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A GENERALIZED METHOD AND ITS APPLICATIONS TO n-DIMENSIONAL FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS IN FRACTAL DOMAIN

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Abstract

A generalized local fractional Riccati differential equation (LFRDE) method, which is a combination of the local fractional Riccati differential equation method and the transformed rational function approach, is presented to obtain the non-differentiable exact traveling wave solutions for n-dimensional fractional partial differential equations. In order to verify the effectiveness of the method, the seventh-order local fractional Sawada–Kotera–Ito equation, the local fractional sine-Gordon equation and the local fractional Kadomtsev–Petviashvili equation in fractal domain are first considered. The obtained results show that the presented method involving fractal special functions, are powerful and effective for obtaining exact solutions of nonlinear fractional partial differential equations in fractal domains.

Keywords: Traveling Wave Solution; Generalized LFRDE; Fractional Partial Differential Equation; Fractal.

1. INTRODUCTION

Fractional calculus has important applications to fractal problems in engineering fields,[1–8] such as the heat transport in fractal media,[9] fractal electrostatics,[10] fractal hydrodynamic problems,[11] fractal Fokker–Planck equations[12] and fractal description of stress and strain in elasticity.[13] Some effective methods have been proposed to describe complex phenomena and fractal behaviors in nature.[1–8] The theory of local fractional derivatives (LFDs) is a powerful mathematical tool for modeling a family of complex problems involving fractal engineering. For examples, the fractal complexity in shallow water surfaces,[14] fractal LC-electric circuit,[15] PDEs,[16–23] ODEs,[24] and inequalities.[25,26] Recently, Yang *et al.*[27] introduced a method by using a local fractional Riccati differential equation to find traveling wave solutions to local fractional two-dimensional Burgers-type equations. However, the non-differentiable exact traveling wave solutions for the seventh-order local fractional Sawada–Kotera–Ito equation, the local fractional sine-Gordon equation and the local fractional Kadomtsev–Petviashvili equation in fractal domain have not been considered.

In this paper, a systematic method for solving n-dimensional fractional partial differential equations is proposed, which is a combination of the local fractional Riccati differential equation method and the transformed rational function approach,[28] then more traveling wave solutions of fractional partial differential equations in general dimensions can be worked out.

This paper is organized as follows. First a basic theory of local fractional calculus is presented in Sec. 2. In Sec. 3, we introduce a generalized local fractional Riccati differential equation method for solving fractional partial differential equations in general dimensions. In Sec. 4, we present practical applications of the proposed algorithm to three fractional partial differential equations. In Sec. 5, some conclusions and discussions are given.

2. PRELIMINARIES

To make our presentations self-contained and easy to understand, we simply recall some basic notations and important properties in the local fractional calculus.[1,14,15,27] Suppose $C_\varepsilon(x,y)$ denotes non-differentiable functions with a fractal dimension $0<\varepsilon<1$.

Definition 2.1. Let $u(\xi)\in C_\varepsilon(x,y)$. Then Yang local fractional derivative (YLFD) of $u(\xi)$ of order ε at $\xi=\xi_0$ is defined as follows[1,14,15,27]:

$$\begin{aligned} D^{(\varepsilon)}u(\xi_0) = u^{(\varepsilon)}(\xi_0) &= \frac{d^\varepsilon u(\xi)}{d\xi^\varepsilon}|_{\xi=\xi_0} \\ &= \lim_{\xi\to\xi_0}\frac{\Delta^\varepsilon(u(\xi)-u(\xi_0))}{(\xi-\xi_0)^\varepsilon}, \end{aligned} \tag{1}$$

where $\Delta^\varepsilon(u(\xi)-u(\xi_0))\cong\Gamma(1+\varepsilon)(u(\xi)-u(\xi_0))$ and $0<\varepsilon<1$.

Then, the following rules hold[1,14,15,27]:

$$D^{(\varepsilon)}[u_1(\xi)\pm u_2(\xi)] = D^{(\varepsilon)}u_1(\xi)\pm D^{(\varepsilon)}u_2(\xi), \tag{2}$$

$$\begin{aligned} &D^{(\varepsilon)}[u_1(\xi)u_2(\xi)] \\ &\quad = [D^{(\varepsilon)}u_1(\xi)]u_2(\xi)+u_1(\xi)[D^{(\varepsilon)}u_2(\xi)], \end{aligned} \tag{3}$$

$$D^{(\varepsilon)}[u_1(\xi)/u_2(\xi)] = \{[D^{(\varepsilon)}u_1(\xi)]u_2(\xi) - u_1(\xi)[D^{(\varepsilon)}u_2(\xi)]\}/u_2^2 \times(\xi), \quad \texttt{provided}\ u_2(\xi) \neq 0. \tag{4}$$

The special functions defined on fractal sets is defined by

Suppose $\Phi : \kappa \to \mathfrak{N}$ be a function defined on a fractal set κ of fractal dimension $0 < \varepsilon < 1$. A real-valued function $\Phi(\xi)$ defined on the fractal set κ: $\Phi(\xi) = \xi^\varepsilon$, where $\xi^\varepsilon \in \kappa$ and $0 < \varepsilon < 1$. Now we find that $\Phi(\xi) = \xi^\varepsilon$ is a Lebesgue–Cantor function and $\lim_{\varepsilon\to 1} \Phi(\xi) = \xi \in R$ with real number set R. Then the Mittag-Leffler function defined on the fractal set κ is presented as[1,14,15,27]

$$E_\varepsilon(\xi^\varepsilon) = \sum_{k=0}^{\infty} \frac{\xi^{k\varepsilon}}{\Gamma(1+k\varepsilon)},$$

where $\xi \in R$ and $0 < \varepsilon < 1$.

In the following, we list YLFDs of special functions on fractal sets[1] (Table 1) and some special functions on fractal sets[1] (Table 2) , which will be applied later.

Table 1 YLFDs of Special Functions.[1]

Special Functions	LFDs
$E_\varepsilon(\xi^\varepsilon)$	$E_\varepsilon(\xi^\varepsilon)$
$\sin_\varepsilon(\xi^\varepsilon)$	$\cos_\varepsilon(\xi^\varepsilon)$
$\cos_\varepsilon(\xi^\varepsilon)$	$-\sin_\varepsilon(\xi^\varepsilon)$

Table 2 Special Functions Defined on Fractal Sets.[1]

Special Functions	Formulas
$\tan_\varepsilon(\xi^\varepsilon)$	$\frac{E_\varepsilon(i^\varepsilon\xi^\varepsilon)-E_\varepsilon(-i^\varepsilon\xi^\varepsilon)}{i^\varepsilon(E_\varepsilon(i^\varepsilon\xi^\varepsilon)+E_\varepsilon(-i^\varepsilon\xi^\varepsilon))}$
$\cot_\varepsilon(\xi^\varepsilon)$	$\frac{i^\varepsilon(E_\varepsilon(i^\varepsilon\xi^\varepsilon)+E_\varepsilon(-i^\varepsilon\xi^\varepsilon))}{E_\varepsilon(i^\varepsilon\xi^\varepsilon)-E_\varepsilon(-i^\varepsilon\xi^\varepsilon)}$
$\tanh_\varepsilon(\xi^\varepsilon)$	$\frac{E_\varepsilon(\xi^\varepsilon)-E_\varepsilon(-\xi^\varepsilon)}{E_\varepsilon(\xi^\varepsilon)+E_\varepsilon(-\xi^\varepsilon)}$
$\coth_\varepsilon(\xi^\varepsilon)$	$\frac{E_\varepsilon(\xi^\varepsilon)+E_\varepsilon(-\xi^\varepsilon)}{E_\varepsilon(\xi^\varepsilon)-E_\varepsilon(-\xi^\varepsilon)}$

