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Resonant multiple wave solutions for a (3 + 1)-dimensional nonlinear evolution equation by linear superposition principle

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ABSTRACT

The linear superposition principle can apply to the construction of resonant multiple wave solutions for a (3 + 1)-dimensional nonlinear evolution equation. Two types of resonant solutions are obtained by the parameterization for wave numbers and frequencies for linear combinations of exponential traveling waves. The resonance phenomena of multiple waves are discussed through the figures for several sample solutions.

1. Introduction

It is known that the Hirota bilinear method is an efficient tool to construct exact solutions of nonlinear evolution equations [1–3], especially the soliton solutions [4–7]. By virtue of the dependent variable transformations, nonlinear equations are transformed into bilinear equations with binary differential operators, and then the perturbation expansion can be used to solve those bilinear equations. The resulting bilinear equations are still nonlinear equations and generally do not obey the principle of linear superposition. However, it has been shown that the linear superposition principle can apply to exponential traveling waves of Hirota bilinear equations, and can form a specific sub-class of solutions from linear combinations of exponential wave solutions [8–11].

Let us consider a Hirota bilinear equation

$$F(D_{x_1}, D_{x_2}, \ldots, D_{x_M})(f \cdot f) = 0, \tag{1}$$

where F is a multivariate polynomial in M variables satisfying

$$F(0, 0, \ldots, 0) = 0, \tag{2}$$

and the Hirota bilinear operators D_{x_j} $(1 \le j \le M)$ are defined by

$$D_x^m D_t^n\, g \cdot f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n g(x,t) f(x',t')|_{x'=x,\, t'=t}, \tag{3}$$

for nonnegative integers m and n. We remark that the Hirota bilinear operators have been generalized in Ref. [12].

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Now Let us fix $N \in \mathbb{N}$ and introduce N wave variables

$$\theta_i = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M, \quad 1 \le i \le N, \tag{4}$$

where $k_{j,i}$'s are all constants, and set a linear combination of N exponential traveling waves

$$f = \sum_{i=1}^{N} \varepsilon_i e^{\theta_i} = \sum_{i=1}^{N} \varepsilon_i e^{k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M}, \tag{5}$$

with all ε_i's as arbitrary constants. Note that we have the following bilinear identity

$$F(D_{x_1}, \ldots, D_{x_M})e^{\theta_i} \cdot e^{\theta_j} = F(k_{1,i} - k_{1,j}, \ldots, k_{M,i} - k_{M,j})e^{\theta_i + \theta_j}, \quad 1 \le i, j \le N. \tag{6}$$

From Eq. (6), it is obvious to find that a linear combination function f defined by Eq. (5) solves the bilinear equation (1) if only if the constants $k_{j,i}$'s satisfy

$$F(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, \ldots, k_{M,i} - k_{M,j}) = 0, \quad 1 \le i < j \le N. \tag{7}$$

Based on the condition (7), we point out that the linear superposition principle can apply to the bilinear equation (1), and allow us to construct specific subspaces of solutions from linear combinations of exponential traveling waves.

In this paper, with the above presented linear superposition principle, we will consider a (3 + 1)-dimensional nonlinear evolution equation

$$3u_{xz} - (2u_t + u_{xxx} - 2uu_x)_y + 2(u_x \partial_x^{-1} u_y)_x = 0, \tag{8}$$

where ∂_x^{-1} stands for an inverse operator $\partial_x = \partial/\partial x$. This (3 + 1)-dimensional nonlinear equation (8) was originally introduced as a model for the study of algebraic-geometrical solutions [13]. Obviously, Eq. (8) possesses the Korteweg–de Vries (KdV) equation as a main term. Therefore, Eq. (8) can be regarded as an extension of the KdV equation, and may be used to study shallow-water waves in nonlinear dispersive models [14]. Although the application of Eq. (8) in physics or other science is not well clear, Eq. (8) admits more abundant soliton structures due to the higher space dimension. Its integrability and large classes of exact solutions have been studied with various methods [13–18], e.g., the soliton, positon, negaton and rational solutions.

In fact, the integral term in Eq. (8) can be removed by introducing the potential

$$u(x, y, z, t) = w(x, y, z, t)_x, \tag{9}$$

and then Eq. (8) is transformed to

$$3w_{xxz} - (2w_{xt} + w_{xxxx} - 2w_x w_{xx})_y + 2(w_{xx} w_y)_x = 0. \tag{10}$$

Through the dependent variable transformation $u = -3(\ln f)_{xx}$ or $w = -3(\ln f)_x$, the (3 + 1)-dimensional nonlinear equation (8) has the following bilinear form

$$\left(3D_x D_z - 2D_y D_t - D_x^3 D_y\right) f \cdot f = 0. \tag{11}$$

In the following sections, we will apply the linear superposition principle to this Hirota bilinear equation (11) and construct specific subspaces of solutions from linear combinations of exponential traveling waves, i.e., resonant multiple wave solutions of Eq. (8).

