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Bi-integrable couplings of a Kaup–Newell type soliton hierarchy and their bi-Hamiltonian structures

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ABSTRACT

A new Kaup–Newell type soliton hierarchy is generated from an asymmetric matrix spectral problem associated with the three-dimensional special linear Lie algebra $sl(2,\mathbb{R})$. Then based on semi-direct sums of matrix Lie algebras consisting of 3×3 block matrix Lie algebras, corresponding bi-integrable couplings of this hierarchy are constructed. Each equation in the resulting system has a bi-Hamiltonian structure furnished by the variational identity, which lead to Liouville integrability.

1. Introduction

Recently, seeking for new integrable systems including soliton hierarchies and integrable couplings forms a pretty important and interesting area of research in mathematical physics. Usually, soliton hierarchies associate with semisimple Lie algebras from the point of view of zero curvature equations and possess many nice properties such as bi-Hamiltonian structures, which lead to Liouville integrability. Classical soliton hierarchies include the Ablowitz–Kaup–Newell–Segur hierarchy, the Kaup–Newell (KN) hierarchy, the Wadati–Konno–Ichikawa hierarchy, the Korteweg–de Vries hierarchy, the Dirac hierarchy, the Boiti–Pempinelli–Tu hierarchy and so on [1–19]. For integrable couplings, zero curvature equations over semi-direct sums of Lie algebras, i.e., non-semisimple Lie algebras, lay the foundation for generating them, which provide valuable new insights into the classification of multi-component integrable systems. Some concrete integrable couplings showed various specific mathematical structures such as block matrix type Lax representations, bi-Hamiltonian structures of triangular form, have been presented in Refs. [20–41]. In addition, the Hamiltonian structures for soliton hierarchies can be established by the trace identity [8–10] or for integrable couplings by the variational identity [16,36].

Firstly, we shall make use of the three-dimensional special linear Lie algebra $sl(2,\mathbb{R})$ to construct a new KN type soliton hierarchy and consider corresponding bi-Hamiltonian structures by means of the trace identity [8–10] in the paper. This Lie algebra consists of 2×2 trace-free matrices, and has the following basis

$$e_1=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad e_2=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad e_3=\begin{bmatrix}0&0\\1&0\end{bmatrix}, \tag{1.1}$$

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with the circular commutator relations

$$[e_1, e_2] = 2e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_3] = -2e_3.$$

Then the matrix loop algebra $\widetilde{\mathrm{sl}}(2,\mathbb{R})$ can be defined by

$$\widetilde{\mathrm{sl}}(2,\mathbb{R}) = \left\{ \sum_{i \geqslant 0} M_i \lambda^{n-i} | M_i \in \mathrm{sl}(2,\mathbb{R}),\ i \geqslant 0,\ n \in \mathbb{Z} \right\}, \tag{1.2}$$

which is the space of all Laurent series in λ with a finite number of non-zero terms of positive powers of λ and coefficient matrices in $\mathrm{sl}(2,\mathbb{R})$. Thus the standard procedure for building soliton hierarchy associated with $\mathrm{sl}(2,\mathbb{R})$ can be given as follows:

Step 1: We need to select an appropriate spectral matrix $U = U(u,\lambda) \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ to form a spatial spectral problem $\phi_x = U(u,\lambda)\phi$, where u denotes a column dependent variable and λ is the spectral parameter.

Step 2: Then, we construct a Laurent series solution $W = W(u,\lambda)$ such as $W = \sum_{k=0}^{\infty} W_k \lambda^{-2k}, W_k \in \mathrm{sl}(2,\mathbb{R})$ to the stationary zero curvature equation $W_x = [U, W]$, based on which one can also prove the localness property for W.

Step 3: Further, we construct suitable temporal spectral problems $\phi_{t_m} = V^{[m]}\phi, m \geqslant 0$ to guarantee that the zero curvature equations $U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0$ can generate a soliton hierarchy $u_{t_m} = K_m(u)$, where $V^{[m]} = (\lambda^m W)_+ + \Delta_m \in \widetilde{\mathrm{sl}}(2,\mathbb{R})$ is derived from W.

Step 4: Finally, in order to show a certain integrability, we construct bi-Hamiltonian structures for the obtained soliton hierarchy, which lead to Liouville integrability by using the trace identity $\frac{\delta}{\delta u}\int \mathrm{tr}(\frac{\partial U}{\partial \lambda} W)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}(\frac{\partial U}{\partial u} W)$.

After completing the construction of the new KN type soliton hierarchy, we consider further work including corresponding bi-integrable couplings and bi-Hamiltonian structures via the variational identity [16,36]. Bi-integrable couplings of a soliton hierarchy $u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots)$ have the following form

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u, u_1), \\ u_{2,t} = S_2(u, u_1, u_2) \end{cases} \tag{1.3}$$

generated from certain non-semisimple Lie algebras, where u, u_1, u_2 denote some column vectors of dependent variables. The system (1.3) is called a nonlinear bi-integrable coupling if at least one of $S_1(u, u_1), S_2(u, u_1, u_2)$ are nonlinear with respect to any sub-vectors u_1, u_2. In this paper, we will introduce the following non-semisimple Lie algebras $\bar{\mathfrak{g}}$ with triangular block matrices [40,41]

$$\bar{\mathfrak{g}} = \{M(A_1, A_2, A_3)\},$$
$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha A_2 \\ 0 & 0 & A_1 \end{bmatrix}, \tag{1.4}$$

to generate bi-integrable couplings (1.3), where A_1, A_2, A_3 are arbitrary square matrices of the same order and α is a non-zero constant. In the following calculation process, we can see that block A_1 is used for generating the initial soliton hierarchy and block A_2, A_3 are used for constructing the supplementary sub-vector fields S_1, S_2. The above non-semisimple Lie algebras $\bar{\mathfrak{g}}$ have two subalgebras

$$\tilde{\mathfrak{g}} = \{M(A_1, 0, 0)\},$$
$$\tilde{\mathfrak{g}}_c = \{M(0, A_2, A_3)\},$$

which form semi-direct sums: $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}} \Subset \tilde{\mathfrak{g}}_c$. The notion of semi-direct sums means that the two subalgebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ satisfy $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_c] \subseteq \tilde{\mathfrak{g}}_c$. We also require the closure property between $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ under the matrix multiplication: $\tilde{\mathfrak{g}}\tilde{\mathfrak{g}}_c, \tilde{\mathfrak{g}}_c\tilde{\mathfrak{g}} \subseteq \tilde{\mathfrak{g}}_c$.