Definition 2.2. Yang local fractional partial derivative (YLFPD) of $u_\varepsilon(\theta,\vartheta)$ of order ε with respect to ϑ is defined as follows[1,14,15,27]:

$$u_{\vartheta,\varepsilon}^{(\varepsilon)}(\theta,\vartheta_0) = \frac{\partial^\varepsilon u_\varepsilon(\theta,\vartheta)}{\partial\vartheta^\varepsilon}|_{\vartheta=\vartheta_0} = \lim_{\vartheta\to\vartheta_0} \frac{\Delta^\varepsilon(u_\varepsilon(\theta,\vartheta) - u_\varepsilon(\theta,\vartheta_0))}{(\vartheta-\vartheta_0)^\varepsilon}, \tag{5}$$

where $\Delta^\varepsilon(u_\varepsilon(\theta,\vartheta) - u_\varepsilon(\theta,\vartheta_0)) \cong \Gamma(1+\varepsilon)(u_\varepsilon(\theta,\vartheta) - u_\varepsilon(\theta,\vartheta_0))$.

Similarly, YLFPD of $u_\varepsilon(\theta,\vartheta)$ of order ε with respect to θ is defined as follows:

$$u_{\vartheta,\varepsilon}^{(\varepsilon)}(\theta_0,\vartheta) = \frac{\partial^\varepsilon u_\varepsilon(\theta,\vartheta)}{\partial\theta^\varepsilon}|_{\theta=\theta_0} = \lim_{\theta\to\theta_0} \frac{\Delta^\varepsilon(u_\varepsilon(\theta,\vartheta) - u_\varepsilon(\theta_0,\vartheta))}{(\theta-\theta_0)^\varepsilon}, \tag{6}$$

where $\Delta^\varepsilon(u_\varepsilon(\theta,\vartheta) - u_\varepsilon(\theta_0,\vartheta)) \cong \Gamma(1+\varepsilon)(u_\varepsilon(\theta,\vartheta) - u_\varepsilon(\theta_0,\vartheta))$.

If $u_\varepsilon(\theta,\vartheta) = u_\varepsilon(\xi)$, then we have

$$\frac{\partial^\varepsilon u_\varepsilon(\theta,\vartheta)}{\partial\theta^\varepsilon} = \frac{\partial^\varepsilon u_\varepsilon}{\partial\xi^\varepsilon}\left(\frac{\partial\xi}{\partial\theta}\right)^\varepsilon, \tag{7}$$

$$\frac{\partial^\varepsilon u_\varepsilon(\theta,\vartheta)}{\partial\vartheta^\varepsilon} = \frac{\partial^\varepsilon u_\varepsilon}{\partial\xi^\varepsilon}\left(\frac{\partial\xi}{\partial\vartheta}\right)^\varepsilon. \tag{8}$$

3. A GENERALIZED LFRDE METHOD

We investigate the following $(n+1)$-dimensional nonlinear fractional partial differential equation

$$P\left(u_\varepsilon(x_1,x_2,\ldots,x_n,t), \frac{\partial^\varepsilon u_\varepsilon}{\partial x_i^\varepsilon}, \frac{\partial^\varepsilon u_\varepsilon}{\partial t^\varepsilon}, \frac{\partial^{2\varepsilon} u_\varepsilon}{\partial x_i^{2\varepsilon}}, \frac{\partial^{2\varepsilon} u_\varepsilon}{\partial t^{2\varepsilon}}, \cdots\right) = 0. \tag{9}$$

Step 1. By making use of a traveling wave transformation

$$\xi^\varepsilon = \sum_{i=1}^{n} l_i^\varepsilon x_i^\varepsilon + m^\varepsilon t^\varepsilon, \tag{10}$$

then Eq. (9) can be translated into

$$P\left(u_\varepsilon, \frac{d^\varepsilon u_\varepsilon}{d\xi^\varepsilon}, \frac{d^{2\varepsilon} u_\varepsilon}{d\xi^{2\varepsilon}}, \ldots\right) = 0. \tag{11}$$

Step 2. Assume that Eq. (9) has a non-differentiable solution in this form

$$u_\varepsilon(\xi) = \sum_{i=0}^{N} a_i\phi_\varepsilon^i(\xi) + \sum_{i=1}^{N} b_i\phi_\varepsilon^{-i}(\xi), \tag{12}$$

where N is a positive integer to be found by balancing the highest-order linear term with the nonlinear term in Eq. (11). This is a rational expression of $\phi_\varepsilon(\xi)$, and thus, it is regarded as a special case of the transformed rational function method. The function $\phi_\varepsilon(\xi)$ satisfies the local fractional Riccati equation (LFRDE)

$$\frac{d^{\varepsilon}\phi_{\varepsilon}(\xi)}{d\xi^{\varepsilon}} = k_0 + k_1\phi_{\varepsilon}(\xi) + k_2\phi_{\varepsilon}^2(\xi), \tag{13}$$

where k_1, k_2, k_3 are constants. Note that in the case $\varepsilon = 1$, the LFRDE (13) is exactly the classical RDE, whose general solution has been presented via (40)–(42) in Ref. 29. Accordingly, the LFRDE (13) has a general solution as follows:

Case 1. For the case $\sum = k_1^2 - 4k_0k_2 > 0$, a non-differentiable solution of the LFRDE is given by

$$\phi_{\varepsilon}(\xi) = \frac{A_0C_0E_{\varepsilon}(-C_0\xi^{\varepsilon}) + B_0D_0E_{\varepsilon}(-D_0\xi^{\varepsilon})}{k_2(A_0E_{\varepsilon}(-C_0\xi^{\varepsilon}) + B_0E_{\varepsilon}(-D_0\xi^{\varepsilon}))},$$

where

$$\begin{aligned} A_0 &= \frac{\alpha}{2} + \frac{\beta(k_1+1) - \frac{\alpha k_1}{2}}{\sqrt{k_1^2 - 4k_0k_2}}, \\ B_0 &= \frac{\alpha}{2} - \frac{\beta(k_1+1) - \frac{\alpha k_1}{2}}{\sqrt{k_1^2 - 4k_0k_2}}, \\ C_0 &= \frac{k_1 + \sqrt{k_1^2 - 4k_0k_2}}{2}, \\ D_0 &= \frac{k_1 - \sqrt{k_1^2 - 4k_0k_2}}{2}, \\ \alpha &= D^{(\varepsilon)}\psi(0), \quad \beta = \psi(0), \\ \phi_{\varepsilon}(\xi) &= -\frac{D^{(\varepsilon)}\psi(\xi)}{k_2\psi(\xi)}. \end{aligned} \tag{14}$$

