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Classifying bilinear differential equations by linear superposition principle

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In this paper, we investigate the linear superposition principle of exponential traveling waves to construct a sub-class of N-wave solutions of Hirota bilinear equations. A necessary and sufficient condition for Hirota bilinear equations possessing this specific sub-class of N-wave solutions is presented. We apply this result to find N-wave solutions to the $(2+1)$-dimensional KP equation, a $(3+1)$-dimensional generalized Kadomtsev–Petviashvili (KP) equation, a $(3+1)$-dimensional generalized BKP equation and the $(2+1)$-dimensional BKP equation. The inverse question, i.e., constructing Hirota Bilinear equations possessing N-wave solutions, is considered and a refined 3-step algorithm is proposed. As examples, we construct two very general kinds of Hirota bilinear equations of order 4 possessing N-wave solutions among which one satisfies dispersion relation and another does not satisfy dispersion relation.

Keywords: Hirota bilinear equations; multi-wave solutions; linear superposition principle of exponential functions; KP equation; BKP equation.

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1. Introduction

It is well known that because of the validity of the linear superposition principle, it is relatively easy to investigate the structure of the solution sets or even exact solutions of linear equations, whether they are linear algebraic equations or differential equations. Of course the linear superposition principle is not valid in the

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nonlinear world and that a lot of researchers are still researching in this area to have a breakthrough. When studying nonlinear differential equations, the first idea that comes to our mind is to transform the nonlinear differential equations to linear ones. Some differential equations which could be transformed into linear differential equations through a change of dependent variable are shown in Ref. 1. Once a nonlinear differential equation has been linearized, it is relatively easy to investigate the structure of its solution set and find its exact solutions. Unfortunately, only a few special classes of nonlinear differential equations can be linearized. To relax this constraint and to find exact solutions for a slightly wider class of nonlinear differential equations, Hirota[1] introduced a type of bilinear differential equations, often called the Hirota form, for which one can find exact solutions by the perturbation method.

By using some independent variable transformations, various nonlinear differential equations of mathematical physics can be transformed into Hirota bilinear equations[1,2] which possess some specific properties and are used to study the solution sets of nonlinear differential equations. Recently, some programs were designed[12–16] and some algorithms[11,19] were proposed on searching for integrable bilinear equations. Based on the Hirota bilinear form, soliton solutions of certain nonlinear differential equations were obtained by the Hirota perturbation technique,[1] the multiple exp-function algorithm, and various other methods.[4–10,18–21]

Hirota's bilinear technique provides a powerful method to investigate and construct the solutions of nonlinear differential equations.[2–9,20,21] In Refs. 6 and 7, a linear superposition principle of exponential traveling waves is analyzed for Hirota bilinear equations and a sub-class of N-wave solutions are constructed by linear combinations of exponential traveling waves. This means the solution sets of some bilinear equations have linear subspaces. The inverse question, that is about generating Hirota bilinear equations possessing the indicated N-wave solutions, is also discussed and an algorithm using weights is also proposed at the same time. Even for some generalized bilinear differential equations,[8–10,17] some similar properties were found and thus N-wave solutions subspace was constructed in the same way.

In this paper, we aim to further investigate the properties of Hirota bilinear differential equations and use them to construct the exponential wave solution sets which are subspaces of the solution sets to Hirota bilinear equation. Additionally, a refined algorithm to generate Hirota bilinear equations possessing the indicated N-wave solutions is presented, which results in compensating the theory that was proposed in Refs. 6 and 7.

2. Linear Superposition Principle for Hirota Bilinear Equations

Suppose $P(x_1, x_2, \ldots, x_M)$ is a multivariate polynomial satisfying

$$P(0, 0, \ldots, 0) = 0 \tag{2.1}$$

and

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M}) f \cdot f = 0 \tag{2.2}$$

is a Hirota bilinear equation, where $D_{x_i}, 1 \leq i \leq M$, are Hirota differential operators which are defined by

$$D_y^p f(y) \cdot g(y) = (\partial_y - \partial_{y'})^p f(y) g(y')|_{y'=y} = \partial_{y'}^p f(y+y') g(y-y')|_{y'=0}, \quad p \geq 1 . \tag{2.3}$$

Let $\mathbf{k}_i = \{k_{1,i}, k_{2,i}, \dots, k_{M,i}\}^T$, $\mathbf{x} = \{x_1, x_2, \dots, x_M\}^T$ and $\eta_i = \mathbf{k}_i \cdot \mathbf{x}$, $1 \leq i \leq N$, where η_i denote the N-wave variables and f_i and $1 \leq i \leq N$ denote N exponential wavefunctions given by

$$f_i = e^{\eta_i} = e^{k_{1,i} x_1 + k_{2,i} x_2 + \cdots + k_{M,i} x_M}, \quad 1 \leq i \leq N . \tag{2.4}$$

Here $k_{j,i}, 1 \leq i \leq N, 1 \leq j \leq M$ are some constants which will be determined later. Denote $P(x_1, x_2, \dots, x_M)$ by $P(\mathbf{x})$. According to the properties of Hirota bilinear operators,[1,2] one can easily conclude that:

(a) For any Hirota bilinear operator, we have

$$D_{x_1}^{k_1} D_{x_2}^{k_2} \cdots D_{x_M}^{k_M} f \cdot f = (-1)^{\sum_{i=1}^M k_i} D_{x_1}^{k_1} D_{x_2}^{k_2} \cdots D_{x_M}^{k_M} f \cdot f . \tag{2.5}$$