The paper is organized as follows. In Section 2, we would like to construct a new KN type soliton hierarchy associated with $\mathrm{sl}(2,\mathbb{R})$. Then bi-Hamiltonian structures are furnished by using the trace identity, and thus all equations in the resulting soliton hierarchy are Liouville integrable. In Section 3, we will use the non-semisimple Lie algebras (1.4) to establish bi-integrable couplings of the new KN type soliton hierarchy. Bi-Hamiltonian structures are shown by means of the variational identity. The last section is devoted to conclusions and discussions.

2. New KN type soliton hierarchy and bi-Hamiltonian structures

In order to present a new KN type soliton hierarchy associated with $\mathrm{sl}(2,\mathbb{R})$, we introduce the following asymmetric matrix spectral problem

$$\phi_x = U(u,\lambda)\phi = \begin{bmatrix} \lambda^2 q & \lambda p \\ \lambda & -\lambda^2 q \end{bmatrix}\phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.1}$$

We solve the stationary zero curvature equation firstly

$$W_x = [U, W], \quad W \in \widetilde{\mathrm{sl}}(2, \mathbb{R}). \tag{2.2}$$

When W is selected to be

$$W = ae_1 + be_2 + ce_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \tag{2.3}$$

we have

$$\begin{cases} a_x = cp\lambda - b\lambda, \\ b_x = 2bq\lambda^2 - 2ap\lambda, \\ c_x = 2a\lambda - 2cq\lambda^2, \end{cases} \tag{2.4}$$

from (2.2). Let us assume that a, b and c are Laurent series in λ given by

$$a = \sum_{i \geqslant 0} a_i \lambda^{-2i}, \quad b = \sum_{i \geqslant 0} b_i \lambda^{-2i-1}, \quad c = \sum_{i \geqslant 0} c_i \lambda^{-2i-1}. \tag{2.5}$$

Substituting (2.5) into (2.4), system (2.4) leads equivalently to

$$\begin{cases} a_{i,x} = pc_i - b_i, \\ b_{i+1} = \frac{p}{q} a_{i+1} + \frac{1}{2q} b_{i,x}, \quad i \geqslant 0, \\ c_{i+1} = \frac{1}{q} a_{i+1} - \frac{1}{2q} c_{i,x}, \end{cases} \tag{2.6}$$

where the initial values are taken as

$$a_0 = 1, \quad b_0 = \frac{p}{q}, \quad c_0 = \frac{1}{q}, \tag{2.7}$$

which are required by the equations on the first powers of λ in (2.4)

$$a_{0,x} = pc_0 - b_0, \quad qb_0 = pa_0, \quad qc_0 = a_0.$$

Moreover, by (2.6), we can obtain an expression for $a_{i+1,x}$,

$$\begin{aligned} a_{i+1,x} &= pc_{i+1} - b_{i+1} \\ &= p\left(\frac{1}{q} a_{i+1} - \frac{1}{2q} c_{i,x}\right) - \left(\frac{p}{q} a_{i+1} + \frac{1}{2q} b_{i,x}\right) \\ &= -\frac{p}{2q} c_{i,x} - \frac{1}{2q} b_{i,x}. \end{aligned} \tag{2.8}$$

Thus we obtain a recursion relation for a_{i+1},

$$a_{i+1} = -\partial^{-1} \frac{p}{2q} \partial c_i - \partial^{-1} \frac{1}{2q} \partial b_i, \quad i \geqslant 0. \tag{2.9}$$

Then, from (2.6) and (2.9), the recursion relations for b_{i+1} and c_{i+1} are as follows

$$\begin{bmatrix} b_{i+1} \\ c_{i+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2q}\partial - \frac{p}{2q}\partial^{-1}\frac{1}{q}\partial & -\frac{p}{2q}\partial^{-1}\frac{p}{q}\partial \\ -\frac{1}{2q}\partial^{-1}\frac{1}{q}\partial & -\frac{1}{2q}\partial - \frac{1}{2q}\partial^{-1}\frac{p}{q}\partial \end{bmatrix} \begin{bmatrix} b_i \\ c_i \end{bmatrix}, \quad i \geqslant 0. \tag{2.10}$$

While using (2.9) and the above recursion relations (2.10), we impose the condition on the constants of integration:

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geqslant 1,$$

to determine the sequence of $\{a_i, b_i, c_i | i \geqslant 1\}$ uniquely. The first two sets can be computed as follows by using Maple

$$\begin{aligned} a_1 &= -\frac{p}{2q^2}, \\ b_1 &= \frac{qp_x - p^2 - pq_x}{2q^3}, \\ c_1 &= \frac{q_x - p}{2q^3}; \\ a_2 &= \frac{1}{8q^4}(3p^2 - 2qp_x), \\ b_2 &= \frac{1}{8q^5}(2p_{xx}q^2 - 6p_x pq - 6p_x q_x q - 2pq_{xx}q + 3p^3 + 6p^2 q_x + 6pq_x^2), \end{aligned}$$

$$c_2 = \frac{1}{8q^5}(-2q_{xx}q + 3p^2 - 6pq_x + 6q_x^2);$$
$$a_3 = -\frac{1}{16q^6}(2p_{xx}q^2 - 4p_xq_xq - 6p_xpq - 4pqq_{xx} + 10pq_x^2 + 5p^3),$$
$$b_3 = \frac{1}{16q^7}(2q^3p_{xxx} - 12q^2q_xp_{xx} - 8q^2pp_x - 2q^2pq_{xxx} - 6q^2p_x^2 - 8q^2p_xq_{xx} + 30qp_xq_x^2 + 40pqp_xq_x + 20pqq_xq_{xx} + 15qp^2p_x + 10qp^2q_{xx} - 30pq_x^3 - 40p^2q_x^2 - 15p^3q_x - 5p^4),$$
$$c_3 = -\frac{1}{16q^7}(2p_{xx}q^2 - 2q^2q_{xxx} - 10p_xq_xq + 20qq_xq_{xx} - 10pqq_{xx} - 30q_x^3 + 40pq_x^2 - 15p^2q_x + 5p^3).$$