2. Linear combinations of exponential traveling wave solutions

According to Eq. (4), let us take

$$\theta_i = k_i x + l_i y + m_i z + \omega_i t, \quad 1 \le i \le N, \tag{12}$$

where all k_i's, l_i's, m_i's and ω_i's are constants to be determined. Substituting Eq. (12) into the linear superposition principle condition (7) corresponding to Eq. (11) shows

$$\begin{aligned} &F(k_i - k_j, l_i - l_j, m_i - m_j, \omega_i - \omega_j) \\ &\quad = 3(k_i - k_j)(m_i - m_j) - 2(l_i - l_j)(\omega_i - \omega_j) - (k_i - k_j)^3(l_i - l_j) \\ &\quad = k_i^3 l_j + 3k_i^2 l_i k_j - 3k_i^2 k_j l_j - 3k_i l_i k_j^2 + 3k_i k_j^2 l_j + l_i k_j^3 - 3k_i m_j - 3m_i k_j \\ &\qquad + 2\omega_i l_j + 2l_i \omega_j - k_i^3 l_i + 3k_i m_i - 2l_i \omega_i - k_j^3 l_j + 3k_j m_j - 2l_j \omega_j = 0. \end{aligned} \tag{13}$$

In order to solve Eq. (13), as in Refs. [8,19], we adopt a kind of the parameterization for wave numbers and frequencies

$$l_j = a\,k_j^{\alpha} \qquad m_j = b\,k_j^{\beta}, \qquad \omega_j = c\,k_j^{\gamma}, \quad 1 \le j \le N, \tag{14}$$

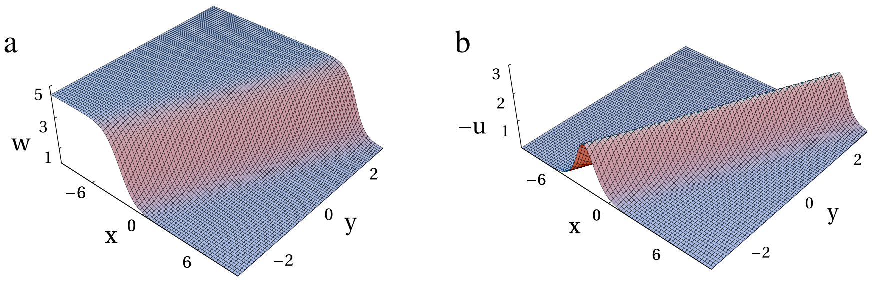

Fig. 1. Single traveling wave and soliton via solutions (19) and (20) with $k_1 = -1.5$, $a = 1$, $t = 2$ and $z = 2$.

where a, b, c are all constants, and α, β, γ are integers which can be positive or negative. By straightforward substitution of Eq. (14), we can directly obtain two sets of solutions of Eq. (13)

$$\left\{k_j = k_j,\ l_j = a\,k_j^{-1},\ m_j = a\,k_j,\ \omega_j = -\frac{1}{2}k_j^3\right\}, \tag{15}$$

and

$$\left\{k_j = k_j,\ l_j = a\,k_j^2,\ m_j = a\,k_j^4,\ \omega_j = k_j^3\right\} \tag{16}$$

with a being an arbitrary constant.

As a result, we can derive two types of resonant multiple wave solutions of Eq. (8) or Eq. (10)

$$f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + a\,k_i^{-1} y + a\,k_i z - \frac{1}{2}k_i^3 t}, \qquad u = -3(\ln f)_{xx}, \qquad w = -3(\ln f)_x, \tag{17}$$

and

$$f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + a\,k_i^2 y + a\,k_i^4 z + k_i^3 t}, \qquad u = -3(\ln f)_{xx}, \qquad w = -3(\ln f)_x. \tag{18}$$

3. Resonant multiple wave solutions

In this section, we take the solution (18) as an illustrative example to analyze the structures of the resulting resonant multiple wave solutions.

Firstly, without loss of generality, we set $\varepsilon_1 = \varepsilon_2 = 1$ and $k_2 = 0$ in the solution (18) with $N = 2$, and obtain the single-front wave as

$$w = -\frac{3\,k_1}{2}\left[1 + \tanh\left(\frac{k_1 x + a\,k_1^2 y + a\,k_1^4 z + k_1^3 t}{2}\right)\right] \tag{19}$$

and

$$u = -\frac{3\,k_1^2}{4}\left[\operatorname{sech}^2\left(\frac{k_1 x + a\,k_1^2 y + a\,k_1^4 z + k_1^3 t}{2}\right)\right]. \tag{20}$$

It is obvious that the solution (19) is the kink-shape traveling wave [20] and the solution (20) is a bell-shape soliton, as illustrated in Fig. 1(a) and (b).

