A τ-Symmetry Algebra of the Generalized Derivative Nonlinear Schrödinger Soliton Hierarchy with an Arbitrary Parameter

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Abstract: A matrix spectral problem is researched with an arbitrary parameter. Through zero curvature equations, two hierarchies are constructed of isospectral and nonisospectral generalized derivative nonlinear schrödinger equations. The resulting hierarchies include the Kaup-Newell equation, the Chen-Lee-Liu equation, the Gerdjikov-Ivanov equation, the modified Korteweg-de Vries equation, the Sharma-Tasso-Olever equation and a new equation as special reductions. The integro-differential operator related to the isospectral and nonisospectral hierarchies is shown to be not only a hereditary but also a strong symmetry of the whole isospectral hierarchy. For the isospectral hierarchy, the corresponding τ-symmetries are generated from the nonisospectral hierarchy and form an infinite-dimensional symmetry algebra with the *K*-symmetries.

Keywords: generalized derivative nonlinear shcrödinger equation; master symmetry; infinite-dimensional Lie algebra

1. Introduction

As we all know, from a matrix spectral problem, we can obtain isospectral and nonisospectral soliton hierarchies [1,2]. The inverse scattering transformation is one of most powerful tools for solving these equations [1,2]. Furthermore, these equations are integrable and they have a lot of integrable characteristics, including the existence of conservation laws (CLs) and infinitely many symmetries [3].

There are many approaches to seek the CLs for $(1 + 1)$-dimensional integrable systems, such as the method based on the non-semisimple Lie algebras to formulate generating functions for conserved densities by the variational identities [4,5], utilizing adjoint symmetries [6,7] and expanding the ratios of eigenfunctions of matrix spectral problems [8,9]. Lax pairs have been used in generating CLs for various evolution equations [8–12]. How to obtain a conservation density is very important from the spectral problem of Lax pairs. Actually, CLs can be derived by taking advantage of the generating conservation density and evolution equation of time [13].

It is another important integrable property that an evolution equation owns *K*-symmetries. We remark that these symmetries do not depend explicitly on space and time variables. Li and Zhu found the τ-symmetries which depend explicitly on space variable by a general method in 1987 [14,15]. They further discovered that these symmetries form a infinite Lie algebra along with *K*-symmetries.

It is interesting that, regarding τ-symmetries as vector fields, we can also generate new symmetries for the evolution equations [16,17]. In fact, all these τ-symmetries are produced from the first degree generators [18]. Later, based on the previous work, a more generalized framework was established on K-symmetries and τ-symmetries of the evolution equations [19,20] by one of the authors (Ma). At the same time, the infinite-dimensional symmetry Lie algebra was also discussed. In last few years, symmetries of discrete soliton hierarchies were also discussed [21–24].

As we know, there are three famous derivative nonlinear schrödinger (DNLS) equations including the Kaup-Newell (KN) equation, the Chen-Lee-Liu (CLL) equation and the Gerdjikov-Ivanov (GI) equation [25–29]. Starting from a matrix spectral problem with an arbitrary parameter, we generate two hierarchies of generalized derivative nonlinear schrödinger (gDNLS) equations including isospectral and nonisospectral hierarchies. The famous three isospectral and nonisospectral DNLS equations mentioned above are deduced by selecting special values of the arbitrary parameter. We show that the nonisospectral gDNLS hierarchy with an arbitrary parameter is the τ-symmetries for the isospectral gDNLS hierarchy with the same parameter. Finally, we prove that the resulting τ-symmetries constitute an infininte-dimensional Lie algebra with K-symmetries.

The paper is organized as follows. In Section 2, the basic notions and notations are discussed. In Section 3, two hierarchies of the isospectral and nonisospectral gDNLS hierarchies with an arbitrary parameter are presented. In Section 4, the K-symmetries and τ-symmetries are deduced, and the two kinds of symmetries form an infinite-dimensional Lie algebra. We conclude the paper in Section 5.

2. Basic Notions

First, we give some basic notions and notations used in this paper [19,24]. Let $\mathscr{L}$ be the set of differential vector functions which map $\mathbb{R}^N \times \mathbb{R} \times \mathcal{L}$ into $\mathcal{L}$, where $\mathbb{R}$ and $\mathbb{C}$ are the real and complex fields, respectively, and $\mathcal{L}$ is one linear topological space over $\mathbb{C}$.