Case 2. For the case $\sum = k_1^2 - 4k_0k_2 < 0$, we have

$$\phi_{\varepsilon}(\xi) = -\frac{A_1C_1E_{\varepsilon}(-C_1\xi^{\varepsilon}) + B_1D_1E_{\varepsilon}(-D_1\xi^{\varepsilon})}{k_2(A_1E_{\varepsilon}(-C_1\xi^{\varepsilon}) + B_1E_{\varepsilon}(-D_1\xi^{\varepsilon}))},$$

where

$$\begin{aligned} A_1 &= \frac{\alpha}{2} + \frac{\beta(k_1+1) - \frac{\alpha k_1}{2}}{i^{\varepsilon}\sqrt{4k_0k_2 - k_1^2}}, \\ B_1 &= \frac{\alpha}{2} - \frac{\beta(k_1+1) - \frac{\alpha k_1}{2}}{i^{\varepsilon}\sqrt{4k_0k_2 - k_1^2}} \\ C_1 &= \frac{k_1 + i^{\varepsilon}\sqrt{4k_0k_2 - k_1^2}}{2}, \\ D_1 &= \frac{k_1 - i^{\varepsilon}\sqrt{4k_0k_2 - k_1^2}}{2}, \end{aligned} \tag{15}$$

with the fractional imaginary i^{ε}.

Case 3. For the case $\sum = k_1^2 - 4k_0k_2 = 0$, the solution reads

$$\phi_{\varepsilon}(\xi) = \alpha E_{\varepsilon}\left(-\frac{k_1}{2}\xi^{\varepsilon}\right) + \left(\beta(k_1+1) - \frac{\alpha k_1}{2}\right) \times \frac{\xi^{\varepsilon}}{\Gamma(\varepsilon+1)}E_{\varepsilon}\left(-\frac{k_1}{2}\xi^{\varepsilon}\right). \tag{16}$$

Step 3. Substituting (12) into (11) and setting the coefficients of $\phi^i(\xi)$ to be zero, one gets an over-determined algebraic system, and directly solving the system leads to its solutions.

Step 4. Then substituting these constants and the solutions of (13) into (12), we obtain a non-differentiable solution of Eq. (9).

4. APPLICATIONS

In the following, we use three examples to verify the efficiency of the presented algorithm.

Example 4.1. We investigate the seventh-order local fractional Sawada–Kotera–Ito equation in 1-fractal-dimensional space

$$\begin{aligned} &\frac{\partial^{\varepsilon}u_{\varepsilon}}{\partial t^{\varepsilon}} + 252u_{\varepsilon}^3\frac{\partial^{\varepsilon}u_{\varepsilon}}{\partial x^{\varepsilon}} + 63\left(\frac{\partial^{\varepsilon}u_{\varepsilon}}{\partial x^{\varepsilon}}\right)^3 \\ &\quad +378u_{\varepsilon}\frac{\partial^{\varepsilon}u_{\varepsilon}}{\partial x^{\varepsilon}}\frac{\partial^{2\varepsilon}u_{\varepsilon}}{\partial x^{2\varepsilon}} + 126u_{\varepsilon}^2\frac{\partial^{3\varepsilon}u_{\varepsilon}}{\partial x^{3\varepsilon}} + 63\frac{\partial^{2\varepsilon}u_{\varepsilon}}{\partial x^{2\varepsilon}} \\ &\quad \times\frac{\partial^{3\varepsilon}u_{\varepsilon}}{\partial x^{3\varepsilon}} + 42\frac{\partial^{\varepsilon}u_{\varepsilon}}{\partial x^{\varepsilon}}\frac{\partial^{4\varepsilon}u_{\varepsilon}}{\partial x^{4\varepsilon}} + 21u_{\varepsilon}\frac{\partial^{5\varepsilon}u_{\varepsilon}}{\partial x^{5\varepsilon}} \\ &\quad + \frac{\partial^{7\varepsilon}u_{\varepsilon}}{\partial x^{7\varepsilon}} = 0. \end{aligned} \tag{17}$$

Taking $i = 1$ in (10), we have the traveling wave transformation

$$\xi^{\varepsilon} = l_1^{\varepsilon}x_1^{\varepsilon} + m^{\varepsilon}t^{\varepsilon}, \tag{18}$$

where

$$\lim_{\varepsilon\to 1}\xi^{\varepsilon} = l_1x_1 + mt. \tag{19}$$

Then from (7) and (8), we have

$$\frac{\partial^{\varepsilon}u_{\varepsilon}}{\partial t^{\varepsilon}} = m^{\varepsilon}\frac{\partial^{\varepsilon}u_{\varepsilon}}{\partial\xi^{\varepsilon}}, \quad \frac{\partial^{j\varepsilon}u_{\varepsilon}}{\partial x_i^{j\varepsilon}} = l_i^{j\varepsilon}\frac{\partial^{j\varepsilon}u_{\varepsilon}}{\partial\xi^{j\varepsilon}}, \quad i, j = 1, 2, \ldots, n. \tag{20}$$

Upon substituting (20) into (17), (17) reduces to

$$m^{\varepsilon}\frac{d^{\varepsilon}u_{\varepsilon}}{d\xi^{\varepsilon}} + 252l_1^{\varepsilon}u_{\varepsilon}^3\frac{d^{\varepsilon}u_{\varepsilon}}{d\xi^{\varepsilon}} + 63l_1^{3\varepsilon}\left(\frac{d^{\varepsilon}u_{\varepsilon}}{d\xi^{\varepsilon}}\right)^3$$

$$+378l_1^{3\varepsilon}u_\varepsilon\frac{d^\varepsilon u_\varepsilon}{d\xi^\varepsilon}\frac{d^{2\varepsilon}u_\varepsilon}{d\xi^{2\varepsilon}}+126l_1^{3\varepsilon}u_\varepsilon^2\frac{d^{3\varepsilon}u_\varepsilon}{d\xi^{3\varepsilon}}$$
$$+63l_1^{5\varepsilon}\frac{d^{2\varepsilon}u_\varepsilon}{d\xi^{2\varepsilon}}\frac{d^{3\varepsilon}u_\varepsilon}{d\xi^{3\varepsilon}}+42l_1^{5\varepsilon}\frac{d^\varepsilon u_\varepsilon}{d\xi^\varepsilon}\frac{d^{4\varepsilon}u_\varepsilon}{d\xi^{4\varepsilon}}$$
$$+21l_1^{5\varepsilon}u_\varepsilon\frac{d^{5\varepsilon}u_\varepsilon}{d\xi^{5\varepsilon}}+l_1^{7\varepsilon}\frac{d^{7\varepsilon}u_\varepsilon}{d\xi^{7\varepsilon}}=0. \tag{21}$$

It is easy to find that $N=2$ by balancing the highest-order linear term with the nonlinear term in Eq. (21), and then we have

$$u_\varepsilon(\xi)=\sum_{i=0}^{2}a_i\phi_\varepsilon^i(\xi)+\sum_{i=1}^{2}b_i\phi_\varepsilon^{-i}(\xi). \tag{22}$$

We substitute (22) and (13) into (21), and then collect the coefficients of $\phi_\varepsilon^i(\xi)$ and equate them to be 0, to obtain the following solutions.