(b) For any Hirota bilinear operator $P(D_{x_1}, D_{x_2}, \dots, D_{x_M})$ and any two N exponential wavefunctions e^{η_i} and e^{η_j},

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M}) e^{\eta_i} \cdot e^{\eta_j} = P(\mathbf{k}_i - \mathbf{k}_j) e^{\eta_i + \eta_j} . \tag{2.6}$$

For any n-degree multivariate polynomial, we have

$$P(x_1, x_2, \dots, x_M) = \sum_{k=1}^{n} P_k(x_1, x_2, \dots, x_M) , \tag{2.7}$$

where $P_k(x_1, x_2, \dots, x_M), 1 \leq k \leq N$, are homogeneous multivariate polynomial of degree k. From (2.5), we conclude that $P_k(D_{x_1}, D_{x_2}, \dots, D_{x_M}) f \cdot f \equiv 0$ if k is an odd number, which means that any Hirota bilinear operator can be written as

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M}) = \sum_{k=1}^{n} P_{2k}(D_{x_1}, D_{x_2}, \dots, D_{x_M}) . \tag{2.8}$$

Let f be a linear combination of the above N exponential wavefunctions such that $f = \sum_{i=1}^N \varepsilon_i f_i = \sum_{i=1}^N \varepsilon_i e^{\eta_i}$, where $\varepsilon_i, 1 \leq i \leq N$, are arbitrary constants. From the analysis above, we obtain

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M}) f \cdot f = 2 \sum_{1 \leq j < i \leq N} \varepsilon_i \varepsilon_j P(\mathbf{k_i} - \mathbf{k_j}) e^{\eta_i + \eta_j} . \tag{2.9}$$

It follows directly that any linear combination of N exponential wavefunctions $e^{\eta_i}, 1 \leq i \leq N$ solves Hirota bilinear equation (1.2) if and only if $P(\mathbf{k}_i - \mathbf{k}_j) = 0$ holds for any $1 \leq j < i \leq N$ (see Theorem 2.1 in Ref. 6). For the convenience of readers, we recall this theorem.

Theorem 1 (Linear Superposition Principle Ref. 6). Let $P(x_1, x_2, \ldots, x_M)$ be a multivariate polynomial of degree $2n$ satisfying (2.1) and the wave variables $\eta_i = \mathbf{k}_i \cdot \mathbf{x}$, $1 \leq i \leq N$. Then $\text{span}\{e^{\eta_1}, \ldots, e^{\eta_N}\}$ is a subset of the solution set of the Hirota bilinear equation (2.1), if and only if

$$P(\mathbf{k_i} - \mathbf{k_j}) = 0 \tag{2.10}$$

for any $1 \leq j < i \leq N$.

This theorem presents a necessary and sufficient condition when Hirota bilinear differential equation (2.1) possesses the linear superposition principle for exponential traveling waves. However, it is usually not easy to determine the coefficients $k_{j,i}, 1 \leq i \leq N, 1 \leq j \leq M$, since (2.10) is a system $N(N-1)/2$ coupled algebraic equations. Fortunately, there is only one algebraic equation to determine the 2-wave solutions, so it might be easy to obtain the linear subspace of 2-wave solution set of the Hirota bilinear equation (2.2) by using (2.10). In addition, some techniques to simplify the calculations to obtain some N-wave subspace of (2.2) have been found. In Ref. 7, by observing some concrete soliton equations, Ma and Fan proposed an algorithm to find some special solutions to these equations which have the relation

$$k_{l,i} = a_l k_i^{n_l}, \quad 1 \leq l \leq M, \quad 1 \leq i \leq N\,, \tag{2.11}$$

where $k_i, 1 \leq i \leq N$, are arbitrary constants, $n_l, 1 \leq l \leq M$, are some integers, $a_l, 1 \leq l \leq M$, are constants and $n_l, a_l, 1 \leq l \leq M$, are determined by the polynomial P. From (2.11), we obtain

$$\begin{aligned} &P(k_{1,i} - k_{1,j}, k_{2,i} - k_{2,j}, \ldots, k_{M,i} - k_{M,j}) \\ &\quad = P(a_1(k_i^{n_1} - k_j^{n_1}), \quad a_2(k_i^{n_2} - k_j^{n_2}), \ldots, a_M(k_i^{n_M} - k_j^{n_M}))\,. \end{aligned} \tag{2.12}$$

In fact, $n_l, 1 \leq l \leq M$ are usually obtained by balancing the powers of the left-hand side of (2.12) and $a_l, 1 \leq l \leq M$ are chosen to make the polynomial $P(a_1(k_i^{n_1} - k_j^{n_1}), a_2(k_i^{n_2} - k_j^{n_2}), \ldots, a_M(k_i^{n_M} - k_j^{n_M})) \equiv 0$ for any values of k_i and k_j. To obtain possible $n_l s, a_l, 1 \leq l \leq M$, the soliton equations are usually required to be higher dimensional equations which may provide more chance to have the subspaces $\text{span}\{e^{\eta_1}, \ldots, e^{\eta_N}\}$ as the subsets of their solution sets. The merit of the algorithm proposed in Ref. 7 is that the $N(N-1)/2$ coupled algebraic equations are reduced into algebraic equations of $a_l, 1 \leq l \leq M$, which simplify the computations a lot and can also be applied to consider the opposite question, namely constructing the Hirota bilinear equations possessing N-wave solutions for arbitrary integer N.