Definition 1. *Let* $K = K(u) = K(x,t,u), S = S(u) = S(x,t,u) \in \mathscr{L}$. *We define*

$$K'[S] = K'(u)[S(u)] = \frac{\partial}{\partial \varepsilon} K(u + \varepsilon S(u))|_{\varepsilon=0} \tag{1}$$

is the Gateaux derivative of $K(u)$ *in the direction* $S(u)$ *with respect to u, where* $K(u) = K(x,t,u)$ *means u is a function of variables* x,t *and* $K(x,t,u)$ *is a certain function of* x,t,u *and its x derivatives, but we abbreviate* $K(x,t,u)$ *as* $K(u)$ *to emphasize u and its x derivatives.*

As we know, with respect to the following product:

$$[\![K,S]\!] = [\![K(u),S(u)]\!] = K'(u)[S(u)] - S'(u)[K(u)] \quad K,S \in \mathscr{L}, \tag{2}$$

$\mathscr{L}$ forms a Lie algebra. Suppose that the differentiable function or a differential vector function $u = u(x,t)$ map $\mathbb{R}^N \times \mathbb{R}$ into $\mathcal{L}$. We take into account the following evolution equation

$$u_t = K(x,t,u) \quad K \in \mathscr{L}. \tag{3}$$

Definition 2. *If a function* $G = G(x,t,u) \in \mathscr{L}$ *satisfies the linearised Equation of* (3)

$$\frac{\mathrm{d}G}{\mathrm{d}t} = K'(u)[G], \tag{4}$$

we call it a symmetry of Equation (3)*, where* d/dt *is the total t-derivative, u is a solution of Equation* (3) *and* $K'(u)[G]$ *is a Gateaux derivative of* $K(u)$ *in the direction G with respect to u.*

Simple calculations show that the equation

$$\frac{\partial G}{\partial t} = [\![K, G]\!]$$ (5)

is equivalent to the linearised Equation (4), where $[\![,]\!]$ is defined as in Equation (2).

Denote the linear operators mapping $\mathscr{L}$ into itself by $L(\mathscr{L})$. Furthermore, we denote all differentiable operators that map $\mathbb{R}^n \times \mathbb{R} \times \mathscr{L}$ into $L(\mathscr{L})$ by $\mathscr{U}$. At the same time, we suppose $\Phi K = \Phi(x, t, u)K$ for $\Phi \in \mathscr{U}, K \in \mathscr{L}, (x, t) \in \mathbb{R}^N \times \mathbb{R}, u \in \mathscr{L}$.

Definition 3. *The Lie derivative $L_K\Phi \in \mathscr{U}$ of Φ with respect to K is*

$$(L_K\Phi) = \Phi'[K] - K'\Phi + \Phi K',$$

where $\Phi \in \mathscr{U}, K \in \mathscr{L}$ and $\Phi'[K]$ is the Gateaux derivative of the operator $\Phi(u)$ in the direction K with respect to u.

Definition 4. *If an operator $\Phi \in \mathscr{U}$ satisfies the following relation*

$$\Phi'[\Phi K]S - \Phi'[\Phi S]K = \Phi(\Phi'[K]S - \Phi'[S]K) \quad K, S \in \mathscr{L},$$

then Φ is called a hereditary symmetry.

Definition 5. *If an operator $\Phi \in \mathscr{U}$ maps one symmetry of Equation (3) into another symmetry of Equation (3), then Φ is a strong symmetry of Equation (3).*

It is easy to verify that the relation

$$\frac{\partial \Phi}{\partial t} + L_K\Phi = 0$$

is equivalent to $\Phi \in \mathscr{U}$ is a strong symmetry of Equation (3).

3. Isospectral and Nonisospcetral Hierarchies of the gDNLS Equations

In this section, from a matrix spectral problem with arbitrary parameter, we constitute the isospectral and nonisospectral gDNLS hierarchies.

Suppose that T is the transpose of a matrix and

$$\varsigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

are the three Pauli matrices. Let us consider the following Lax pair [30,31]

$$\phi_x = M\phi, \quad M = \begin{pmatrix} -\frac{1}{2}(\eta^2 - \beta qr) & \eta q \\ \eta r & \frac{1}{2}(\eta^2 - \beta qr) \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

and its time evolution

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$

where η is a spectral parameter and $q = q(x, t), r = r(x, t)$ are potential functions. We assume that $q(x, t)$ and $r(x, t)$ are smooth functions of variables t and x; and their derivatives of any order with respect to x vanish rapidly as $x \to \infty$. The compatibility condition, the zero curvature equation, reads

$$M_t - N_x + [M, N] = 0,$$

which yields

$$\frac{1}{2}\beta(qr)_t = A_x + \eta(rB - qC) + \eta\eta_t, \tag{6a}$$

$$\eta q_t = B_x + 2\eta q A + (\eta^2 - \beta qr)B - \eta_t q, \tag{6b}$$

$$\eta r_t = C_x - 2\eta r A - (\eta^2 - \beta qr)C - \eta_t r. \tag{6c}$$

Multiplying Equations (6b) and (6c) by βr and βq, respectively, and adding the results together, we get

$$\eta\beta(qr)_t = \beta r B_x + \beta q C_x + \beta(\eta^2 - \beta qr)(rB - qC) - 2\beta qr\eta_t. \tag{7}$$