Family 1.

$$a_0=-\frac{8}{3}k_0k_2l_1^{2\varepsilon}-\frac{1}{3}k_1^2l_1^{2\varepsilon},\quad a_1=a_2=0,$$
$$b_1=-4k_0k_1l_1^{2\varepsilon},\quad b_2=-4k_0^2l_1^{2\varepsilon},$$
$$m^\varepsilon=-\frac{4}{3}l_1^{7\varepsilon}(64k_0^3k_2^3-48k_0^2k_1^2k_2^2+12k_0k_1^4k_2-k_1^6).$$

Thus, from the above results and (14)–(16), the non-differentiable solution of Eq. (17) becomes

$$u_{\varepsilon,1}(x_1,t)=-\frac{8}{3}k_0k_2l_1^{2\varepsilon}-\frac{1}{3}k_1^2l_1^{2\varepsilon}-\frac{4k_0k_1l_1^{2\varepsilon}}{\phi_\varepsilon(\xi)}-\frac{4k_0^2l_1^{2\varepsilon}}{\phi_\varepsilon^2(\xi)}. \tag{23}$$

(I) For $\sum=k_1^2-4k_0k_2>0$,

$$\phi_\varepsilon(\xi)=\frac{A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))}{k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)))}. \tag{24}$$

(II) For $\sum=k_1^2-4k_0k_2<0$,

$$\phi_\varepsilon(\xi)=-\frac{A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))}{k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)))}. \tag{25}$$

(III) For $\sum=k_1^2-4k_0k_2=0$,

$$\phi_\varepsilon(\xi)=\alpha E_\varepsilon\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)\right)+\left(\beta(k_1+1)-\frac{\alpha k_1}{2}\right)\frac{l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon}{\Gamma(\varepsilon+1)}E_\varepsilon\times\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)\right). \tag{26}$$

Family 2.

$$a_1=a_2=0,\quad b_1=-2k_0k_1l_1^{2\varepsilon},\quad b_2=-2k_0^2l_1^{2\varepsilon},$$
$$m^\varepsilon=-l_1^\varepsilon(608k_0^3k_2^3l_1^{6\varepsilon}+216k_0^2k_1^2k_2^2l_1^{6\varepsilon}+30k_0k_1^4k_2l_1^{6\varepsilon}+k_1^6l_1^{6\varepsilon}+1344a_0k_0^2k_2^2l_1^{4\varepsilon}+336a_0k_0k_1^2k_2l_1^{4\varepsilon}+21a_0k_1^4l_1^{4\varepsilon}+1008a_0^2k_0k_2l_1^{2\varepsilon}+126a_0^2k_1^2l_1^{2\varepsilon}+252a_0^3).$$

Then, we have

$$u_{\varepsilon,2}(x_1,t)=a_0-\frac{2k_0k_1l_1^{2\varepsilon}}{\phi_\varepsilon(\xi)}-\frac{2k_0^2l_1^{2\varepsilon}}{\phi_\varepsilon^2(\xi)}. \tag{27}$$

(I) For $\sum=k_1^2-4k_0k_2>0$,

$$\phi_\varepsilon(\xi)=\frac{A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))}{k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)))}. \tag{28}$$

(II) For $\sum=k_1^2-4k_0k_2<0$,

$$\phi_\varepsilon(\xi)=-\frac{A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))}{k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon))+B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)))}. \tag{29}$$

(III) For $\sum=k_1^2-4k_0k_2=0$,

$$\phi_\varepsilon(\xi)=\alpha E_\varepsilon\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)\right)+\left(\beta(k_1+1)-\frac{\alpha k_1}{2}\right)\frac{l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon}{\Gamma(\varepsilon+1)}E_\varepsilon\times\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon+m^\varepsilon t^\varepsilon)\right). \tag{30}$$

Family 3.

$$a_1 = -2k_1k_2l_1^{2\varepsilon}, \quad a_2 = -2k_2^2l_1^{2\varepsilon}, \quad b_1 = b_2 = 0,$$
$$m^\varepsilon = -l_1^\varepsilon(608k_0^3k_2^3l_1^{6\varepsilon} + 216k_0^2k_1^2k_2^2l_1^{6\varepsilon} + 30k_0k_1^4k_2l_1^{6\varepsilon} + k_1^6l_1^{6\varepsilon} + 1344a_0k_0^2k_2^2l_1^{4\varepsilon} + 336a_0k_0k_1^2k_2l_1^{4\varepsilon} + 21a_0k_1^4l_1^{4\varepsilon} + 1008a_0^2k_0k_2l_1^{2\varepsilon} + 126a_0^2k_1^2l_1^{2\varepsilon} + 252a_0^3).$$

Then, we get

$$u_{\varepsilon,3}(x_1,t) = a_0 - 2k_1k_2l_1^{2\varepsilon}\phi_\varepsilon(\xi) - 2k_2^2l_1^{2\varepsilon}\phi_\varepsilon^2(\xi). \tag{31}$$

(I) For $\sum = k_1^2 - 4k_0k_2 > 0$,

$$\phi_\varepsilon(\xi) = \frac{A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon))}{k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)))}. \tag{32}$$

(II) For $\sum = k_1^2 - 4k_0k_2 < 0$,

$$\phi_\varepsilon(\xi) = -\frac{A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon))}{k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)))}. \tag{33}$$

(III) For $\sum = k_1^2 - 4k_0k_2 = 0$,

$$\phi_\varepsilon(\xi) = \alpha E_\varepsilon\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)\right) + \left(\beta(k_1+1) - \frac{\alpha k_1}{2}\right)\frac{l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon}{\Gamma(\varepsilon+1)}E_\varepsilon \times \left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)\right). \tag{34}$$

Family 4.

$$a_0 = -\frac{8}{3}k_0k_2l_1^{2\varepsilon} - \frac{1}{3}k_1^2l_1^{2\varepsilon}, \quad a_1 = -4k_1k_2l_1^{2\varepsilon},$$
$$a_2 = -4k_2^2l_1^{2\varepsilon}, \quad b_1 = b_2 = 0, \quad m^\varepsilon = -\frac{4}{3}l_1^{7\varepsilon} \times (64k_0^3k_2^3 - 48k_0^2k_1^2k_2^2 + 12k_0k_1^4k_2 - k_1^6).$$

Thus, we have

$$u_{\varepsilon,4}(x_1,t) = -\frac{8}{3}k_0k_2l_1^{2\varepsilon} - \frac{1}{3}k_1^2l_1^{2\varepsilon} - 4k_1k_2l_1^{2\varepsilon}\phi_\varepsilon(\xi) - 4k_2^2l_1^{2\varepsilon}\phi_\varepsilon^2(\xi). \tag{35}$$

(I) For $\sum = k_1^2 - 4k_0k_2 > 0$,

$$\phi_\varepsilon(\xi) = \frac{A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon))}{k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)))}. \tag{36}$$

(II) For $\sum = k_1^2 - 4k_0k_2 < 0$,

$$\phi_\varepsilon(\xi) = -\frac{A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon))}{k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)))}. \tag{37}$$

(III) For $\sum = k_1^2 - 4k_0k_2 = 0$,

$$\phi_\varepsilon(\xi) = \alpha E_\varepsilon\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)\right) + \left(\beta(k_1+1) - \frac{\alpha k_1}{2}\right)\frac{l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon}{\Gamma(\varepsilon+1)}E_\varepsilon \times \left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)\right). \tag{38}$$

Family 5.