When $N \geq 3$, the solution (17) or (18) represents the resonance of multiple waves with the development of time. In Fig. 2, two, three and four traveling waves along the same propagation directions respectively coalesce into one large wave in their interaction region of the (x, y)-plane, the amplitude of which amounts to their initial amplitudes. With the increase of N, various combinations of the coalescence mechanism can occur when multiple traveling waves have different or same propagation directions.

Fig. 3 respectively displays the resonant phenomena of two, three and four traveling solitons with the development of time. In three resonance cases, interacting solitons can produce a big soliton, whose amplitude is large than the summation of the amplitudes of solitons before resonance in the interaction region. With the increase of number of solitons, the interaction will become more complicated and can exhibit web-like structures. It has been shown that the soliton resonance behaviors can also occur in many high-dimensional equations like the $(2+1)$-dimensional Boussinesq equation [21], the Kadomtsev–Petviashvili equation [22] and the coupled Kadomtsev–Petviashvili system [23].

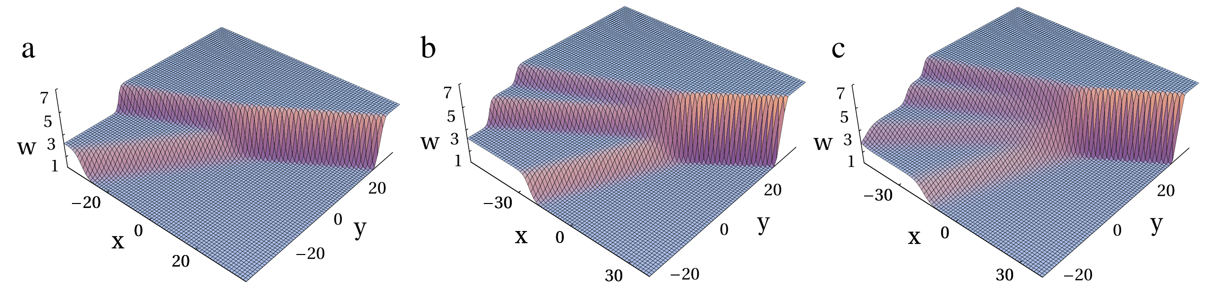

Fig. 2. The resonant two-, three- and four-wave solutions (18). The related parameters are, respectively, chosen as: $a = 1, t = 5, z = 2$ and (a) $N = 3, k_1 = -1.5, k_2 = -0.75, k_3 = 0, \varepsilon_j = 1\ (1 \le j \le 3)$; (b) $N = 4, k_1 = -1.5, k_2 = -0.75, k_3 = -2,\ k_4 = 0, \varepsilon_j = 1\ (1 \le j \le 4)$; (c) $N = 5, k_1 = -1.5, k_2 = -0.75, k_3 = -2,\ k_4 = -1, k_5 = 0, \varepsilon_j = 1\ (1 \le j \le 5)$.

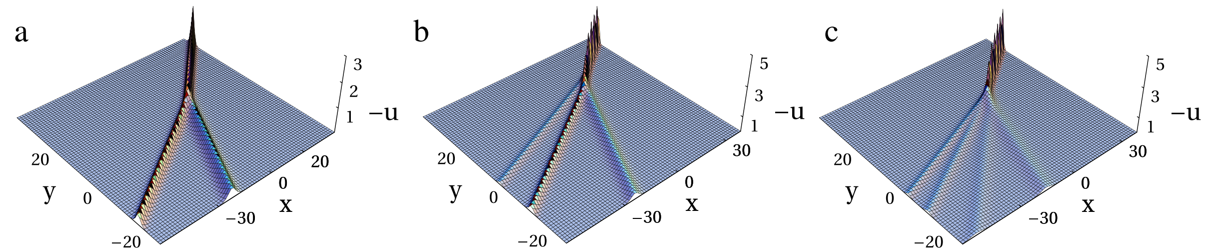

Fig. 3. The resonant two-, three- and four-wave solutions (18). The related parameters are the same as in Fig. 2.

4. Conclusions

In this paper, we have discussed the linear superposition principle of exponential traveling waves for Hirota bilinear equations. Applying the linear superposition principle to a (3 + 1)-dimensional nonlinear evolution equation, we have presented two sets of the resonant multiple wave solutions by the parameterization for wave numbers and frequencies of exponential traveling waves. We have discussed the resonance behaviors of multiple waves through the figures for several sample solutions.

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