 1. \tag{3.3}$$

We can prove this statement by the mathematical induction. Actually, under the induction assumption for $l = n$ and the recursion relation (2.4), one can compute

$$b_{n+1}^* = ic_{n,x}^* + p^*a_n^* = ib_{n,x} + qa_n = c_{n+1}, \\ a_{n+1,x}^* = -i(p^*c_{n+1}^* - q^*b_{n+1}^*) = i(pc_{n+1} - qb_{n+1}) = a_{n+1,x}.$$

Therefore, one obtains

$$(V(\lambda^*))^\dagger = CV(\lambda)C^{-1}, \tag{3.4}$$

and further

$$((U_t - V_x + i[U,V])(\lambda^*))^\dagger = C(U_t - V_x + i[U,V])(\lambda)C^{-1}. \tag{3.5}$$

This implies that the potential reduction (3.2) is compatible with the zero curvature representation (2.7) of the integrable system (2.8).

Obviously, the reduced fourth-order integrable equation is given by

$$ip_t = -p_{xxxx}^* + \frac{5}{2}(p^*)^2p_{xx}^* + \frac{5}{2}p^*(p_x^*)^2 + \frac{3}{2}p^2p_{xx}^* \\ +3pp_xp_x^* + pp_{xx}p^* - \frac{1}{2}p_x^2p^* - \frac{3}{8}p^*\left[p^2 + (p^*)^2\right]^2. \tag{3.6}$$

This is a new integrable equation associated with Lax pairs from the Lie algebra so(3,ℝ). It inherits a Hamiltonian structure from the integrable system (2.8). The infinitely many symmetries and conservation laws for the integrable system (2.8) are reduced to infinitely many ones for the above scalar integrable equation (3.6).

We can take another specific reduction for the spectral matrix

$$(U(\lambda^*))^\dagger = CU(\lambda)C^{-1},\ C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{3.7}$$

where † denotes the Hermitian transpose again. This leads to another potential reduction

$$p = -q^*. \tag{3.8}$$

With this potential reduction, one has

$$a_l^* = a_l,\ b_l^* = -c_l,\ l \geq 1. \tag{3.9}$$

Note that the differential polynomial on the left-hand side of the first equation in the integrable system (2.8) is odd with respect to q and even with respect to p, and the differential polynomial on the left-hand side of the second equation in the integrable system (2.8) is odd with respect to p and even with respect to q. Therefore, the reduced fourth-order equation under (3.8) has just a different sign from the above integrable equation (3.6).

This feature for soliton equations associated with so(3, $\mathbb{R}$) is different from the one for soliton equations associated with sl(2, $\mathbb{R}$). In the case of sl(2,$\mathbb{R}$), there are two unequivalent integrable reductions: focusing and defocusing.

4 Conclusion and remarks

We have constrcuted a scalar fourth-order integrable equation from a zero curvature equation associated with the special orthogonal Lie algebra so(3, $\mathbb{R}$). The presented integrable equation is shown to possess a bi-Hamiltonian structure and infinitely many symmetries and conservation laws.

It is known that the nonlinear Schrödinger (NLS) equation is used as a master envelop equation for information transfer in optical fibers [6]. This model equation is generated under the assumption that we take the second-order Taylor expansion of the wave number around the carrier frequency and the electric field intensity. A third-order Taylor expansion of the wave number leads to a third-order integrable perturbed NLS equation, which determines nonlinear optical solitons efficiently [19,20]. Such solitons are formulated, basically due to the existence of high-order conservation laws. Our integrable equation of fourth-order could provide an effective model equation for exploring nonlinear dispersive waves under the fourth-order Taylor expansion of the wave number.

There are many other interesting problems for integrable equations associated with the special orthogonal Lie algebra so(3, $\mathbb{R}$). For exmaple, what are general structures of Darboux transformations? How can one formulate Riemann-Hilbert problems and inverse scattering transforms?

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