Theorem 2. *For Hirota bilinear equation* (2.2), *there exist some integers* $n_l, 1 \leq l \leq M$, *and some constants* $a_l, 1 \leq l \leq M$, *such that*

$$P(a_1(x^{n_1} - y^{n_1}), \quad a_2(x^{n_2} - y^{n_2}), \ldots, a_M(x^{n_M} - y^{n_M})) \equiv 0 \tag{2.13}$$

for any x *and* y *if and only if for arbitrary integer* N, $l_i, 1 \leq i \leq M$, *and any nonzero constants* $m_i, i = 1, 2, \ldots, N$, *with* $m_i \neq m_j$ *for any* $i \neq j$, span$\{e^{\eta_0-\eta_1},\dots,e^{\eta_0-\eta_N},e^{\eta_0}\}$ *and* span$\{e^{\eta_0},e^{\eta_0+\eta_1},\dots,e^{\eta_0+\eta_N}\}$ *are two subspaces of the solution set of the Hirota bilinear equation* (2.2)*, where* $\eta_0=\sum_{j=1}^M l_i x_j$ *and*

$$\eta_i=\sum_{j=1}^M a_j m_i^{n_j} x_j,\quad i=1,2,\dots,N\,. \tag{2.14}$$

From the above theorem, we deduce that the Hirota bilinear equation (2.2) has an abundant of N-wave solutions for any integer N if there exist integers $n_l, 1\le l\le M$, and constants $a_l, 1\le l\le M$, such that (2.13) holds for any x and y. Notice that the left-hand side of (2.13) is a polynomial of the variables x and y, so (2.13) holds for any x and y if and only if every coefficient of this polynomial is zero. By this observation, we may obtain the undetermined integers $n_l, 1\le l\le M$, and constants $a_l, 1\le l\le M$. In Refs. 6 and 7, some Hirota bilinear equations such as the (3+1)-dimensional Kadomtsev–Petviashvili (KP) equation, the (3+1)-dimensional Jimbo–Miwa equation and the (3+1)-dimensional BKP equation were studied and some of their N-wave solutions were obtained. By this algorithm, for any arbitrary N, another subspace of solution set of the (3+1)-dimensional Jimbo–Miwa equation was found in Ref. 20. Next, we present some examples to illustrate this algorithm.

3. Applications to Some Hirota Bilinear Equations

3.1. *The* $(2+1)$*-dimensional KP equation*

The $(2+1)$-dimensional KP equation is given by[4,5,19]

$$(u_{xxx}+6uu_x+u_t)_x+3d^2u_{yy}=0\,, \tag{3.1}$$

which can be transformed into the Hirota bilinear equation

$$\left(\frac{(D_x^4+D_xD_t+3d^2D_y^2)f\cdot f}{f^2}\right)_{xx}=0 \tag{3.2}$$

through the dependent variable transformation $u=2(\ln f)_{xx}$. Then the solutions of the Hirota bilinear equation

$$(D_x^4+D_xD_t+3d^2D_y^2)f\cdot f=0 \tag{3.3}$$

are in the solution set of (3.2). Solving (3.3), one obtains the solutions to KP equation (3.1).

According to Theorem 2.2, we search for $a_i, i=1,2,3$, and $n_i, i=1,2,3$, such that

$$a_1^4(x^{n_1}-y^{n_1})^4+a_1a_3(x^{n_1}-y^{n_1})(x^{n_3}-y^{n_3})+3d^2a_2^2(x^{n_2}-y^{n_2})^2\equiv 0\,. \tag{3.4}$$

By precise analysis of (3.4), we can possibly obtain $a_i, i=1,2,3$, if (n_1,n_2,n_3) is chosen to be $(1,2,3)$. For $(n_1,n_2,n_3)=(1,2,3)$, comparing the coefficients of (2.20), one obtains the equations of $a_i, i=1,2,3$, as

$$\begin{cases} a_1^4+a_1a_3+3d^2a_2^2=0\,,\\ -2a_1^4+a_1a_3+6d^2a_2^2=0\,. \end{cases} \tag{3.5}$$

Solving (3.5), we obtain $a_2 = \pm a_1^2/d, a_3 = -4a_1^3$ and a_1 is a free parameter. Consequently, letting $a_1 = 1$, one obtains a group of N-wave solution sets of the $(2+1)$-dimensional KP equation (3.1) as

$$u = 2(\ln f)_{xx}, \quad f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + \delta \frac{k_i^2}{d} y - 4k_i^3 t}, \tag{3.6}$$

where $\delta \in \{-1, 1\}, N$ is an arbitrary integer, k_i and $\varepsilon_i, i = 1, \ldots, N$, are arbitrary constants.

3.2. *A* $(3+1)$*-dimensional generalized KP equation*

In this subsection, we study the multi-wave solutions of $(3+1)$-dimensional generalized KP equation[9]

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0, \tag{3.7}$$

which can be transformed into the Hirota bilinear equation

$$(D_x^3 D_y + D_x D_t + D_y D_t - D_z^2) f \cdot f = 0 \tag{3.8}$$

through the dependent variable transformation $u = 2(\ln f)_x$.