We point out that the localness of the sequence of $\{a_i, b_i, c_i | i \geqslant 3\}$ can be shown by the mathematical induction, and thus, all the functions $\{a_i, b_i, c_i | i \geqslant 1\}$ are differential functions. Since from the stationary zero curvature Eq. (2.2), we can compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}(W^2) = 2\mathrm{tr}(WW_x) = 2\mathrm{tr}(W[U, W]) = 0$$

and hence, due to $\mathrm{tr}(W^2) = 2(a^2 + bc)$, we can obtain

$$a^2 + bc = (a^2 + bc)|_{u=0} = 1.$$

Thus a balance of coefficients of λ^{-2i-2} tells that

$$a_{i+1} = -\frac{1}{2}\left(\sum_{\substack{k+l=i+1\\k,l\geqslant 1}} a_k a_l + \sum_{\substack{k+l=i\\k,l\geqslant 0}} b_k c_l\right), \quad i \geqslant 2.$$

So based on the recursion relation (2.9) and the last two equations in (2.6), all the functions $\{a_i, b_i, c_i | i \geqslant 3\}$ are differential functions in p and q by applying the mathematical induction. It means that they are all local.

Now, from the recursion relations in (2.4), we have

$$(\lambda(\lambda^{2m+1}W)_+)_x - [U, \lambda(\lambda^{2m+1}W)_+] = \lambda b_{m,x}e_2 + \lambda c_{m,x}e_3,$$

where P_+ denotes the polynomial part of P. This is not the same type matrix as U_{t_m}:

$$U_{t_m} = \lambda^2 q_{t_m} e_1 + \lambda p_{t_m} e_2.$$

So we choose the suitable Lax matrices and modification terms as follows

$$\begin{aligned} V^{[m]} &= \lambda(\lambda^{2m+1}W)_+ + \Delta_m \\ &= \begin{bmatrix} \sum_{i=0}^m a_i\lambda^{2m-2i+2} + \lambda^2 f_m & \sum_{i=0}^m b_i\lambda^{2m-2i+1} + \lambda g_m \\ \sum_{i=0}^m c_i\lambda^{2m-2i+1} + \lambda h_m & -\sum_{i=0}^m a_i\lambda^{2m-2i+2} - \lambda^2 f_m \end{bmatrix}, \quad m \geqslant 0, \end{aligned} \tag{2.11}$$

to guarantee the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^m] = 0, \quad m \geqslant 0, \tag{2.12}$$

present a soliton hierarchy. After a complicated calculation from the recursion relations (2.6) and (2.9) by using Maple, the modification terms are determined

$$\begin{cases} h_m = -c_m, \\ f_m = -qc_m, \\ g_m = -pc_m, & m \geqslant 0. \\ p_{t_m} = b_{m,x} - (pc_m)_x, \\ q_{t_m} = -(qc_m)_x, \end{cases} \tag{2.13}$$

Thus, we obtain a new KN type soliton hierarchy

$$u_{t_m} = K_m = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = \begin{bmatrix} b_{m,x} - (pc_m)_x \\ -(qc_m)_x \end{bmatrix}, \quad m \geqslant 1, \tag{2.14}$$

which are all local, because of the localness of the sequence of $\{a_i, b_i, c_i | i \geqslant 1\}$. The first two nonlinear systems in this soliton hierarchy are

$$\begin{cases} p_{t_1} = \frac{1}{2q^4}(p_{xx}q^2 - 4p_xq_xq - 2pq_{xx}q + 6pq_x^2), \\ q_{t_1} = \frac{1}{2q^3}(p_xq - qq_{xx} - 2pq_x + 2q_x^2) \end{cases}$$

and

$$\begin{cases} p_{t_2} = \frac{1}{4q^6}\left[q^3p_{xxx} - 3q^2(p_x^2 + p_xq_{xx} + pp_{xx} + 2q_xp_{xx}) + 24pqp_xq_x + 12qp_xq_x^2 + 6qp^2q_{xx} - 30p^2q_x^2\right], \\ q_{t_2} = \frac{1}{4q^5}\left[q^2q_{xxx} - 3q(p_xp - p_xq_x - pq_{xx} + 3q_xq_{xx}) + 6p^2q_x - 12pq_x^2 + 12q_x^3\right]. \end{cases}$$

Next, we shall use the trace identity to construct Hamiltonian structure for the soliton hierarchy (2.14). It is direct to compute

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 2\lambda q & p \\ 1 & -2\lambda q \end{bmatrix}, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial \lambda}\right) = 4aq\lambda + cp + b,$$
$$\frac{\partial U}{\partial p} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial p}\right) = c\lambda,$$
$$\frac{\partial U}{\partial q} = \begin{bmatrix} \lambda^2 & 0 \\ 0 & -\lambda^2 \end{bmatrix}, \quad \mathrm{tr}\left(W\frac{\partial U}{\partial q}\right) = 2a\lambda^2.$$

Then, the trace identity [8–10]

$$\frac{\delta}{\delta u}\int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda}W\right)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}\left(\frac{\partial U}{\partial u}W\right), \tag{2.15}$$

becomes

$$\frac{\delta}{\delta u}\int (4aq\lambda + cp + b)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{bmatrix} c\lambda \\ 2a\lambda^2 \end{bmatrix}.$$