Substituting Equation (7) into Equation (6a) multiplied by 2η, and then integrating both sides, we have

$$2\eta A = 2\eta A_0 - \beta\partial^{-1}(r, q)(\varsigma\partial + \beta qre)\begin{pmatrix} B \\ -C \end{pmatrix} - (2 - \beta)\partial^{-1}(r, q)\begin{pmatrix} B \\ -C \end{pmatrix}\eta^2 - 2(\beta\partial^{-1}qr + \eta^2 x)\eta_t, \tag{8}$$

where A_0 is a integration constant. Finally, we obtain

$$\eta\begin{pmatrix} q \\ r \end{pmatrix}_t = L_1 L_2 \begin{pmatrix} B \\ -C \end{pmatrix} + \eta^2 L_3 \begin{pmatrix} B \\ -C \end{pmatrix} + 2\eta A_0 \begin{pmatrix} q \\ -r \end{pmatrix} - \eta_t L_1 \begin{pmatrix} q \\ r \end{pmatrix}$$
$$- 2\eta^2\eta_t \begin{pmatrix} xq \\ -xr \end{pmatrix}, \tag{9}$$

from Equations (6b) and (6c) by using Equation (8), where

$$L_1 = e + \beta\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r, q), \quad L_2 = -(\varsigma\partial + \beta qre),$$

$$L_3 = e - (2 - \beta)\begin{pmatrix} q \\ -r \end{pmatrix}\partial^{-1}(r, q).$$

Setting $\eta_t = 0$, $A_0 = \frac{1}{2}(-1)^n\eta^{2n}$,

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{n}(-1)^{n-j}\begin{pmatrix} b_j \\ c_j \end{pmatrix}\eta^{2(n-j)+1},$$

and comparing the coefficients of the same power of η in Equation (9), we obtain

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L_1 L_2 \begin{pmatrix} b_n \\ -c_n \end{pmatrix}, \tag{10a}$$

$$\begin{pmatrix} b_1 \\ -c_1 \end{pmatrix} = L_3^{-1}\begin{pmatrix} q \\ -r \end{pmatrix}, \tag{10b}$$

$$\begin{pmatrix} b_{j+1} \\ -c_{j+1} \end{pmatrix} = L_3^{-1}L_1 L_2 \begin{pmatrix} b_j \\ -c_j \end{pmatrix}, \quad (j = 1, 2, \cdots, n - 1). \tag{10c}$$

Through simple iteration, we arrive at the isospectral flow

$$u_t = K_n = \Phi^n \begin{pmatrix} q \\ -r \end{pmatrix},$$ (11)

where n is a positive integer and

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$ (12)

with

$$\Phi_{11} = \partial + (2 - \beta)qr + (2 - \beta)q_x\partial^{-1}r - \beta q\partial^{-1}r_x + 2\beta(1 - \beta)q\partial^{-1}r^2q,$$
$$\Phi_{12} = (2 - \beta)q_x\partial^{-1}q + \beta q\partial^{-1}q_x + 2(1 - \beta)q^2 + 2\beta(1 - \beta)q\partial^{-1}q^2r,$$
$$\Phi_{21} = (2 - \beta)r_x\partial^{-1}r + \beta r\partial^{-1}r_x + 2(1 - \beta)r^2 - 2\beta(1 - \beta)r\partial^{-1}r^2q,$$
$$\Phi_{22} = -\partial + (2 - \beta)qr + (2 - \beta)r_x\partial^{-1}q - \beta r\partial^{-1}q_x - 2\beta(1 - \beta)r\partial^{-1}q^2r.$$

Taking $u^T = (q, r)$, we can express Φ in vector forms

$$\Phi = -\varsigma\partial + \frac{1}{2}\beta u^T\delta ue + (2 - \beta)u_x\partial^{-1}u^T\delta + \beta\varsigma u\partial^{-1}u_x^T\varsigma\delta + 2(1 - \beta)uu^T\delta$$
$$+ \beta(\beta - 1)\varsigma u\partial^{-1}u^T\delta u^T\delta u.$$

Analogously, we can obtain the nonisospectral flow

$$u_t = \sigma_n = \Phi^{n-1} \begin{pmatrix} xq_x + \frac{1}{2}q \\ xr_x + \frac{1}{2}r \end{pmatrix},$$ (13)

by taking $\eta_t = \frac{1}{2}(-1)^{n-1}\eta^{2n-1}$, $A_0 = 0$.

The system in Equation (11) is integrable. It has infinite conservation laws and Hamilton structure. We prove that it is integrable in Liouville sense and gives the relation between infinite conservation laws and the Hamilton structure somewhere else.

The systems in Equations (11) and (13) are generalized systems. Many celebrated integrable systems can be derived from them by selecting different values of the arbitrary parameter β.

For example, when $n = 2$, (11) becomes

$$q_t = q_{xx} + 2(2 - \beta)qrq_x + 2(1 - \beta)q^2r_x + \beta(1 - \beta)q^3r^2,$$
$$r_t = -r_{xx} + 2(2 - \beta)qrr_x + 2(1 - \beta)r^2q_x - \beta(1 - \beta)r^3q^2.$$ (14)

We can deduce the three celebrated isospectral DNLS equations from it (Equation (14) has n-soliton and Wronski solutions with an arbitrary parameter, which will be shown in another paper).

- The isospectral Kaup–Newell (KN) equation

$$q_t = q_{xx} + 2(q^2r)_x,$$
$$r_t = -r_{xx} + 2(qr^2)_x,$$ (15)

  with $\beta = 0$.