According to Theorem 2.2, we search for $a_i, i = 1, 2, 3, 4$, and $n_i, i = 1, 2, 3, 4$, such that

$$\begin{aligned} &a_1^3 a_2 (x^{n_1} - y^{n_1})^3 (x^{n_2} - y^{n_2}) + a_1 a_4 (x^{n_1} - y^{n_1})(x^{n_4} - y^{n_4}) \\ &\quad + a_2 a_4 (x^{n_2} - y^{n_2})(x^{n_4} - y^{n_4}) - a_3^2 (x^{n_3} - y^{n_3})^2 \equiv 0. \end{aligned} \tag{3.9}$$

From (3.9), we can possibly obtain $a_i, i = 1, 2, 3, 4$, if (n_1, n_2, n_3, n_4) is chosen as $(1, 1, 2, 3)$. Thus, comparing the coefficients of (3.9), one obtains the following equations of $a_i, i = 1, 2, 3, 4$:

$$\begin{cases} a_1^3 a_2 + (a_1 + a_2) a_4 - a_3^2 = 0, \\ -2a_1^3 a_2 + (a_1 + a_2) a_4 - 2a_3^2 = 0. \end{cases} \tag{3.10}$$

Solving (3.10), we obtain $a_3^2 = -3a_1^3 a_2, a_4 = -4a_1^3/(a_1 + a_2)$, where a_1 and a_2 satisfy $a_1 a_2 < 0$ with $a_1 + a_2 \neq 0$. Consequently, letting $a_1 = 1$, one obtains a group of N-wave solution sets of the $(2+1)$-dimensional KP equation (3.7) as

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + a_2 k_i y + \delta \sqrt{-3a_2} k_i^2 z - \frac{4a_2}{1+a_2} k_i^3 t}, \tag{3.11}$$

where $\delta \in \{-1, 1\}, N$ is an arbitrary integer, k_i and $\varepsilon_i, i = 1, \ldots, N$, are arbitrary constants.

3.3. *A* $(3+1)$*-dimensional generalized BKP equation and the* $(2+1)$*-dimensional BKP equation*

A generalized form of BKP equation was proposed and studied in Ref. 7 which is given by

$$u_{zt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0\,. \tag{3.12}$$

When $z = y$, (3.12) is reduced to the BKP equation

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0\,. \tag{3.13}$$

Through the dependent variable transformation $u = 2(\ln f)_x$, Eqs. (3.12) and (3.13) can be transformed into the Hirota bilinear equations

$$(D_z D_t - D_x^3 D_y + 3D_x^2) f \cdot f = 0 \tag{3.14}$$

and

$$(D_y D_t - D_x^3 D_y + 3D_x^2) f \cdot f = 0\,, \tag{3.15}$$

respectively.

For Eq. (3.14), we search for $a_i, i = 1,2,3,4$, and $n_i, i = 1,2,3,4$, such that

$$a_3 a_4 (x^{n_3} - y^{n_3})(x^{n_4} - y^{n_4}) - a_1^3 a_2 (x^{n_1} - y^{n_1})^3 (x^{n_2} - y^{n_2}) + 3a_1^2 (x^{n_1} - y^{n_1})^2 \equiv 0\,. \tag{3.16}$$

From the above equation, one can possibly obtain $a_i, i = 1,2,3,4$, if (n_1, n_2, n_3, n_4) is chosen as $(1,-1,-1,3)$ or $(1,-1,3,-1)$.

For the case when $(n_1, n_2, n_3, n_4) = (1,-1,-1,3)$, comparing the coefficients of (3.16), one obtains the following equations of $a_i, i = 1,2,3,4$:

$$\begin{cases} -a_3 a_4 + a_1^3 a_2 = 0\,, \\ -a_3 a_4 - 2a_1^3 a_2 + 3a_1^2 = 0\,. \end{cases} \tag{3.17}$$

Solving (3.17), we obtain $a_2 = 1/a_1, a_4 = a_1^2/a_3$, where a_1 and a_3 are arbitrary nonzero constants. Consequently, letting $a_1 = 1$, a set of N-wave solution sets for the $(3+1)$-dimensional generalized BKP equation (3.12) is obtained as

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + k_i^{-1} y + a_3 k_i^{-1} z + \frac{1}{a_3} k_i^3 t}, \tag{3.18}$$

where N is an arbitrary integer, k_i and $\varepsilon_i, i = 1,\ldots,N$, are arbitrary constants, which was obtained in Ref. 5.

However, for the case $(n_1, n_2, n_3, n_4) = (1,-1,3,-1)$, by similar analysis or the symmetry of the variables z and t, another set of solutions can be derived as

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + k_i^{-1} y + a_3 k_i^{-1} t + \frac{1}{a_3} k_i^3 z}. \tag{3.19}$$

Now for the Hirota bilinear equation (3.15), we search for $a_i, i = 1, 2, 3$, and $n_i, i = 1, 2, 3$, such that

$$a_2a_3(x^{n_2}-y^{n_2})(x^{n_3}-y^{n_3})-a_1^3a_2(x^{n_1}-y^{n_1})^3(x^{n_2}-y^{n_2})+3a_1^2(x^{n_1}-y^{n_1})^2\equiv 0. \tag{3.20}$$

We choose $(n_1, n_2, n_3) = (1, -1, 3)$ for possible solutions of $a_i, i = 1, 2, 3$ and comparing the coefficients of (3.20), one obtains the equations for $a_i, i = 1, 2, 3$, given by

$$\begin{cases} a_2a_3 - a_1^3a_2 = 0\,, \\ a_2a_3 + 2a_1^3a_2 - 3a_1^2 = 0\,. \end{cases} \tag{3.21}$$

Solving (3.21), by letting $a_1 = 1$, one obtains $a_2 = 1, a_3 = 1$. Consequently, we obtain a group of N-wave solution sets of the $(2 + 1)$-dimensional BKP equation (3.13) as

$$u = 2(\ln f)_x, \quad f = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + k_i^{-1} y + k_i^3 t}\,, \tag{3.22}$$

where N is an arbitrary integer, k_i and $\varepsilon_i, i = 1, \ldots, N$, are arbitrary constants.