Substituting (2.5) into above equation and balancing coefficients of λ^{-2m-1} in this equality, it leads to

$$\frac{\delta}{\delta u}\int (4qa_{m+1} + pc_m + b_m)\mathrm{d}x = (\gamma - 2m)\begin{bmatrix} c_m \\ 2a_{m+1} \end{bmatrix}, \quad m \geqslant 0.$$

The identity with $m = 1$ gives $\gamma = 0$, and thus we obtain

$$\frac{\delta}{\delta u}\mathcal{H}_m = \begin{bmatrix} c_m \\ 2a_{m+1} \end{bmatrix}, \quad m \geqslant 0, \tag{2.16}$$

with the Hamiltonian functionals being defined by

$$\mathcal{H}_0 = \int \frac{p}{q}\mathrm{d}x,$$
$$\mathcal{H}_m = -\int \left(\frac{4qa_{m+1} + pc_m + b_m}{2m}\right)\mathrm{d}x, \quad m \geqslant 1.$$

Further, from recursion relation (2.6), we can obtain

$$\begin{aligned} b_{m,x} &= 2qb_{m+1} - 2pa_{m+1} \\ &= 2q(pc_{m+1} - a_{m+1,x}) - 2pa_{m+1} \\ &= 2pqc_{m+1} - 2qa_{m+1,x} - 2pa_{m+1} \\ &= 2p(a_{m+1} - \frac{1}{2}c_{m,x}) - 2pa_{m+1} - 2qa_{m+1,x} \\ &= -pc_{m,x} - 2qa_{m+1,x}. \end{aligned}$$

Consequently, the soliton hierarchy (2.14) has the Hamiltonian structure

$$u_{t_m} = K_m = \begin{bmatrix} b_{m,x} - (pc_m)_x \\ -(qc_m)_x \end{bmatrix} = J\begin{bmatrix} c_m \\ 2a_{m+1} \end{bmatrix}, \quad m \geqslant 1, \tag{2.17}$$

where the Hamiltonian operator is defined by

$$J = \begin{bmatrix} -p\partial - \partial p & -q\partial \\ -\partial q & 0 \end{bmatrix}. \tag{2.18}$$

From the recursion relation (2.6) and (2.9), it is obvious that

$$\frac{\delta \mathcal{H}_m}{\delta u} = \Psi\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geqslant 1, \tag{2.19}$$

where

$$\Psi = \begin{bmatrix} -\frac{1}{2q}\partial & \frac{1}{2q} \\ \frac{1}{2}\partial^{-1}\frac{1}{q}\partial\frac{p}{q}\partial + \frac{1}{2}\partial^{-1}\frac{p}{q}\partial\frac{1}{q}\partial & \frac{1}{2}\partial^{-1}\frac{1}{q}\partial^2 - \frac{1}{2}\partial^{-1}\frac{1}{q}\partial\frac{p}{q} - \frac{1}{2}\partial^{-1}\frac{p}{q}\partial\frac{1}{q} \end{bmatrix}. \tag{2.20}$$

It is a direct computation that all members in the soliton hierarchy (2.14) are bi-Hamiltonian

$$u_{t_m} = K_m = J\frac{\delta\mathcal{H}_m}{\delta u} = M\frac{\delta\mathcal{H}_{m-1}}{\delta u}, \quad m \geqslant 1, \tag{2.21}$$

where

$$M = J\Psi = \begin{bmatrix} 0 & -\frac{1}{2}\partial^2 \\ \frac{1}{2}\partial^2 & -\frac{1}{2}\partial \end{bmatrix}. \tag{2.22}$$

Then from $K_{m+1} = \Phi K_m, m \geqslant 1$, and $J\Psi = \Phi J$, we obtain a common hereditary recursion operator for the soliton hierarchy (2.14)

$$\Phi = \Psi^\dagger = \begin{bmatrix} \partial\frac{1}{2q} & -\frac{1}{2}\partial\frac{p}{q}\partial\frac{1}{q}\partial^{-1} - \frac{1}{2}\partial\frac{1}{q}\partial\frac{p}{q}\partial^{-1} \\ \frac{1}{2q} & -\frac{1}{2}\partial^2\frac{1}{q}\partial^{-1} - \frac{p}{2q}\partial\frac{1}{q}\partial^{-1} - \frac{1}{2q}\partial\frac{p}{q}\partial^{-1} \end{bmatrix}, \tag{2.23}$$

where $\Psi^\dagger$ denotes the conjugate operator of Ψ.

Upon observation of the bi-Hamiltonian structures (2.21) and differential orders of the sequence $\{a_i, b_i, c_i | i \geqslant 1\}$, we can state that the soliton hierarchy (2.14) is Liouville integrable [37–41]. Every member in the hierarchy (2.14) possesses infinitely many independent commuting conserved functionals

$$\{\mathcal{H}_k, \mathcal{H}_l\}_J := \int \left(\tfrac{\delta\mathcal{H}_k}{\delta u}\right)^T J\tfrac{\delta\mathcal{H}_l}{\delta u}\,\mathrm{d}x = 0, \quad k, l \geqslant 0,$$
$$\{\mathcal{H}_k, \mathcal{H}_l\}_M := \int \left(\tfrac{\delta\mathcal{H}_k}{\delta u}\right)^T M\tfrac{\delta\mathcal{H}_l}{\delta u}\,\mathrm{d}x = 0, \quad k, l \geqslant 0$$

and symmetries

$$[K_k, K_l] := K_k'(u)[K_l] - K_l'(u)[K_k] = 0, \quad k, l \geqslant 0,$$

where K' denotes the Gateaux derivative.

3. Bi-integrable couplings and bi-Hamiltonian structures

In this section, we consider corresponding bi-integrable couplings and bi-Hamiltonian structures for the obtained KN type soliton hierarchy (2.14). Bi-integrable couplings will be constructed directly by means of non-semisimple Lie algebras which are used for generating enlarged zero curvature equations. Then bi-Hamiltonian structures are furnished by the variational identity [16,36], which lead to Liouville integrability for the new obtained bi-integrable couplings.