$$a_0 = -\frac{8}{3}k_0k_2l_1^{2\varepsilon} - \frac{1}{3}k_1^2l_1^{2\varepsilon}, \quad a_1 = -2k_1k_2l_1^{2\varepsilon},$$
$$a_2 = -2k_2^2l_1^{2\varepsilon}, \quad b_1 = -2k_0k_1l_1^{2\varepsilon}, \quad b_2 = -2k_0^2l_1^{2\varepsilon},$$
$$m^\varepsilon = -\frac{4}{3}l_1^{7\varepsilon}(64k_0^3k_2^3 - 48k_0^2k_1^2k_2^2 + 12k_0k_1^4k_2 - k_1^6).$$

Thus, we get

$$u_{\varepsilon,5}(x_1,t) = -\frac{8}{3}k_0k_2l_1^{2\varepsilon} - \frac{1}{3}k_1^2l_1^{2\varepsilon} - 2k_1k_2l_1^{2\varepsilon}\phi_\varepsilon(\xi) - 2k_2^2l_1^{2\varepsilon}\phi_\varepsilon^2(\xi) - \frac{2k_2^2l_1^{2\varepsilon}}{\phi_\varepsilon(\xi)} - \frac{2k_0k_1l_1^{2\varepsilon}}{\phi_\varepsilon^2(\xi)}. \tag{39}$$

(I) For $\sum = k_1^2 - 4k_0k_2 > 0$,

$$\phi_\varepsilon(\xi) = \frac{A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon))}{k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)))}. \tag{40}$$

(II) For $\sum = k_1^2 - 4k_0k_2 < 0$,

$$\phi_\varepsilon(\xi) = -\frac{A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon))}{k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)) + B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)))}. \tag{41}$$

(III) For $\sum = k_1^2 - 4k_0k_2 = 0$,

$$\begin{aligned}\phi_\varepsilon(\xi) &= \alpha E_\varepsilon\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)\right) \\ &\quad + \left(\beta(k_1+1) - \frac{\alpha k_1}{2}\right)\frac{l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon}{\Gamma(\varepsilon+1)}E_\varepsilon \\ &\quad \times \left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + m^\varepsilon t^\varepsilon)\right).\end{aligned} \tag{42}$$

It is necessary to illustrate the characteristics of the exact traveling wave solution of Eq. (17) shown in Fig. 1. Figure 1a represents the exact traveling wave solution when $A_0 = C_0 = D_0 = 1, B_0 = 2, k_2 = 3$, and $l_1^\varepsilon = m^\varepsilon = 1$ in (24). Figure 1b explains the exact traveling wave solution when $\alpha = 50, \beta = 50, k1 = -3$ and $l_1^\varepsilon = m^\varepsilon = 1$ in (42).

Example 4.2. We discuss the local fractional sine-Gordon equation in 2-fractal-dimensional space

$$\frac{\partial^{2\varepsilon}u_\varepsilon}{\partial t^{2\varepsilon}} - \frac{\partial^{2\varepsilon}u_\varepsilon}{\partial x_1^{2\varepsilon}} - \frac{\partial^{2\varepsilon}u_\varepsilon}{\partial x_2^{2\varepsilon}} + r^2\mathtt{sin}u_\varepsilon = 0. \tag{43}$$

Taking $i = 2$ in (10), we have the traveling wave transformation

$$\xi^\varepsilon = l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon, \tag{44}$$

where

$$\lim_{\varepsilon\to 1}\xi^\varepsilon = l_1x_1 + l_2x_2 + mt. \tag{45}$$

Upon substituting (20) into (43), (43) reduces to the following local fractional differential equation:

$$(m^{2\varepsilon} - l_1^{2\varepsilon} - l_2^{2\varepsilon})\frac{d^{2\varepsilon}u_\varepsilon}{d\xi^{2\varepsilon}} + r^2\mathtt{sin}u_\varepsilon = 0. \tag{46}$$

Let $v_\varepsilon = e^{iu_\varepsilon}$, and then (46) becomes

$$\begin{aligned}&2(m^{2\varepsilon} - l_1^{2\varepsilon} - l_2^{2\varepsilon})\left(v_\varepsilon\frac{d^{2\varepsilon}v_\varepsilon}{d\xi^{2\varepsilon}} - \left(\frac{d^\varepsilon v_\varepsilon}{d\xi^\varepsilon}\right)^2\right) \\ &\quad + r^2(v_\varepsilon^3 - v_\varepsilon) = 0.\end{aligned} \tag{47}$$

It is easy to find that $N = 2$, and thus we have

$$v_\varepsilon(\xi) = \sum_{i=0}^{2}a_i\phi_\varepsilon^i(\xi) + \sum_{i=1}^{2}b_i\phi_\varepsilon^{-i}(\xi). \tag{48}$$

We substitute (48) and (13) into (47), then collect the coefficients of $\phi_\varepsilon^i(\xi)$ and equate them to be 0, to obtain the following solutions.

Family 1.

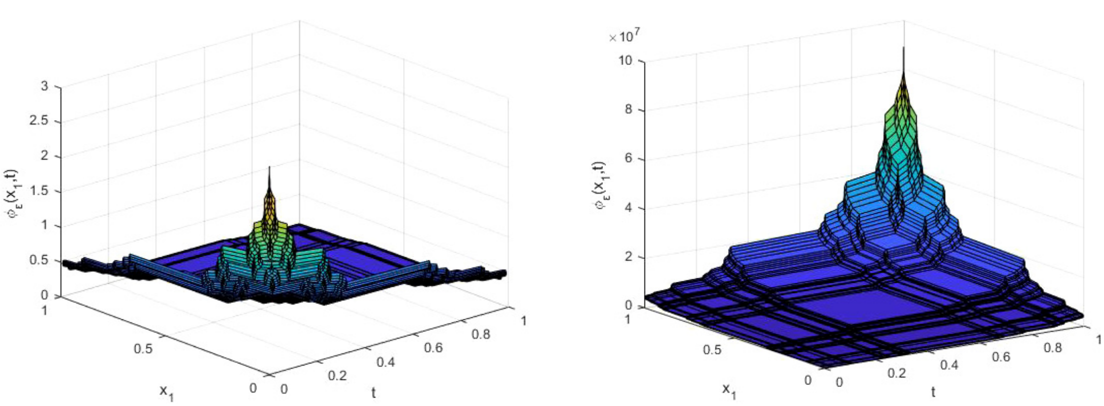
$$a_0 = \frac{k_1^2}{k_1^2 - 4k_0k_2}, \quad a_1 = a_2 = 0,$$

$$b_1 = \frac{4k_0k_1}{k_1^2 - 4k_0k_2}, \quad b_2 = \frac{4k_0^2}{k_1^2 - 4k_0k_2},$$

$$l_1^\varepsilon = \pm\sqrt{\frac{-4m^{2\varepsilon}k_0k_2 + m^{2\varepsilon}k_1^2 + 4l_2^{2\varepsilon}k_0k_2 - l_2^{2\varepsilon}k_1^2 + r^2}{k_1^2 - 4k_0k_2}}.$$

Thus, we get

$$\begin{aligned}&u_{\varepsilon,1}(x_1, x_2, t) \\ &\quad = -i\mathtt{ln}\left(\frac{k_1^2}{k_1^2 - 4k_0k_2} + \frac{4k_0k_1}{(k_1^2 - 4k_0k_2)\phi_\varepsilon(\xi)}\right. \\ &\qquad \left. + \frac{4k_0^2}{(k_1^2 - 4k_0k_2)\phi_\varepsilon^2(\xi)}\right).\end{aligned} \tag{49}$$



(a) (b)

Fig. 1 The exact traveling wave solution for the seventh-order local fractional Sawada–Kotera–Ito equation for different parameters.