4. Construction of Hirota Bilinear Equations Possessing *N*-wave Solutions

In this section, we apply Theorem 2.2 of Sec. 2 to systematize the algorithm proposed in Refs. 6 and 7 for constructing Hirota bilinear equations that possess N-wave solutions that could be formed by linear combinations of exponential waves. In Ref. 6, Ma and Fan used $(1, n_2, \ldots, n_M)$ as weights to construct multivariate polynomial $P(x_1, x_2, \ldots, x_M)$ of degree $2n$ to obtain the Hirota bilinear equations possessing N-wave solutions. Their main idea was to construct each term whose degree could be a combination of $1, n_2, \ldots, n_M$. For example, by weight $(1, 1, 2, 3)$, the terms of weight 4 can be $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$, where m_1, m_2, m_3, m_4 are nonnegative integers and $m_1 + m_2 + 2m_3 + 3m_4 = 4$. Stimulated by their work, and by using Theorem 2.2, we now systematize the algorithm to construct the Hirota bilinear equations of order $2n$ possessing N-wave solutions.

4.1. *A systematic algorithm to construct Hirota bilinear equations*

We perform the following three steps to construct Hirota bilinear equations (2.2) of order $2n$ possessing N-wave solutions which are any linear combinations of $e^{\eta_1}, \ldots, e^{\eta_N}$, where

$$\eta_i = a_1k_i^{n_1}x_1 + a_2k_i^{n_2}x_2 + \cdots + a_Mk_i^{n_M}x_M, \quad 1 \le i \le N\,, \tag{4.1}$$

$n_1, n_2, \ldots, n_M$, are integers possessing no nontrivial common factor and $k_is, 1 \le i \le N$, can be arbitrary constants. Here $n_1, n_2, \ldots, n_M$ are supposed to be positive in order to obtain the Hirota bilinear equations (2.2) which satisfy the dispersion relation. Without loss of generality, we suppose that $n_1 \leq n_2 \leq \cdots \leq n_M$.

Step 1: Suppose that the multivariate polynomials of weight W are given by

$$P_{2n}^W(x_1, x_2, \ldots, x_M) = \sum_{\substack{n_1m_1+n_2m_2+\cdots+n_Mm_M=W \\ m_1+m_2+\cdots+m_M=2i\,(1\leq i\leq n)}} a_{m_1m_2\cdots m_M}^W x_1^{m_1} x_2^{m_2} \cdots x_M^{m_M}\,, \tag{4.2}$$

where $a_{m_1m_2\cdots m_M}^W$ are undetermined constants, $n_1m_1 + n_2m_2 + \cdots + n_Mm_M = W$, $m_1 + m_2 + \cdots + m_M = 2i, 1 \leq i \leq n$ and $2 \leq W \leq 2nm_M$.

Step 2: Determine $a_{m_1m_2\cdots m_M}^W$ for each possible weight W.

As we discussed in Sec. 2, the multivariate polynomials $P_{2n}^W(x_1, x_2, \ldots, x_M)$, corresponding to Hirota bilinear equations (2.2) possess N-wave solutions which are any linear combinations of $e^{\eta_1}, \ldots, e^{\eta_N}$, if and only if

$$P_{2n}^W(a_1(k_i^{n_1} - k_j^{n_1}), a_2(k_i^{n_2} - k_j^{n_2}), \ldots, a_M(k_i^{n_M} - k_j^{n_M})) = 0\,. \tag{4.3}$$

It is easy to see that the left-hand side of (4.3) is a homogeneous polynomial of k_i and k_j of order W. Comparing the coefficients of (4.3), we obtain the equations $a_{m_1m_2\cdots m_M}^W$, $n_1m_1+n_2m_2+\cdots+n_Mm_M = W$, $m_1+m_2+\cdots+m_M = 2i, 1 \leq i \leq n$, from which $a_{m_1m_2\cdots m_M}^W$ are determined.

Step 3: Make linear combination of all the P_{2n}^W. Let

$$P_{2n}(x_1, x_2, \ldots, x_M) = \sum_W \varepsilon_W P_{2n}^W(x_1, x_2, \ldots, x_M)\,, \tag{4.4}$$

then we obtain the Hirota bilinear equations (2.2) of order $2n$ defined by the polynomial $P_{2n}(x_1, x_2, \ldots, x_M)$.

4.2. *Hirota bilinear equations of degree* 4 *and weights* $(1, 2, 2, 3)$

As an example of our refined algorithm, we follow these three steps to construct Hirota bilinear equations of degree 4 and weights $(1, 2, 2, 3)$ in this subsection.

(1) By simple analysis, one knows that the possible weight W should be in the set $\{W : 3 \leq W \leq 11, W \text{ is integer}\}$. Next, we study each possible case of the weight W one by one. To compare with the results of Example 2 in Ref. 7, we study the case of weight 4 first.