3.1. Bi-integrable couplings of (2.14)

We proceed to construct bi-integrable couplings of the new KN type soliton hierarchy (2.14). An enlarged spectral matrix is chosen as

$$\overline{U} = \overline{U}(\bar{u}, \lambda) = M(U, U_1, U_2), \quad \bar{u} = (p, q, r_1, s_1, r_2, s_2)^T, \tag{3.1}$$

where U is defined as in (2.1) and

$$U_i = U_i(u_i) = \begin{bmatrix} \lambda^2 s_i & \lambda r_i \\ 0 & -\lambda^2 s_i \end{bmatrix}, \quad u_i = \begin{bmatrix} r_i \\ s_i \end{bmatrix}, \quad i = 1, 2, \tag{3.2}$$

where r_1, s_1, r_2 and s_2 are new dependent variables.

To solve the corresponding enlarged stationary zero curvature equation

$$\overline{W}_x = [\overline{U}, \overline{W}], \tag{3.3}$$

we choose a solution of the following form

$$\overline{W} = \overline{W}(\bar{u}, \lambda) = M(W, W_1, W_2) \in \bar{\mathfrak{g}}, \tag{3.4}$$

where W is given by (2.3) and W_1, W_2 are assumed to be

$$W_1 = \begin{bmatrix} e & f \\ g & -e \end{bmatrix} = \begin{bmatrix} \Sigma_{i\geqslant 0} e_i \lambda^{-2i} & \Sigma_{i\geqslant 0} f_i \lambda^{-2i-1} \\ \Sigma_{i\geqslant 0} g_i \lambda^{-2i-1} & -\Sigma_{i\geqslant 0} e_i \lambda^{-2i} \end{bmatrix},$$
$$W_2 = \begin{bmatrix} e' & f' \\ g' & -e' \end{bmatrix} = \begin{bmatrix} \Sigma_{i\geqslant 0} e'_i \lambda^{-2i} & \Sigma_{i\geqslant 0} f'_i \lambda^{-2i-1} \\ \Sigma_{i\geqslant 0} g'_i \lambda^{-2i-1} & -\Sigma_{i\geqslant 0} e'_i \lambda^{-2i} \end{bmatrix}. \tag{3.5}$$

Then Eq. (3.3) is equivalent to

$$\begin{cases} W_x = [U, W], \\ W_{1,x} = [U, W_1] + [U_1, W], \\ W_{2,x} = [U, W_2] + [U_2, W] + \alpha[U_1, W_1]. \end{cases} \tag{3.6}$$

Substituting (3.5) into (3.6), we get (2.4),

$$\begin{cases} e_x = cr_1\lambda + gp\lambda - f\lambda, \\ f_x = 2bs_1\lambda^2 + 2fq\lambda^2 - 2ar_1\lambda - 2ep\lambda, \\ g_x = -2cs_1\lambda^2 - 2gq\lambda^2 + 2e\lambda \end{cases} \tag{3.7}$$

and

$$\begin{cases} e'_x = \alpha g r_1\lambda + cr_2\lambda + g'p\lambda - f'\lambda, \\ f'_x = 2bs_2\lambda^2 + 2f'q\lambda^2 - 2ar_2\lambda - 2e'p\lambda + 2\alpha s_1 f\lambda^2 - 2\alpha r_1 e\lambda, \\ g'_x = -2cs_2\lambda^2 - 2g'q\lambda^2 + 2e'\lambda - 2\alpha g s_1\lambda^2. \end{cases} \tag{3.8}$$

Equivalently, the systems (3.7) and (3.8) lead to the recursion relations

$$\begin{cases} e_{i,x} = r_1 c_i + pg_i - f_i, \\ f_{i,x} = 2s_1 b_{i+1} + 2qf_{i+1} - 2r_1 a_{i+1} - 2pe_{i+1}, \\ g_{i,x} = -2s_1 c_{i+1} - 2qg_{i+1} + 2e_{i+1} \end{cases} \tag{3.9}$$

and

$$\begin{cases} e'_{i,x} = \alpha r_1 g_i + r_2 c_i + pg'_i - f'_i, \\ f'_{i,x} = 2qf'_{i+1} - 2pe'_{i+1} + 2\alpha s_1 f_{i+1} - 2\alpha r_1 e_{i+1} + 2s_2 b_{i+1} - 2r_2 a_{i+1}, \\ g'_{i,x} = -2\alpha s_1 g_{i+1} - 2s_2 c_{i+1} - 2qg'_{i+1} + 2e'_{i+1}. \end{cases} \tag{3.10}$$

We take the initial values

$$e_0 = 0, \quad f_0 = \frac{r_1 q - s_1 p}{q^2}, \quad g_0 = -\frac{s_1}{q^2},$$
$$e'_0 = 0, \quad f'_0 = \frac{-\alpha s_1 r_1 q + \alpha s_1^2 p - s_2 pq + r_2 q^2}{q^3}, \quad g'_0 = \frac{\alpha s_1^2 - s_2 q}{q^3} \tag{3.11}$$

and assume the constants of integration as zero, the recursion relations (3.9) and (3.10) uniquely generate the sequences of $\{e_i, f_i, g_i | i \geqslant 1\}, \{e'_i, f'_i, g'_i | i \geqslant 1\}$. The first set of functions in the two sequences are computed as follows by Maple