(I) For $\sum = k_1^2 - 4k_0k_2 > 0$,

$$\phi_\varepsilon(\xi) = \frac{\begin{array}{l} A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{50}$$

(II) For $\sum = k_1^2 - 4k_0k_2 < 0$,

$$\phi_\varepsilon(\xi) = -\frac{\begin{array}{l} A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{51}$$

Similarly, the exact traveling wave solution of Eq. (43) is shown in Fig. 2. Figure 2a displays the exact traveling wave solution when $A_1 = C_1 = D_1 = 1, B_1 = 2, k_2 = -3$, and $l_2^\varepsilon = m^\varepsilon = 1, l_1^\varepsilon = 0$ in (51). Figure 2b explains the exact traveling wave solution when $A_1 = C_1 = D_1 = 1, B_1 = 2, k_2 = -3$, and $l_1^\varepsilon = 2, m^\varepsilon = 1, l_2^\varepsilon = 0$ in (51).

Family 2.

$$a_0 = -\frac{k_1^2}{k_1^2 - 4k_0k_2}, \quad a_1 = a_2 = 0,$$

$$b_1 = -\frac{4k_0k_1}{k_1^2 - 4k_0k_2}, \quad b_2 = -\frac{4k_0^2}{k_1^2 - 4k_0k_2},$$

$$l_1^\varepsilon = \pm\sqrt{\frac{\begin{array}{c} -4m^{2\varepsilon}k_0k_2 + m^{2\varepsilon}k_1^2 + 4l_2^{2\varepsilon}k_0k_2 \\ - l_2^{2\varepsilon}k_1^2 - r^2 \end{array}}{k_1^2 - 4k_0k_2}}.$$

Thus, we get

$$\begin{aligned} u_{\varepsilon,2}(x_1, x_2, t) &= -i\ln\left(-\frac{k_1^2}{k_1^2 - 4k_0k_2} - \frac{4k_0k_1}{(k_1^2 - 4k_0k_2)\phi_\varepsilon(\xi)}\right. \\ &\quad \left. - \frac{4k_0^2}{(k_1^2 - 4k_0k_2)\phi_\varepsilon^2(\xi)}\right). \end{aligned} \tag{52}$$

(I) For $\sum = k_1^2 - 4k_0k_2 > 0$,

$$\phi_\varepsilon(\xi) = \frac{\begin{array}{l} A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{53}$$

(II) For $\sum = k_1^2 - 4k_0k_2 < 0$,

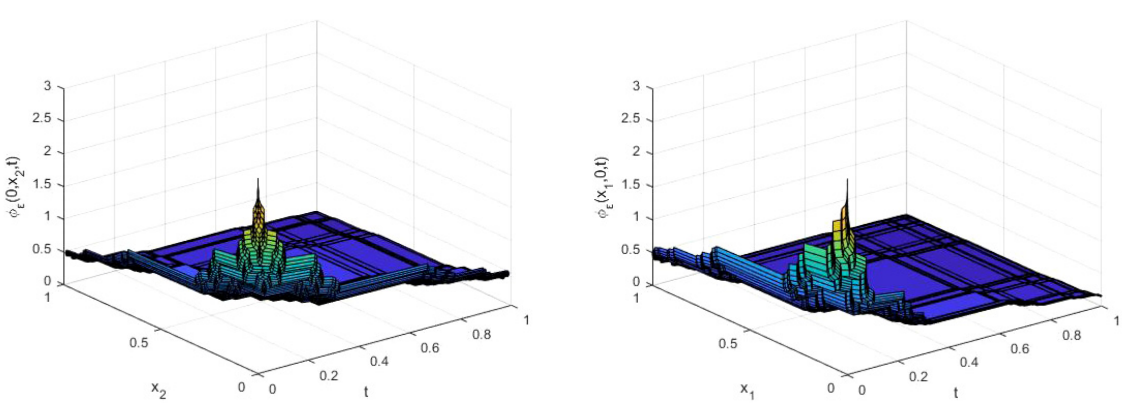
$$\phi_\varepsilon(\xi) = -\frac{\begin{array}{l} A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon)) \\ \quad + B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{54}$$

Family 3.

$$a_0 = \frac{k_1^2}{k_1^2 - 4k_0k_2}, \quad a_1 = \frac{4k_1k_2}{k_1^2 - 4k_0k_2},$$

$$a_2 = \frac{4k_2^2}{k_1^2 - 4k_0k_2}, \quad b_1 = b_2 = 0,$$

$$l_1^\varepsilon = \pm\sqrt{\frac{\begin{array}{c} -4m^{2\varepsilon}k_0k_2 + m^{2\varepsilon}k_1^2 + 4l_2^{2\varepsilon}k_0k_2 \\ - l_2^{2\varepsilon}k_1^2 + r^2 \end{array}}{k_1^2 - 4k_0k_2}}.$$



(a) (b)

Fig. 2 The exact traveling wave solution for the local fractional sine-Gordon equation for different parameters.

Thus, we get

$$\begin{aligned}u_{\varepsilon,3}(x_1,x_2,t)&\\ &= -i\ln\left(\frac{k_1^2}{k_1^2-4k_0k_2}+\frac{4k_1k_2}{k_1^2-4k_0k_2}\phi_\varepsilon(\xi)\right.\\ &\quad\left.+\frac{4k_2^2}{k_1^2-4k_0k_2}\phi_\varepsilon^2(\xi)\right).\end{aligned} \tag{55}$$

(I) For $\sum = k_1^2-4k_0k_2>0$,

$$\phi_\varepsilon(\xi)=\frac{\begin{array}{c}A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\end{array}}{\begin{array}{c}k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon)))\end{array}}. \tag{56}$$

(II) For $\sum = k_1^2-4k_0k_2<0$,

$$\phi_\varepsilon(\xi)=-\frac{\begin{array}{c}A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\end{array}}{\begin{array}{c}k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon)))\end{array}}. \tag{57}$$

Family 4.

$$a_0=-\frac{k_1^2}{k_1^2-4k_0k_2},\quad a_1=-\frac{4k_1k_2}{k_1^2-4k_0k_2},$$

$$a_2=-\frac{4k_2^2}{k_1^2-4k_0k_2},\quad b_1=b_2=0,$$

$$l_1^\varepsilon=\pm\sqrt{\frac{\begin{array}{c}-4m^{2\varepsilon}k_0k_2+m^{2\varepsilon}k_1^2+4l_2^{2\varepsilon}k_0k_2\\ -l_2^{2\varepsilon}k_1^2-r^2\end{array}}{k_1^2-4k_0k_2}}.$$