- The isospectal Chen–Lee–Liu (CLL) equation

$$q_t = q_{xx} + 2qrq_x,$$
$$r_t = -r_{xx} + 2qrr_x,$$ (16)

  with $\beta = 1$.

- The isospectral Gerdjikov–Ivanov (GI) equation

$$
\begin{aligned}
q_t &= q_{xx} - 2q^2 r_x - 2q^3 r^2, \\
r_t &= -r_{xx} - 2r^2 q_x + 2q^2 r^3,
\end{aligned}
\tag{17}
$$

with $\beta = 2$.

When $n = 2$, Equation (13) becomes

$$
\begin{aligned}
q_t =& xq_{xx} + 2(2 - \beta)xqrq_x + \frac{3}{2}q_x + \beta q \partial^{-1} q_x r \\
&+ 2(1 - \beta)(xr_x + r)q^2 + \beta(1 - \beta)(q^3 r^2 x + q \partial^{-1} q^2 r^2), \\
r_t =& -xr_{xx} + 2(2 - \beta)xqrr_x - \frac{3}{2}r_x + \beta r \partial^{-1} r_x q \\
&+ 2(1 - \beta)(xq_x + q)r^2 - \beta(1 - \beta)(r^3 q^2 x + r \partial^{-1} q^2 r^2).
\end{aligned}
\tag{18}
$$

We can also deduce the three celebrated nonisospectral DNLS equations. For example,

- The nonisospectral KN equation

$$
\begin{aligned}
q_t &= xq_{xx} + 2x(q^2 r)_x + \frac{3}{2}q_x + 2q^2 r, \\
r_t &= -xr_{xx} + 2x(qr^2)_x - \frac{3}{2}r_x + 2qr^2,
\end{aligned}
$$

with $\beta = 0$.

- The nonisospectral CLL equation

$$
\begin{aligned}
q_t &= xq_{xx} + 2xqrq_x + \frac{3}{2}q_x + q \partial^{-1} q_x r, \\
r_t &= -xr_{xx} + 2xqrr_x - \frac{3}{2}r_x + r \partial^{-1} qr_x,
\end{aligned}
$$

with $\beta = 1$.

- The nonisospectral GI equation

$$
\begin{aligned}
q_t &= xq_{xx} - 2xq^2 r_x - 2xq^3 r^2 + \frac{3}{2}q_x + 2q \partial^{-1} q_x r - 2q^2 r - 2q \partial^{-1} q^2 r^2, \\
r_t &= -xr_{xx} - 2xr^2 q_x + 2xq^2 r^3 - \frac{3}{2}r_x + 2r \partial^{-1} qr_x - 2qr^2 + 2r \partial^{-1} q^2 r^2,
\end{aligned}
$$

with $\beta = 2$.

When $n = 3$, Equation (11) becomes

$$
\begin{aligned}
q_t =& q_{xxx} + 3(2 - \beta)(q_x q)_x r + 6(1 - \beta)qq_x r_x + 3(6 - 6\beta + \beta^2)q^2 q_x r^2 \\
&+ 6(1 - \beta)(2 - \beta)q^3 rr_x + 2\beta(1 - \beta)(2 - \beta)q^4 r^3, \\
r_t =& r_{xxx} - 3(2 - \beta)q(rr_x)_x - 6(1 - \beta)q_x rr_x + 3(6 - 6\beta + \beta^2)q^2 r^2 r_x \\
&+ 6(1 - \beta)(2 - \beta)qq_x r^3 - 2\beta(1 - \beta)(2 - \beta)q^3 r^4.
\end{aligned}
\tag{19}
$$

Through taking the different values of β and r, we can get the following different types of equations.

- The modified Korteweg-de Vries (mKdV) equation

$$
q_t = q_{xxx} - 6q^2 q_x
$$

with $\beta = 2$ and $r = 1$.

- The Sharma-Tasso-Olever (STO) equation

$$q_t = q_{xxx} + 3(qq_x)_x + 3q^2 q_x$$

  with $\beta = 1$ and $r = 1$.

- A new equation

$$q_t = q_{xxx} - 6i|q|^2 q_{xx} - 6iq^* q_x^2 - 6iq|q_x|^2 - 18|q|^4 q_x - 12q^2|q|^2 q_x \tag{20}$$

  with $\beta = 0$ and $r = -iq$.

  As far as we known, Equation (20) is a new integrable equation with fifth-order nonlinear term. Since it is a special case of the system in Equation (11), it has infinite conservation laws and Hamilton Structure. It also has two symmetries and these symmetries have an infinite dimensional Lie algebra structure.

  If we replace η with $i\lambda$, as the process given in [32], we can restrict Equations (14) and (19) to nonlocal soliton systems. Recent research shows that these nonlocal soliton systems are not only integrable [33] but also can be solved by inverse scattering transformation [32,34].