$$e_1 = -\frac{r_1 q - 2s_1 p}{2q^3},$$
$$g_1 = -\frac{r_1 q - qs_{1,x} + 3s_1 q_x - 3s_1 p}{2q^4},$$
$$f_1 = \frac{r_{1,x}q^2 - r_1 q_x q - 2r_1 pq - 2qp_x s_1 - qps_{1,x} + 3ps_1 q_x + 3s_1 p^2}{2q^4},$$
$$e'_1 = -\frac{-2\alpha s_1 r_1 q + 3\alpha s_1^2 p + r_2 q^2 - 2s_2 pq}{2q^4},$$
$$g'_1 = -\frac{-6\alpha s_1^2 q_x + 3\alpha s_1 q s_{1,x} - q^2 s_{2,x} + 3qs_2 q_x + 6\alpha s_1^2 p - 3\alpha s_1 r_1 q - 3s_2 pq + r_2 q^2}{2q^5},$$
$$f'_1 = -\frac{1}{2q^5}[(\alpha r_1 q^2 - 3\alpha s_1 pq)s_{1,x} + 2\alpha s_1 q^2 r_{1,x} - q^3 r_{2,x} + pq^2 s_{2,x} + (2s_2 q^2 - 3\alpha q s_1^2)p_x$$
$$+ (r_2 q^2 - 3s_2 pq + 6\alpha p s_1^2 - 3\alpha q s_1 r_1)q_x + 6\alpha s_1^2 p^2 - 3p^2 q s_2 + \alpha r_1^2 q^2 + 2r_2 pq^2 - 6\alpha pq s_1 r_1].$$

Further, we introduce the enlarge Lax matrices

$$\overline{V}^{[m]} = M(V^{[m]}, V_1^{[m]}, V_2^{[m]}) \in \bar{\mathfrak{g}}, \quad m \geqslant 0, \tag{3.12}$$

where $V^{[m]}$ is defined as in (2.11) and

$$V_1^{[m]} = \begin{bmatrix} \sum_{i=0}^{m} e_i\lambda^{2m-2i+2} + \lambda^2\varepsilon_m & \sum_{i=0}^{m} f_i\lambda^{2m-2i+1} + \lambda\delta_m \\ \sum_{i=0}^{m} g_i\lambda^{2m-2i+1} + \lambda\eta_m & -\sum_{i=0}^{m} e_i\lambda^{2m-2i+2} - \lambda^2\varepsilon_m \end{bmatrix}, \quad m \geqslant 0,$$
$$V_2^{[m]} = \begin{bmatrix} \sum_{i=0}^{m} e'_i\lambda^{2m-2i+2} + \lambda^2\varepsilon'_m & \sum_{i=0}^{m} f'_i\lambda^{2m-2i+1} + \lambda\delta'_m \\ \sum_{i=0}^{m} g'_i\lambda^{2m-2i+1} + \lambda\eta'_m & -\sum_{i=0}^{m} e'_i\lambda^{2m-2i+2} - \lambda^2\varepsilon'_m \end{bmatrix}, \quad m \geqslant 0. \tag{3.13}$$

Then, the enlarged zero curvature equations

$$\overline{U}_{t_m} = \overline{V}_x^{[m]} + [\overline{U}, \overline{V}^{[m]}], \quad m \geqslant 0, \tag{3.14}$$

give

$$\begin{cases} U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \\ U_{1,t_m} - V_{1,x}^{[m]} + [U, V_1^{[m]}] + [U_1, V^{[m]}] = 0, \\ U_{2,t_m} - V_{2,x}^{[m]} + [U, V_2^{[m]}] + [U_2, V^{[m]}] + \alpha[U_1, V_1^{[m]}] = 0. \end{cases} \tag{3.15}$$

All above equations determine the KN type soliton hierarchy (2.14) of the bi-integrable couplings

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r_1 \\ s_1 \\ r_2 \\ s_2 \end{bmatrix}_{t_m} = \overline{K}_m(\bar{u}) = \begin{bmatrix} K_m(u) \\ S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix} = \begin{bmatrix} b_{m,x} - (pc_m)_x \\ -(qc_m)_x \\ f_{m,x} - (pg_m + r_1c_m)_x \\ -(s_1c_m + qg_m)_x \\ f'_{m,x} - (pg'_m + r_2c_m + \alpha r_1 g_m)_x \\ -(s_2c_m + qg'_m + \alpha s_1 g_m)_x \end{bmatrix}. \tag{3.16}$$

3.2. Bi-Hamiltonian structures and Liouville integrability

When the associated matrix Lie algebras are semisimple, the Hamiltonian structure can be established by the trace identity. However, when the associated matrix Lie algebras are non-semisimple, the variational identity [16,36] provides an effective approach to furnish the Hamiltonian structure of soliton equations. We shall search for bi-Hamiltonian structures by the variational identity:

$$\frac{\delta}{\delta\bar{u}}\int\left\langle\frac{\partial\overline{U}}{\partial\lambda}, \overline{W}\right\rangle dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle\frac{\partial\overline{U}}{\partial\bar{u}}, \overline{W}\right\rangle. \tag{3.17}$$

The key step of constructing non-degenerate, symmetric and ad-invariant bilinear form on non-semisimple matrix loop algebras $\bar{\mathfrak{g}}$ is to transform the semi-direct sums $\bar{\mathfrak{g}}$ into a vector form (see [37,40]).

First, Defining a mapping

$$\sigma : \bar{\mathfrak{g}}(\lambda) \to \mathbb{R}^9, \quad A \mapsto (a_1, \ldots, a_9)^T, \tag{3.18}$$

where

$$A = M(A_1, A_2, A_3) \in \bar{\mathfrak{g}}(\lambda), \quad A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix}, \quad i = 1, 2, 3. \tag{3.19}$$

This mapping σ induces a Lie algebraic structure on $\mathbb{R}^9$. The Lie bracket $[\cdot,\cdot]$ on $\mathbb{R}^9$ can be computed as follows

$$[a, b]^T = a^T R(b), \quad a = (a_1, \ldots, a_9)^T, \quad b = (b_1, \ldots, b_9)^T \in \mathbb{R}^9, \tag{3.20}$$

where

$$R(b) = M(R_1, R_2, R_3), \quad R_i = \begin{bmatrix} 0 & 2b_{3i-1} & -2b_{3i} \\ b_{3i} & -2b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2b_{3i-2} \end{bmatrix}, \quad i = 1, 2, 3. \tag{3.21}$$

The mapping (3.18) is a Lie algebra isomorphism between the two Lie algebras.