Thus, we get

$$\begin{aligned}u_{\varepsilon,4}(x_1,x_2,t)&\\ &= -i\ln\left(-\frac{k_1^2}{k_1^2-4k_0k_2}-\frac{4k_1k_2}{k_1^2-4k_0k_2}\phi_\varepsilon(\xi)\right.\\ &\quad\left.-\frac{4k_2^2}{k_1^2-4k_0k_2}\phi_\varepsilon^2(\xi)\right).\end{aligned} \tag{58}$$

(I) For $\sum = k_1^2-4k_0k_2>0$,

$$\phi_\varepsilon(\xi)=\frac{\begin{array}{c}A_0C_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_0D_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\end{array}}{\begin{array}{c}k_2(A_0E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_0E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon)))\end{array}}. \tag{59}$$

(II) For $\sum = k_1^2-4k_0k_2<0$,

$$\phi_\varepsilon(\xi)=-\frac{\begin{array}{c}A_1C_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_1D_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\end{array}}{\begin{array}{c}k_2(A_1E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon))\\ +B_1E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+m^\varepsilon t^\varepsilon)))\end{array}}. \tag{60}$$

Example 4.3. We discuss the local fractional Kadomtsev–Petviashvili equation in 3-fractal-dimensional space

$$\begin{aligned}&\frac{\partial^\varepsilon}{\partial x_1^\varepsilon}\left(\frac{\partial^\varepsilon u_\varepsilon}{\partial t^\varepsilon}+6u_\varepsilon\frac{\partial^\varepsilon u_\varepsilon}{\partial x_1^\varepsilon}+\frac{\partial^{3\varepsilon}u_\varepsilon}{\partial x_1^{3\varepsilon}}\right)\\ &\quad-3\frac{\partial^{2\varepsilon}u_\varepsilon}{\partial x_2^{2\varepsilon}}-3\frac{\partial^{2\varepsilon}u_\varepsilon}{\partial x_3^{2\varepsilon}}=0.\end{aligned} \tag{61}$$

Taking $i=3$ in (10), we have the traveling wave transformation

$$\xi^\varepsilon=l_1^\varepsilon x_1^\varepsilon+l_2^\varepsilon x_2^\varepsilon+l_3^\varepsilon x_3^\varepsilon+m^\varepsilon t^\varepsilon, \tag{62}$$

where

$$\lim_{\varepsilon\to1}\xi^\varepsilon=l_1x_1+l_2x_2+l_3+mt. \tag{63}$$

Substituting (20) into (61), then (61) reduces to

$$\begin{aligned}&l_1^{4\varepsilon}\frac{d^{4\varepsilon}u_\varepsilon}{d\xi^{4\varepsilon}}+(l_1^\varepsilon m^\varepsilon-3(l_2^{2\varepsilon}+l_3^{2\varepsilon}))\frac{d^{2\varepsilon}u_\varepsilon}{d\xi^{2\varepsilon}}\\ &\quad+6l_1^{2\varepsilon}\frac{d^\varepsilon}{d\xi^\varepsilon}\left(u_\varepsilon\frac{d^\varepsilon u_\varepsilon}{d\xi^\varepsilon}\right)=0.\end{aligned} \tag{64}$$

Applying the inverse local fractional derivative operator of (64) with respect to the integration constants taken to be zero twice, then we get

$$\frac{d^{2\varepsilon}u_\varepsilon}{d\xi^{2\varepsilon}}+\frac{l_1^\varepsilon m^\varepsilon-3l_2^{2\varepsilon}-3l_3^{2\varepsilon}}{l_1^{4\varepsilon}}u_\varepsilon+\frac{3}{l_1^{2\varepsilon}}u_{2\varepsilon}^2=0. \tag{65}$$

It is easy to find that $N=2$, and then we have

$$v_\varepsilon(\xi)=\sum_{i=0}^{2}a_i\phi_\varepsilon^i(\xi)+\sum_{i=1}^{2}b_i\phi_\varepsilon^{-i}(\xi). \tag{66}$$

We substitute (66) and (13) into (65), and then collect the coefficients of $\phi_\varepsilon^i(\xi)$ and equate them to be 0, to obtain the following solutions.

Family 1.

$$a_0=-\frac{1}{3}l_1^{2\varepsilon}m^{2\varepsilon}(k_1^2+2k_0k_2),\quad a_1=-2l_1^{2\varepsilon}m^{2\varepsilon}k_1k_2,$$

$$a_2 = -2l_1^{2\varepsilon} m^{2\varepsilon} k_2^2 \quad b_1 = b_2 = 0,$$

$$l_2^\varepsilon = \pm\sqrt{\frac{4}{3} l_1^{4\varepsilon} m^{2\varepsilon} k_0 k_2 - \frac{1}{3} l_1^{4\varepsilon} m^{2\varepsilon} k_1^2 + \frac{1}{3} l_1^\varepsilon m^\varepsilon - l_3^{2\varepsilon}}.$$

Thus, we get

$$\begin{aligned} &u_{\varepsilon,1}(x_1, x_2, x_3, t) \\ &\quad = -\frac{1}{3} l_1^{2\varepsilon} m^{2\varepsilon}(k_1^2 + 2k_0 k_2) - 2l_1^{2\varepsilon} m^{2\varepsilon} k_1 k_2 \phi_\varepsilon(\xi) \\ &\qquad - 2l_1^{2\varepsilon} m^{2\varepsilon} k_2^2 \phi_\varepsilon^2(\xi). \end{aligned} \tag{67}$$

(I) For $\sum = k_1^2 - 4k_0 k_2 > 0$,

$$\phi_\varepsilon(\xi) = \frac{\begin{array}{l} A_0 C_0 E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_0 D_0 E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon \\ \quad + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_0 E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_0 E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon \\ \quad + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{68}$$

(II) For $\sum = k_1^2 - 4k_0 k_2 < 0$,

$$\phi_\varepsilon(\xi) = -\frac{\begin{array}{l} A_1 C_1 E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_1 D_1 E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon \\ \quad + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_1 E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_1 E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon \\ \quad + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{69}$$

(III) For $\sum = k_1^2 - 4k_0 k_2 = 0$,

$$\begin{aligned} \phi_\varepsilon(\xi) = {} & \alpha E_\varepsilon\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)\right) \\ & + \left(\beta(k_1 + 1) - \frac{\alpha k_1}{2}\right) \\ & \times \frac{l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon}{\Gamma(\varepsilon + 1)} E_\varepsilon \\ & \times \left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)\right). \end{aligned} \tag{70}$$

Family 2.

$$a_0 = -\frac{1}{3} l_1^{2\varepsilon} m^{2\varepsilon}(k_1^2 + 2k_0 k_2), \quad a_1 = a_2 = 0,$$

$$b_1 = -2l_1^{2\varepsilon} m^{2\varepsilon} k_1 k_2, \quad b_2 = -2l_1^{2\varepsilon} m^{2\varepsilon} k_2^2,$$

$$l_2^\varepsilon = \pm\sqrt{\frac{4}{3} l_1^{4\varepsilon} m^{2\varepsilon} k_0 k_2 - \frac{1}{3} l_1^{4\varepsilon} m^{2\varepsilon} k_1^2 + \frac{1}{3} l_1^\varepsilon m^\varepsilon - l_3^{2\varepsilon}}.$$