4. A τ-Symmetry Algebra of the gDNLS Soliton Hierarchy

In this section, we prove that the obtained isospectral flow in Equation (11) and nonisospectral flow in Equation (13) are symmetries of Equation (11) and they have an infinite dimensional algebra structure. For the two types of symmetries related to the recursion operator Φ, firstly we prove the integro-differential operator Φ is not only a hereditary symmetry operator but also a strong symmetry operator.

Lemma 1. *For arbitrary* $f, g \in C^\infty$*, let*

$$
\begin{aligned}
w_1 =& 2(1-\beta)\varsigma(gu^T\delta f_x - fu^T\delta g_x) - \beta(g_x^T\varsigma\delta uf - f_x^T\varsigma\delta ug) + \beta\varsigma u(f_x^T\delta g - g_x^T\delta f) \\
& - (2-\beta)\varsigma(f_x u^T\delta g - g_x u^T\delta f) + 2(1-\beta)u(f_x^T\varsigma\delta g - g_x^T\varsigma\delta f).
\end{aligned}
$$

Then, we have $w_1 = 0$.

Proof. Simple calculations show that, for arbitrary $f, g, u \in C^\infty$, the following identity relations

$$
\begin{aligned}
u^T\varsigma\delta f_x &= -f_x^T\varsigma\delta u, \quad u^T\delta f_x = f_x^T\delta u, \quad fu_x^T\varsigma\delta g - gu_x^T\varsigma\delta f = \varsigma fu_x^T\delta g - \varsigma gu_x^T\delta f, \\
\varsigma f_x u^T\delta g &- \varsigma gu^T\delta f_x = uf_x^T\varsigma\delta g, \quad \varsigma gu^T\delta f_x - \varsigma uf_x^T\delta g = f_x^T\varsigma\delta ug - uf_x^T\varsigma\delta g,
\end{aligned}
$$

hold. Using the above relations, we conclude that $w_1 = 0$. □

Lemma 2. *For arbitrary* $f, g \in C^\infty$*, let*

$$w_3 = (2-\beta)^2 w_{35} + \beta(2-\beta)\left(w_{36} + \frac{1}{2}w_{33}\right) + 2\beta(1-\beta)(w_{34} + 3w_{32}) - 2\beta^2 w_{31},$$

where

$$
\begin{aligned}
w_{31} =& \varsigma u\partial^{-1} f^T \varsigma\delta gu^T\delta u_x, \quad w_{32} = \varsigma u\partial^{-1} u_x^T\varsigma\delta fu^T\delta g - \varsigma u\partial^{-1} u_x^T\varsigma\delta gu^T\delta f, \\
w_{33} =& u^T\delta uu^T\delta gf - u^T\delta uu^T\delta fg - \varsigma u\partial^{-1} f_x^T\varsigma\delta gu^T\delta u + \varsigma u\partial^{-1} g_x^T\varsigma\delta fu^T\delta u,
\end{aligned}
$$

$$w_{34} = \varsigma u \partial^{-1} u^T \delta f g_x^T \varsigma \delta u - \varsigma u \partial^{-1} u^T \delta g f_x^T \varsigma \delta u - \varsigma u \partial^{-1} u^T \varsigma \delta g u_x^T \delta f + \varsigma u \partial^{-1} u^T \varsigma \delta f u_x^T \delta g$$
$$+ f u^T \delta g u^T \delta u - g u^T \delta f u^T \delta u - \varsigma u \partial^{-1} u^T \varsigma \delta g u^T \delta f_x + \varsigma u \partial^{-1} u^T \varsigma \delta f u^T \delta g_x,$$
$$w_{35} = u_x \partial^{-1} u^T \delta f_x \partial^{-1} u^T \delta g - u_x \partial^{-1} u^T \delta g_x \partial^{-1} u^T \delta f - u_x u^T \delta f \partial^{-1} u^T \delta g$$
$$+ u_x u^T \delta g \partial^{-1} u^T \delta f - u_x \partial^{-1} u_x^T \delta g \partial^{-1} u^T \delta f + u_x \partial^{-1} u_x^T \delta f \partial^{-1} u^T \delta g,$$
$$w_{36} = \varsigma u \partial^{-1} u_x^T \varsigma \delta f_x \partial^{-1} u^T \delta g - \varsigma u \partial^{-1} u_x^T \varsigma \delta g_x \partial^{-1} u^T \delta f - \varsigma u u_x^T \varsigma \delta f \partial^{-1} u^T \delta g$$
$$+ \varsigma u u_x^T \varsigma \delta g \partial^{-1} u^T \delta f - \varsigma u \partial^{-1} u_{xx}^T \varsigma \delta g \partial^{-1} u^T \delta f + \varsigma u \partial^{-1} u_{xx}^T \varsigma \delta f \partial^{-1} u^T \delta g.$$

Then, we have $w_3 = 0$.