Then, we define a bilinear form [37,40] on $\mathbb{R}^9$ as follow

$$\langle a,b\rangle = a^T F b, \tag{3.22}$$

where F is a constant matrix. Obviously the symmetric property and ad-invariance property of F

$$\langle a,b\rangle = \langle b,a\rangle, \quad \langle a,[b,c]\rangle = \langle [a,b],c\rangle, \tag{3.23}$$

require that

$$F^T = F, \quad F(R(b))^T = -R(b)F, \quad b\in\mathbb{R}^9. \tag{3.24}$$

Solving the system (3.24) by Maple, we obtain

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{3.25}$$

where η_1, η_2, η_3 are arbitrary constants. The bilinear form on the semi-direct sums $\bar{\mathfrak{g}}(\lambda)$ of the two Lie subalgebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_c$ is defined as

$$\langle A,B\rangle_{\bar{\mathfrak{g}}(\lambda)} = \langle \sigma(A),\sigma(B)\rangle_{\mathbb{R}^9} = (a_1,\ldots,a_9)F(b_1,\ldots,b_9)^T, \tag{3.26}$$

where $A=\sigma^{-1}((a_1,\ldots,a_9)^T)\in\bar{\mathfrak{g}}(\lambda), B=\sigma^{-1}((b_1,\ldots,b_9)^T)\in\bar{\mathfrak{g}}(\lambda)$.

Because of the isomorphism of σ, the bilinear form (3.26) is also symmetric and ad-invariant:

$$\langle A,B\rangle_{\bar{\mathfrak{g}}(\lambda)} = \langle B,A\rangle_{\bar{\mathfrak{g}}(\lambda)}, \quad \langle A,[B,C]\rangle_{\bar{\mathfrak{g}}(\lambda)} = \langle [A,B],C\rangle_{\bar{\mathfrak{g}}(\lambda)}, \quad A,B,C\in\bar{\mathfrak{g}}(\lambda). \tag{3.27}$$

But this kind of bilinear form is not of Killing type, since the enlarged matrix loop algebras $\bar{\mathfrak{g}}(\lambda)$ are not semisimple. A bilinear form (3.26) is non-degenerate if and only if the determinant of F is not zero, namely,

$$\det(F) = 8\alpha^3\eta_3^9 \neq 0, \tag{3.28}$$

thus, η_3 and α should be non-zero constants.

Then, according to the definition of bilinear form, we can compute by Maple

$$\left\langle \overline{W}, \frac{\partial\overline{U}}{\partial\lambda}\right\rangle_{\bar{\mathfrak{g}}(\lambda)} = (4aq\lambda + cp + b)\eta_1 + (4eq\lambda + gp + f + 4as_1\lambda + cr_1)\eta_2 + (4e'q\lambda + g'p + f' + 4\alpha es_1\lambda + \alpha gr_1 + 4as_2\lambda + cr_2)\eta_3$$

and

$$\left\langle \overline{W}, \frac{\partial\overline{U}}{\partial\bar{u}}\right\rangle_{\bar{\mathfrak{g}}(\lambda)} = (c\lambda\eta_1 + g\lambda\eta_2 + g'\lambda\eta_3, 2a\lambda^2\eta_1 + 2e\lambda^2\eta_2 + 2e'\lambda^2\eta_3, c\lambda\eta_2 + \alpha g\lambda\eta_3, 2a\lambda^2\eta_2 + 2\alpha e\lambda^2\eta_3, c\lambda\eta_3, 2a\lambda\eta_3)^T,$$

From the formula

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle \overline{W},\overline{W}\rangle|, \tag{3.29}$$

we can determine the parameter $\gamma = 0$. Consequently, by using the variational identity, we obtain the Hamiltonian structure for the hierarchy (3.16) of the bi-integrable couplings

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \quad m \geqslant 0, \tag{3.30}$$

with the Hamiltonian functionals

$$\bar{\mathcal{H}}_m = \int \left[\frac{(4qa_{m+1} + pc_m + b_m)\eta_1 + (4qe_{m+1} + pg_m + f_m + 4s_1a_{m+1} + r_1c_m)\eta_2}{2m} + \frac{(4qe'_{m+1} + pg'_m + f'_m + 4\alpha e_{m+1}s_1 + \alpha g_m r_1 + 4a_{m+1}s_2 + c_m r_2)\eta_3}{2m}\right] dx, \quad m \geqslant 0 \tag{3.31}$$

and the Hamiltonian operator

$$\bar{J} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & J_1 \\ \mathbf{0} & J_1 & J_2 \\ J_1 & J_2 & J_3 \end{bmatrix}, \tag{3.32}$$

where

$$J_1=\begin{bmatrix}\frac{-p\partial-\partial p}{\alpha\eta_3} & -\frac{q\partial}{\alpha\eta_3}\\ -\frac{\partial q}{\alpha\eta_3} & 0\end{bmatrix},$$

$$J_2=\begin{bmatrix}\frac{(p\partial+\partial p)\eta_2}{\alpha\eta_3^2}-\frac{r_1\partial+\partial r_1}{\eta_3} & \frac{q\partial\eta_2}{\alpha\eta_3^2}-\frac{s_1\partial}{\eta_3}\\ \frac{\partial q\eta_2}{\alpha\eta_3^2}-\frac{\partial s_1}{\eta_3} & 0\end{bmatrix},$$

$$J_3=\begin{bmatrix}\frac{(p\partial+\partial p)\eta_1}{\eta_3^2}-\frac{(p\partial+\partial p)\eta_2^2}{\alpha\eta_3^3}+\frac{(r_1\partial+\partial r_1)\eta_2}{\eta_3^2}-\frac{r_2\partial+\partial r_2}{\eta_3} & \frac{q\partial\eta_1}{\eta_3^2}-\frac{q\partial\eta_2^2}{\alpha\eta_3^3}+\frac{s_1\partial\eta_2}{\eta_3^2}-\frac{s_2\partial}{\eta_3}\\ \frac{\partial q\eta_1}{\eta_3^2}-\frac{\partial q\eta_2^2}{\alpha\eta_3^3}+\frac{\partial s_1\eta_2}{\eta_3^2}-\frac{\partial s_2}{\eta_3} & 0\end{bmatrix}.$$

Here $\mathbf{0}$ is zero square matrix of the same order as J_1, J_2, J_3 and $\eta_3\neq 0, \alpha, \eta_1, \eta_2$ are arbitrary constants.