Thus, we get

$$\begin{aligned} &u_{\varepsilon,1}(x_1, x_2, x_3, t) \\ &\quad = -\frac{1}{3} l_1^{2\varepsilon} m^{2\varepsilon}(k_1^2 + 2k_0 k_2) - \frac{2l_1^{2\varepsilon} m^{2\varepsilon} k_1 k_2}{\phi_\varepsilon(\xi)} \\ &\qquad - \frac{2l_1^{2\varepsilon} m^{2\varepsilon} k_2^2}{\phi_\varepsilon^2(\xi)}. \end{aligned} \tag{71}$$

(I) For $\sum = k_1^2 - 4k_0 k_2 > 0$,

$$\phi_\varepsilon(\xi) = \frac{\begin{array}{l} A_0 C_0 E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_0 D_0 E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon \\ \quad + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_0 E_\varepsilon(-C_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_0 E_\varepsilon(-D_0(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon \\ \quad + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{72}$$

(II) For $\sum = k_1^2 - 4k_0 k_2 < 0$,

$$\phi_\varepsilon(\xi) = -\frac{\begin{array}{l} A_1 C_1 E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_1 D_1 E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon \\ \quad + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)) \end{array}}{\begin{array}{l} k_2(A_1 E_\varepsilon(-C_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon \\ \quad + m^\varepsilon t^\varepsilon)) + B_1 E_\varepsilon(-D_1(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon \\ \quad + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon))) \end{array}}. \tag{73}$$

Similarly, the exact traveling wave solution of Eq. (61) is shown in Fig. 3. Figure 3a exhibits the exact traveling wave solution when $A_1 = C_1 = D_1 = 1, B_1 = 2, k_2 = -3$, and $l_3^\varepsilon = 3, m^\varepsilon = 4, l_1^\varepsilon = l_2^\varepsilon = 0$ in (73). Figure 3b explains the exact traveling wave solution when $A_1 = B_1 = 2, C_1 = D_1 = 1, k_2 = -9$, and $l_2^\varepsilon = 5, m^\varepsilon = 3, l_1^\varepsilon = l_3^\varepsilon = 0$ in (73). Figure 3c displays the exact traveling wave solution when $A_1 = B_1 = C_1 = 1, D_1 = 2, k_2 = -6$, and $l_1^\varepsilon = 2, m^\varepsilon = 3, l_2^\varepsilon = l_3^\varepsilon = 0$ in (73).

(III) For $\sum = k_1^2 - 4k_0 k_2 = 0$,

$$\begin{aligned} \phi_\varepsilon(\xi) = {} & \alpha E_\varepsilon\left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)\right) \\ & + \left(\beta(k_1 + 1) - \frac{\alpha k_1}{2}\right) \\ & \times \frac{l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon}{\Gamma(\varepsilon + 1)} E_\varepsilon \\ & \times \left(-\frac{k_1}{2}(l_1^\varepsilon x_1^\varepsilon + l_2^\varepsilon x_2^\varepsilon + l_3^\varepsilon x_3^\varepsilon + m^\varepsilon t^\varepsilon)\right). \end{aligned} \tag{74}$$

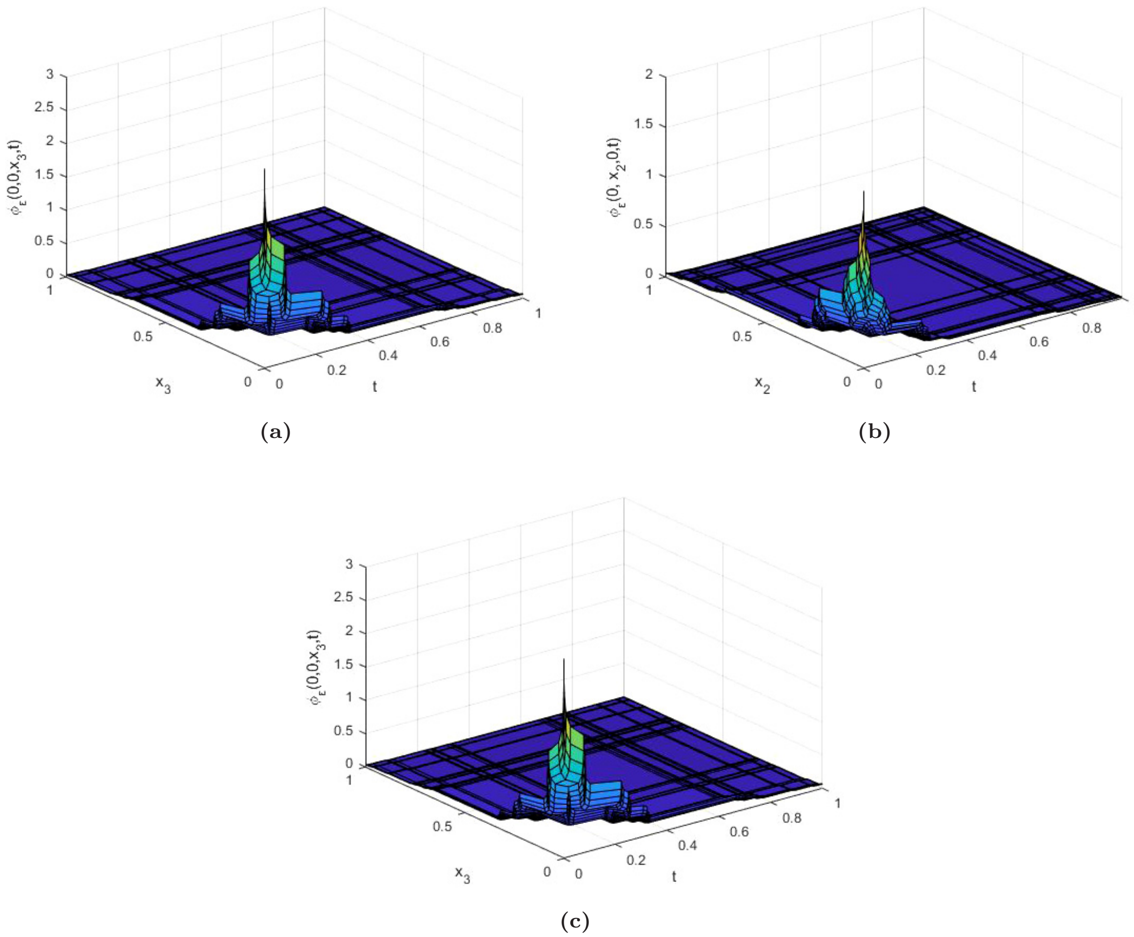


Fig. 3 The exact traveling wave solution for the local fractional Kadomtsev–Petviashvili equation for different parameters.

5. CONCLUSIONS

In this paper, a generalized LFRDE method was introduced to construct non-differentiable traveling wave solutions of factional differential equations in general dimensions, and applied to the seventh-order local fractional Sawada–Kotera–Ito equation in 1-fractal-dimensional space, the local fractional Sine-Gordon equation in 2-fractal-dimensional space and the local fractional Kadomtsev–Petviashvili equation in 3-fractal-dimensional space.

It is expected that the algorithm in the proposed method could be applied to solve other fractional partial differential equations. More generally, we can have an algorithm via Riccati equation, similar to the multiple exp-function method[30] or based on generalized bilinear forms,[31–33] to get multiple wave solutions and lump solutions to fractional partial differential equations.

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