Proof. It is easy to verify that

$$w_{35} = 0, \quad \frac{1}{2} w_{33} = w_{34} = w_{36} = w_{31} = -w_{32}.$$

Through simple calculation, we conclude that $w_3 = 0$. $\quad \square$

Lemma 3. *For arbitrary $f, g \in C^\infty$, let*

$$w_5 = \beta(1 - \beta)(2 - \beta)(w_{51} + w_{52} + w_{53} + w_{54}),$$

where

$$w_{51} = \varsigma u (\partial^{-1} u_x^T \delta f u^T \delta u \partial^{-1} u^T \delta g - \partial^{-1} u_x^T \delta g u^T \delta u \partial^{-1} u^T \delta f),$$
$$w_{52} = 2 \varsigma u (\partial^{-1} u^T \delta f u_x^T \delta u \partial^{-1} u^T \delta g - \partial^{-1} u^T \delta g u_x^T \delta u \partial^{-1} u^T \delta f),$$
$$w_{53} = \varsigma u (\partial^{-1} u^T \delta g_x u^T \delta u \partial^{-1} u^T \delta f - \partial^{-1} u^T \delta f_x u^T \delta u \partial^{-1} u^T \delta g),$$
$$w_{54} = \varsigma u (u^T \delta g u^T \delta g u^T \delta u \partial^{-1} u^T \delta f - u^T \delta g u^T \delta f u^T \delta u \partial^{-1} u^T \delta g).$$

Then, we have $w_5 = 0$.

Proof. It is easy to verify that
$$w_{51} + w_{52} + w_{53} + w_{54} = 0,$$

and so $w_5 = 0$. $\quad \square$

Theorem 1. *The integro-differential operator Φ is hereditary.*

Proof. Noticing that $u^T \varsigma \delta u = 0$, we have

$$\Phi(\Phi'[f]g - \Phi'[g]f) - (\Phi'[\Phi f]g - \Phi'[\Phi g]f) = w_1 + w_3 + w_5$$

with arbitrary $f, g \in C^\infty$. By Lemmas 1, 2 and 3, we know that

$$\Phi(\Phi'[f]g - \Phi'[g]f) = (\Phi'[\Phi f]g - \Phi'[\Phi g]f).$$

$\square$

The hereditary operator Φ ensures that the properties of strong operators can be inherited from one equation to the next equation in Equation (11), i.e., if Φ is a strong symmetry operator of the first equation in the system in Equation (11), it is also the strong symmetry operator of the every equation in the system in Equation (11).

Lemma 4. *The integro-differential operator* Φ *satisfies*

$$\Phi'[K_m] = [K'_m, \Phi] \tag{21}$$

and it is a strong symmetry of the gDNLS soliton hierarchy in Equation (11).

Proof. It is easy to verify

$$\begin{aligned}
\Phi'[-\varsigma u] =& (-\varsigma u)' \Phi - \Phi(-\varsigma u)' \\
=& \beta(1-\beta)\varsigma u \partial^{-1} u^T \varsigma \delta u^T \delta u - (2-\beta)\varsigma u_x \partial^{-1} u^T \delta - 2(1-\beta)u u^T \varsigma \delta \\
& - (2-\beta)u_x \partial^{-1} u^T \varsigma \delta - \beta u \partial^{-1} u_x^T \varsigma \delta - \beta \varsigma u \partial^{-1} u_x^T \delta - 2(1-\beta)\varsigma u u^T \delta \\
& - \beta(\beta-1)u \partial^{-1} u^T \delta u^T \delta u.
\end{aligned}$$

The above equality and Theorem 1 show that Φ is a hereditary and strong symmetry operator for the equation $u_t = K_0$. Thus, Φ is a strong symmetry operator for $u_t = \Phi^m K_0 = K_m$ and the proof has been complete. $\square$

Lemma 5.

$$\Phi'[\sigma_n] + \Phi \sigma'_n - \sigma'_n \Phi = \Phi^n, \quad n = 1, 2, \cdots. \tag{22}$$

Proof. When $n = 1$, we have

$$\Phi'[\sigma_1] + \Phi \sigma'_1 - \sigma'_1 \Phi = \Phi$$

through simple calculation. Assume that

$$\Phi'[\sigma_m] + \Phi \sigma'_m - \sigma'_m \Phi = \Phi^m,$$

we have

$$\begin{aligned}
& (\Phi^{m+1} - \Phi \sigma'_{m+1} + \sigma'_{m+1} \Phi) f \\
=& \Phi^{m+1} f - \Phi(\Phi \sigma_m)' f + (\Phi \sigma_m)' \Phi f \\
=& \Phi^{m+1} f - \Phi(\Phi'[f]\sigma_m + \Phi \sigma'_m[f]) + \Phi'[\Phi f]\sigma_m + \Phi \sigma'_m[\Phi f] \\
=& \Phi^{m+1} f - \Phi \Phi'[f]\sigma_m - \Phi(\sigma'_m \Phi + \Phi^m - \Phi'[\sigma_m])f + \Phi'[\Phi f]\sigma_m + \Phi \sigma'_m[\Phi f] \\
=& \Phi \Phi'[\sigma_m] f + \Phi'[\Phi f]\sigma_m - \Phi \Phi'[f]\sigma_m \\
=& \Phi'[\Phi \sigma_m] f \\
=& \Phi'[\sigma_{m+1}] f,
\end{aligned}$$

where the fact that Φ is a hereditary symmetry operator has been used. Thus, Equation (22) holds for any positive integer n. $\square$

Lemma 5 gives a relation between the recursion operator Φ and τ-symmetries. It is very important in the following computation.