(1.1) The case of weight 4

The general form of the polynomial $P_4^4(x_1, x_2, x_3, x_4)$ of degree 4 and weight 4 is given by

$$P_4^4 = c_1x_1^4 + c_4x_1x_4 + c_5x_2^2 + c_6x_2x_3 + c_7x_3^2\,. \tag{4.5}$$

Then

$$P_4^4(a_1(k_i - k_j),\ \ a_2(k_i^2 - k_j^2),\ \ a_3(k_i^2 - k_j^2),\ \ a_4(k_i^3 - k_j^3)) = 0\,. \tag{4.6}$$

Comparing the coefficients of the same orders of $k_{1,i}$ and $k_{1,j}$ on the left-hand side of (3.5) and letting $a_1 = 1$, one obtains

$$\begin{cases} c_4 a_4 + 4c_1 = 0\,, \\ c_6 a_2 a_3 + c_5 a_2^2 + c_7 a_3^2 = 3c_1\,, \end{cases} \tag{4.7}$$

which are equal to the first and third equations of (4.21) in Ref. 7 with $c_2 = c_3 = 0$. Consequently, any linear combinations of $e^{\eta_1}, \ldots, e^{\eta_N}$ solves

$$(c_1 D_{x_1}^4 + c_4 D_{x_1} D_{x_4} + c_5 D_{x_2}^2 + c_6 D_{x_2} D_{x_3} + c_7 D_{x_3}^2) f \cdot f = 0 \tag{4.8}$$

if $a_2, a_3, a_4, c_1, c_4, c_5, c_6, c_7$ satisfy (3.6), where

$$\eta_i = k_i x_1 + a_2 k_i^2 x_2 + a_3 k_i^2 x_3 + a_4 k_i^3 x_4, \quad 1 \le i \le N\,. \tag{4.9}$$

Here N is any integer and $k_i, 1 \le i \le N$, are any arbitrary numbers (any two of them are usually chosen to be different). Note that (4.7) are linear algebraic equations of c_1, c_4, c_5, c_6, c_7, and have nonzero solutions for any a_2, a_3 and a_4. This means that for any given a_2, a_3 and a_4, we can construct the Hirota bilinear equations having the subset $\mathrm{span}\{e^{\eta_1}, \ldots, e^{\eta_N}\}$ in their N-wave solution sets.

On the other hand, for any given c_1, c_4, c_5, c_6 and c_7, i.e., for a given Hirota bilinear equation (4.8), if there exist a_2, a_3 and a_4 which solve (3.6), then $\mathrm{span}\{e^{\eta_1}, \ldots, e^{\eta_N}\}$ is a subset of the solution set of (4.8). By careful analysis of the algebraic equation (4.7), we obtain the sufficient conditions for c_1, c_4, c_5, c_6 and c_7 to make sure that there exist real a_2, a_3 and a_4 which solve (4.7). Thus, one obtains some sufficient conditions for Hirota bilinear equation having N-wave solutions. For details, see Ref. 7.

Here, we point out that the two terms x^2y and x^2z are not included in (4.5) because the degrees of these terms are odd numbers, that is to say that only the case $c_2 = c_3 = 0$ of the Example 2 in Ref. 7 is meaningful. For the Example 3 in Ref. 7, following the same analysis as above, the two terms x^2z and y^2z need not be considered, that is, only the case $c_6 = c_7 = 0$ of the Example 3 in Ref. 7 is meaningful.

(1.2) The case of weight 3

For the weights $(1, 2, 2, 3)$ the general form of the polynomial $P_4^3(x_1, x_2, x_3, x_4)$ of weight 3 is given by

$$P_4^3 = d_1^{(3)} x_1 x_2 + d_2^{(3)} x_1 x_3\,. \tag{4.10}$$

By the same analysis as above, we obtain

$$d_1^{(3)} a_2 + d_2^{(3)} a_3 = 0\,, \tag{4.11}$$

that is, any linear combination of $e^{\eta_1}, \ldots, e^{\eta_N}$ solves

$$(d_1^{(3)} D_{x_1} D_{x_2} + d_2^{(3)} D_{x_1} D_{x_3}) f \cdot f = 0 \tag{4.12}$$

if $d_1^{(3)}$ and $d_2^{(3)}$ satisfy (4.11), where $\eta_i s, 1 \leq i \leq N$, are defined by (4.9). Obviously, (4.11) is a linear equation in $d_1^{(3)}$ and $d_2^{(3)}$, so for any given a_2 and a_3, $d_1^{(3)}$ and $d_2^{(3)}$ can be obtained from (4.11).

(1.3) The case of weight 5

For the weights $(1, 2, 2, 3)$ the general form of the polynomial $P_4^5(x_1, x_2, x_3, x_4)$ of weight 5 is given by

$$P_4^5 = (d_1^{(5)} x_2 + d_2^{(5)} x_3)(k_1^{(5)} x_4 + k_2^{(5)} x_1^3) \,. \tag{4.13}$$

By the same analysis as above, we obtain

$$d_1^{(5)} a_2 + d_2^5 a_3 = 0 \tag{4.14}$$

with $k_1^{(5)}$ and $k_2^{(5)}$ being real numbers.