Theorem 2. *The two flows K_m in Equation* (11) *and σ_m in Equation* (13) *deduced from the matrix spectral problem of the gDNLS soliton equation satisfy the following commutator relations:*

$$[\![K_m, K_n]\!] = 0, \tag{23a}$$

$$[\![K_m, \sigma_n]\!] = m K_{m+n-1}, \tag{23b}$$

$$[\![\sigma_m, \sigma_n]\!] = (m-n)\sigma_{m+n-1}, \quad m, n = 1, 2, \cdots. \tag{23c}$$

In other words, these two flows form an infinite-dimensional Lie algebra.

Proof. We only prove Equation (23b), as the other two identities can be proven similarly.

Firstly, we prove the following identity

$$[\![K_m, \sigma_1]\!] = mK_m. \tag{24}$$

By simple calculation, we have

$$
\begin{aligned}
[\![K_0, \sigma_1]\!] &= K_0'[\sigma_1] - \sigma_1'[K_0] \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} xq_x + \frac{q}{2} \\ xr_x + \frac{r}{2} \end{pmatrix} - \begin{pmatrix} x\partial + \frac{1}{2} & 0 \\ 0 & x\partial + \frac{1}{2} \end{pmatrix} \begin{pmatrix} q \\ -r \end{pmatrix} \\
&= 0.
\end{aligned}
$$

Thus, Equation (24) is right in the case of $m = 0$.

Suppose that Equation (24) is true in the case of $m - 1$, i.e.,

$$[\![K_{m-1}, \sigma_1]\!] = (m-1)K_{m-1},$$

and then

$$
\begin{aligned}
[\![K_m, \sigma_1]\!] &= [\![\Phi K_{m-1}, \sigma_1]\!] \\
&= \Phi'[\sigma_1]K_{m-1} + \Phi K_{m-1}'[\sigma_1] - \sigma_1'[K_m] \\
&= (\Phi + \sigma_1'\Phi - \Phi\sigma_1')K_{m-1} + \Phi K_{m-1}'[\sigma_1] - \sigma_1'[K_m] \\
&= K_m + \Phi[\![K_{m-1}, \sigma_1]\!] \\
&= mK_m.
\end{aligned}
$$

Thus, for any positive integer m, Equation (24) holds.

Finally, let us think about the general identity in Equation (23b) by using the similar method. We already know that the identity in Equation (23b) is true in the case of $n = 1$ from Equation (24). Suppose that Equation (23b) is true in the case of $n - 1$, i.e.,

$$[\![K_m, \sigma_{n-1}]\!] = mK_{m+n-2}.$$

It is easy to find that

$$
\begin{aligned}
[\![K_m, \sigma_n]\!] &= [\![K_m, \Phi\sigma_{n-1}]\!] \\
&= K_m'[\Phi\sigma_{n-1}] - \Phi'[K_m]\sigma_{n-1} - \Phi\sigma_{n-1}'[K_m] \\
&= K_m'[\Phi\sigma_{n-1}] - (K_m'\Phi - \Phi K_m')\sigma_{n-1} - \Phi\sigma_{n-1}'[K_m] \\
&= \Phi[\![K_m, \sigma_{n-1}]\!] \\
&= mK_{m+n-1}.
\end{aligned}
$$

$\square$

Theorem 2 tells us that the isospectral and nonisospectral flows obtained in Section 3 are K-symmetries and τ-symmetries of Equation (11), respectively. They constitute an infinite dimensional Lie algebra.

Corollary 1. *It is interesting that the vector field $\sigma_2(u)$ is actually a master symmetry, i.e., the other symmetries can be generated by it through the following relations:*

$$K_{s+1} = \frac{1}{s}[\![K_s, \sigma_2]\!],$$

$$\sigma_{s+1} = \frac{1}{s-2}[\![\sigma_s, \sigma_2]\!] \quad s \neq 2.$$

Finally, let us consider combinations of the two kinds of symmetries K_m, σ_n and the time variable t. We define new functions τ_0^m and τ_n^m as

$$\tau_0^m = mtK_m + \sigma_1, \tag{25a}$$

$$\tau_n^m = \Phi^n \tau_0^m = mtK_{m+n} + \sigma_{n+1}. \tag{25b}$$

Theorem 3. *The new functions τ_n^m are all symmetries of Equation* (11)*, i.e.,*

$$(\tau_n^m)_t = K_m'[\tau_n^m], \quad m = 0, 1, 2, \cdots ; n = 1, 2, \cdots . \tag{26}$$

Proof. Since we have already know that Φ is a strong symmetry operator of Equation (11), it just proves

$$(\tau_0^m)_t = K_m'[\tau_0^m].$$

Through Equation (25a), is is easy to find

$$
\begin{aligned}
(\tau_0^m)_t &= mK_m + mtK'[u_t] + \sigma_1'[u_t] \\
&= mK_m + mtK'[K_m] + \sigma_1'[K_m] \\
&= mtK'[K_m] + K_m'[\sigma_1] \\
&= K_m'[\tau_0^m].
\end{aligned}
$$

$\square$

Theorem 3 shows that τ_n^m is another kind symmetry of Equation (11). In fact, it is the special linear combination of K-symmetry and τ-symmetry.