(1.4) The case of weight 6

The general form of the polynomial $P_4^6(x_1, x_2, x_3, x_4)$ of weight 6 is given by

$$P_4^6 = (d_1^{(6)} x_2^2 + d_2^{(6)} x_3^2 + d_3^{(6)} x_2 x_3) x_1^2 \,, \tag{4.15}$$

where $d_i^{(6)}, i = 1, 2, 3$, are determined by the equation

$$d_1^{(6)} a_2^2 + d_2^{(6)} a_3^2 + d_3^{(6)} a_2 a_3 = 0. \tag{4.16}$$

(1.5) The case of weight 7

The general form of the polynomial $P_4^7(x_1, x_2, x_3, x_4)$ of weight 7 is given by

$$P_4^7 = (d_1^{(7)} x_2 + d_2^{(7)} x_3) x_1^2 x_4 + (d_3^{(7)} x_2^3 + d_4^{(7)} x_3^3 + d_5^{(7)} x_2^2 x_3 + d_6^{(7)} x_2 x_3^2) x_1 \,, \tag{4.17}$$

where $d_i^{(7)}, i = 1, \ldots, 6$, are determined by the equations

$$\begin{cases} d_1^{(7)} a_2 + d_2^{(7)} a_3 = 0 \,, \\ d_3^{(7)} a_2^3 + d_4^{(7)} a_3^3 + d_5^{(7)} a_2^2 a_3 + d_6^{(7)} a_2 a_3^2 = 0 \,. \end{cases} \tag{4.18}$$

(1.6) The case of weight 8

The general form of the polynomial $P_4^8(x_1, x_2, x_3, x_4)$ of weight 8 is given by

$$\begin{aligned} P_4^8 = {} & (d_1^{(8)} x_2^2 + d_2^{(8)} x_3^2 + d_3^{(8)} x_2 x_3) x_1 x_4 + d_4^{(8)} x_2^4 + d_5^{(8)} x_3^4 + d_6^{(8)} x_2^3 x_3 \\ & + d_7^{(8)} x_2^2 x_3^2 + d_8^{(8)} x_2 x_3^3 \,, \end{aligned} \tag{4.19}$$

where $d_i^{(8)}, i = 1, \ldots, 6$, are determined by the equations

$$\begin{cases} d_1^{(8)} a_2^2 + d_2^{(8)} a_3^2 + d_3^{(8)} a_2 a_3 = 0 \,, \\ d_4^{(8)} a_2^4 + d_5^{(8)} a_3^4 + d_6^{(8)} a_2^3 a_3 + d_7^{(8)} a_2^2 a_3^2 + d_8^{(8)} a_2 a_3^3 = 0 \,. \end{cases} \tag{4.20}$$

(1.7) The case of weight 9

The general form of the polynomial $P_4^9(x_1, x_2, x_3, x_4)$ of weight 9 is given by

$$P_4^9 = (d_1^{(9)} x_2 + d_2^{(9)} x_3) x_1 x_4^2 + (d_3^{(9)} x_2^3 + d_4^{(9)} x_3^3 + d_5^{(9)} x_2^2 x_3 + d_6^{(9)} x_2 x_3^2) x_4 \,, \tag{4.21}$$

where $d_i^{(9)}, i = 1, \ldots, 6$, are determined by the equations

$$\begin{cases} d_1^{(9)} a_2 + d_2^{(9)} a_3 = 0 \,, \\ d_3^{(9)} a_2^3 + d_4^{(9)} a_3^3 + d_5^{(9)} a_2^2 a_3 + d_6^{(9)} a_2 a_3^2 = 0 \,. \end{cases} \tag{4.22}$$

(1.8) The case of weight 10

The general form of the polynomial $P_4^{10}(x_1, x_2, x_3, x_4)$ of weight 10 is given by

$$P_4^{10} = (d_1^{(10)} x_2^2 + d_2^{(10)} x_3^2 + d_3^{(10)} x_2 x_3) x_4^2 \,, \tag{4.23}$$

where $d_i^{(10)}, i = 1, 2, 3$, are determined by the equation

$$d_1^{(10)} a_2^2 + d_2^{(10)} a_3^2 + d_3^{(10)} a_2 a_3 = 0 \,. \tag{4.24}$$

(1.9) The case of weight 11

The general form of the polynomial $P_4^{11}(x_1, x_2, x_3, x_4)$ of weight 11 is given by

$$P_4^{11} = (d_1^{(11)} x_2 + d_2^{(11)} x_3) x_4^3 \,, \tag{4.25}$$

where $d_i^{(11)}, i = 1, 2$, are determined by the equation

$$d_1^{(11)} a_2 + d_2^{(11)} a_3 = 0 \,. \tag{4.26}$$

(2) Make linear combination of all the $P_4^W, W = 3, \ldots, 11$

Let

$$P_4(x_1, x_2, x_3, x_4) = \sum_{W=3}^{11} \varepsilon_W P_4^W (x_1, x_2, x_3, x_4) \,, \tag{4.27}$$

where $P_4^W, W = 3, \ldots, 11$, are defined by (4.10), (4.5), (4.13), (4.15), (4.23) and (4.25) respectively and $\varepsilon_W, W = 3, \ldots, 11$, are any arbitrary constants. Then we obtain a very general Hirota bilinear equations (2.2) of order 4 defined by the polynomial (4.27), which is a Hirota bilinear equation of order 4 possessing $\text{span}\{e^{\eta_1}, \ldots, e^{\eta_N}\}$ as a subset of its solution set.

4.3. *Hirota bilinear equations not satisfying dispersion relation*

The Hirota bilinear equations constructed in Sec. 4.2 satisfy the dispersion relation. In fact, this algorithm is also applied in constructing Hirota bilinear equations not satisfying dispersion relation. As an example, we construct Hirota bilinear equations of degree 4 with weights $(1, -1, -1, 3)$. Here, to save space, we just show the construction of the Hirota bilinear equations of order 4 with weight 4.