Theorem 4. K_m *and* τ_m^n, $m \geq 0$ *are two sets of symmetries of every equation in the isospectral gDNLS hierarchy. They satisfy the following relations*

$$[\![K_m, K_n]\!] = 0, \tag{27a}$$

$$[\![K_m, \tau_n^l]\!] = mK_{m+n-1}, \tag{27b}$$

$$[\![\tau_l^m, \tau_n^m]\!] = (l - n)\tau_{l+n-1}^m. \tag{27c}$$

Thus, these symmetries also form an infinite-dimensional Lie algebra.

Proof. It is easy to find that

$$
\begin{aligned}
[\![\tau_l^m, \tau_n^m]\!] &= [\![mtK_{m+l} + \sigma_{l+1}, mtK_{m+n} + \sigma_{n+1}]\!] \\
&= mt\{[\![K_{m+l}, \sigma_{n+1}]\!] + [\![\sigma_{l+1}, K_{m+n}]\!]\} + [\![\sigma_{l+1}, \sigma_{n+1}]\!] \\
&= mt\{(m+l)K_{m+n+l} - (m+n)K_{m+n+l}\} + (l-n)\sigma_{n+l+1} \\
&= (l-n)(mtK_{m+n+l} + \sigma_{n+l+1}) \\
&= (l-n)\tau_{n+l}^m
\end{aligned}
$$

by Equations (23) and (25b). Thus, Equation (27c) holds and the other two equalities can be proven similarly. $\square$

τ-symmetries should exist for nonlocal integrable equations developed based on the IST theory [32,34]. Evolution equations generated from τ-symmetry vector fields are normally integrable

based on the IST, but there may not be infinitely many conservation laws for them. Only the first several (normally two) τ-symmetries are local, and all others in the τ-symmetry hierarchy are not local.

The conclusions of Theorems 1–4 and Lemma 4 are generalized because of the arbitrary parameter β. If we take the special values of the parameter β, the conclusions are also true.

For example, when $\beta = 0$, the recursion operator Φ becomes

$$\Phi_0 = -\varsigma\partial + 2u_x\partial^{-1}u^T\delta + 2uu^T\delta.$$

Φ_0 is a hereditary and strong symmetry operator of the KN Equation (15) and it has two sets of symmetries

$$K_m = \Phi_0^m(-\varsigma u), \quad \tau_n^m = mt\Phi_0^{m+n}(-\varsigma u) + \Phi_0^n(xu_x + \frac{1}{2}u) \quad m = 0, 1, 2, \cdots, \quad n = 1, 2, \cdots.$$

When $\beta = 1$, the recursion operator Φ becomes

$$\Phi_1 = -\varsigma\partial + \frac{1}{2}u^T\delta ue + u_x\partial^{-1}u^T\delta + \varsigma u\partial^{-1}u_x^T\varsigma\delta.$$

Φ_1 is a hereditary and strong symmetry operator of the CLL Equation (16) and it has two sets of symmetries

$$K_m = \Phi_1^m(-\varsigma u), \quad \tau_n^m = mt\Phi_1^{m+n}(-\varsigma u) + \Phi_1^n(xu_x + \frac{1}{2}u) \quad m = 0, 1, 2, \cdots, \quad n = 1, 2, \cdots.$$

When $\beta = 2$, the recursion operator Φ becomes

$$\Phi_2 = -\varsigma\partial + u^T\delta ue + 2\varsigma u\partial^{-1}u_x^T\varsigma\delta - 2uu^T\delta + 2\varsigma u\partial^{-1}u^T\delta u^T\delta u.$$

Φ_2 is a hereditary and strong symmetry operator of the GI Equation (17) and it has two sets of symmetries

$$K_m = \Phi_2^m(-\varsigma u), \quad \tau_n^m = mt\Phi_2^{m+n}(-\varsigma u) + \Phi_2^n(xu_x + \frac{1}{2}u) \quad m = 0, 1, 2, \cdots, \quad n = 1, 2, \cdots.$$

5. Conclusions

In general, the integrability of the gDNLS equation is researched in this paper. The isospectral and nonisospectral hierarchies are derived from its Lax pair. Through taking different values of the arbitrary parameter, the gDNLS equation could be reduced to the KN equation, the CLL equation, the GI equation, the mKdV equation, the STO equation and so on. Two kinds of symmetries, *K*-symmetries and τ-symmetries, are deduced from the corresponding isospectral and nonisospectral hierarchies. The recursion operator Φ related to these symmetries with an arbitrary parameter is proven to be a hereditary and strong symmetry operator of the whole isospectral hierarchy. Finally, we find that those two kinds of symmetries form an infinite-dimensional Lie algebra.

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