The general form of the polynomial $P_4^4(x_1, x_2, x_3, x_4)$ of degree 4 and weight 4 is given by

$$P_4^4 = [c_1 x_1 + (c_2 x_2 + c_3 x_3) x_1^2] x_4 + (c_4 x_2^2 + c_5 x_3^2 + c_6 x_2 x_3) x_4^2 + c_7 x_1^4 \,. \quad (4.28)$$

Then for any given $a_i \neq 0, 1 \leq i \leq M, \eta_i, 1 \leq i \leq N$, by the same analysis as in Sec. 4.2, we obtain

$$\begin{aligned} &a_4[c_1 a_1 (k_i - k_j) + (c_2 a_2 + c_3 a_3) a_1^2 (k_i^{-1} - k_j^{-1})(k_i - k_j)^2](k_i^3 - k_j^3) \\ &\quad + (c_4 a_2^2 + c_5 a_3^2 + c_6 a_2 a_3) a_4^2 (k_i^{-1} - k_j^{-1})^2 (k_i^3 - k_j^3)^2 + c_7 a_1^4 (k_i - k_j)^4 = 0 \,. \end{aligned} \quad (4.29)$$

Clearly, (4.29) holds for any arbitrary k_i and k_j if and only if

$$\begin{cases} (c_4 a_2^2 + c_5 a_3^2 + c_6 a_2 a_3) a_4^2 = 0 \,, \\ c_7 a_1^4 = 0 \,, \\ (c_2 a_2 + c_3 a_3) a_4 = 0 \,, \\ a_4 c_1 a_1 = 0 \,. \end{cases} \quad (4.30)$$

From (4.30), we conclude that for $c_1 = c_7 = 0$ and any $c_i, 2 \leq i \leq 6$ satisfying

$$\begin{cases} c_4 a_2^2 + c_5 a_3^2 + c_6 a_2 a_3 = 0 \,, \\ c_2 a_2 + c_3 a_3 = 0 \,, \end{cases} \quad (4.31)$$

the polynomial P_4^4 defined by (4.28), i.e.,

$$(c_2 D_{x_1}^2 D_{x_2} D_{x_4} + c_3 D_{x_1}^2 D_{x_3} D_{x_4} + c_4 D_{x_2}^2 D_{x_4}^2 + c_5 D_{x_3}^2 D_{x_4}^2 + c_6 D_{x_2} D_{x_3} D_{x_4}^2) f \cdot f = 0 \quad (4.32)$$

determines a Hirota bilinear equation possessing N-wave subspace. For other cases, one can obtain the Hirota bilinear equations possessing N-wave subspace by the same process. We omit these here.

5. Conclusions

In this paper, we have shown that the linear superposition principle which does not generally apply to nonlinear equations might apply to some kind of exponential wave solutions of Hirota bilinear equation. The necessary and sufficient conditions that guarantee the existence of the exponential wave solution subspaces were presented. Based on the necessary and sufficient conditions, an algorithm to construct the Hirota bilinear equations possessing exponential wave solution subspaces were obtained.

A natural question is whether there are any other kinds of nonlinear equations to which the linear superposition principle applies. In fact, Ma[8] has given a positive answer by presenting a generalized Hirota bilinear equations. According to the results obtained in this paper and the conclusions in Refs. 6–10, it is reasonable to believe that there are classes of nonlinear equations that the linear superposition principle can apply to the subsets of their wave solutions which may be other kinds of solutions besides exponential wave solutions. However, it remains an open question what are the characteristics of these bilinear equations possessing subspaces.

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References

1. R. Hirota, *The Direct Method in Solution Theory* (Cambridge University Press, Cambridge, 2004).
2. J. Hietarinta, *Phys. AUC* **15**, 31 (2005).
3. E. G. Khamis and A. Gammal, *Phys. Lett. A* **376**, 2422 (2012).
4. A. M. Wazwaz, *Appl. Math. Comput.* **190**, 633 (2007).
5. A. M. Wazwaz, *Appl. Math. Comput.* **204**, 227 (2008).
6. W. X. Ma *et al.*, *Appl. Math. Comput.* **218**, 7174 (2012).
7. W. X. Ma and E. Fan, *Comput. Math. Appl.* **61**, 950 (2011).
8. W. X. Ma, *Stud. Nonlinear Sci.* **2**(4), 140 (2011).
9. W. X. Ma, *Rep. Math. Phys.* **72**, 41 (2013).
10. W. X. Ma, *Front. Math. China* **8**, 1139 (2013).
11. Z. Zhou, J. Fu and Z. Li, *Appl. Math. Comput.* **183**, 872 (2006).
12. Z. Zhou, J. Fu and Z. Li, *Appl. Math. Comput.* **217**(1), 92 (2010).
13. X. Yang and H. Ruan, *Commun. Theor. Phys.* **52**, 801 (2009).
14. J. Hietarinta, *J. Math. Phys.* **28**, 2094 (1987).
15. J. Hietarinta, *J. Math. Phys.* **28**, 2586 (1987).
16. Q. Miao *et al.*, *Compt. Phys. Comm.* **185**(1), 357 (2014).
17. X. Lu and J. Li, *Nonlinear Dyn.* **77**, 135 (2014).
18. Y. Zhang and Y. Song *et al.*, *Nonlinear Dyn.* **68**, 445 (2012).
19. Y. Ye *et al.*, *Appl. Math. Comput.* **218**, 2200 (2011).
20. L. Zhang and C. M. Khalique, *Abs. Appl. Anal.* **2014**, 963852 (2014).
21. L. Luo and E. Fan, *Phys. Lett. A* **374**, 3001 (2010).