Next we consider recursion relations of the bi-integrable couplings by taking the following form

$$\overline{K}_m=\bar{\Phi}\overline{K}_{m-1}=\begin{bmatrix}\Phi & \mathbf{0} & \mathbf{0}\\ \Phi_1 & \Phi & \mathbf{0}\\ \Phi_2 & \alpha\Phi_1 & \Phi\end{bmatrix}\overline{K}_{m-1},\quad m\geqslant 1. \tag{3.33}$$

With the aid of symbolic computation by Maple, we can compute

$$\Phi=\begin{bmatrix}\partial\frac{1}{2q} & -\frac{1}{2}\partial\frac{p}{q}\partial\frac{1}{q}\partial^{-1}-\frac{1}{2}\partial\frac{1}{q}\partial\frac{p}{q}\partial^{-1}\\ \frac{1}{2q} & -\frac{1}{2}\partial^2\frac{1}{q}\partial^{-1}-\frac{p}{2q}\partial\frac{1}{q}\partial^{-1}-\frac{1}{2q}\partial\frac{p}{q}\partial^{-1}\end{bmatrix},$$

$$\Phi_1=\begin{bmatrix}-\partial\frac{s_1}{2q^2} & \partial\frac{s_1}{q}\left(\frac{p}{2q}\partial\frac{1}{q}\partial^{-1}+\frac{1}{2q}\partial\frac{p}{q}\partial^{-1}\right)-\partial\frac{r_1}{2q}\partial\frac{1}{q}\partial^{-1}+\partial\frac{p}{2q}\partial\frac{s_1}{q^2}\partial^{-1}-\partial\frac{1}{2q}\partial\frac{r_1}{q}\partial^{-1}+\partial\frac{1}{2q}\partial\frac{ps_1}{q^2}\partial^{-1}\\ -\frac{s_1}{2q^2} & \frac{s_1}{q}\left(\frac{p}{2q}\partial\frac{1}{q}\partial^{-1}+\frac{1}{2q}\partial\frac{p}{q}\partial^{-1}\right)-\frac{r_1}{2q}\partial\frac{1}{q}\partial^{-1}+\frac{p}{2q}\partial\frac{s_1}{q^2}\partial^{-1}-\frac{1}{2q}\partial\frac{r_1}{q}\partial^{-1}+\frac{1}{2q}\partial\frac{ps_1}{q^2}\partial^{-1}+\partial^2\frac{s_1}{2q^2}\partial^{-1}\end{bmatrix},$$

$$\Phi_2=\begin{bmatrix}\Phi_{21} & \Phi_{22}\\ \Phi_{23} & \Phi_{24}\end{bmatrix},$$

with the entries of Φ_2 being defined as

$$\begin{aligned}\Phi_{21}&=-\partial\frac{s_2}{2q^2},\\ \Phi_{22}&=\partial\frac{s_2}{2q^2}\partial\frac{p}{q}\partial^{-1}+\partial\frac{s_2p}{2q^2}\partial\frac{1}{q}\partial^{-1}+\alpha\partial\frac{r_1}{2q}\partial\frac{s_1}{q^2}\partial^{-1}-\partial\frac{r_2}{2q}\partial\frac{1}{q}\partial^{-1}+\partial\frac{ps_2}{2q^2}\partial\frac{1}{q}\partial^{-1}+\partial\frac{p}{2q}\partial\frac{s_2}{q^2}\partial^{-1}\\ &\quad+\alpha\partial\frac{p}{2q}\partial\frac{s_1^2}{q^2}\partial^{-1}+\alpha\partial\frac{1}{2q}\partial\frac{r_1s_1}{q}\partial^{-1}-\partial\frac{1}{2q}\partial\frac{r_2}{q}\partial^{-1}-\alpha\partial\frac{1}{2q}\partial\frac{ps_1^2}{q^3}\partial^{-1}+\partial\frac{1}{2q}\partial\frac{ps_2}{q^2}\partial^{-1}-\partial\frac{ps_2}{2q^2}\partial^{-1},\\ \Phi_{23}&=-\frac{s_2}{2q^2},\\ \Phi_{24}&=\frac{s_2}{2q^2}\partial\frac{p}{q}\partial^{-1}+\frac{s_2p}{2q^2}\partial\frac{1}{q}\partial^{-1}+\alpha\frac{r_1}{2q}\partial\frac{s_1}{q^2}\partial^{-1}-\frac{r_2}{2q}\partial\frac{1}{q}\partial^{-1}+\frac{ps_2}{2q^2}\partial\frac{1}{q}\partial^{-1}+\frac{p}{2q}\partial\frac{s_2}{q^2}\partial^{-1}-\frac{1}{2q}\partial\frac{r_2}{q}\partial^{-1}\\ &\quad+\alpha\frac{p}{2q}\partial\frac{s_1^2}{q^2}\partial^{-1}+\alpha\frac{1}{2q}\partial\frac{r_1s_1}{q}\partial^{-1}-\alpha\frac{1}{2q}\partial\frac{ps_1^2}{q^3}\partial^{-1}+\frac{1}{2q}\partial\frac{ps_2}{q^2}\partial^{-1}-\frac{ps_2}{2q^2}\partial^{-1}+\partial^2\frac{s_2}{2q^2}\partial^{-1}-\alpha\partial^2\frac{s_1^2}{2q^2}\partial^{-1}.\end{aligned}$$

It is direct and lengthy to show that $\bar{\Phi}$ is hereditary by Maple, $\bar{J}$ and $\overline{M}=\bar{\Phi}\bar{J}$ constitute a Hamiltonian pair [40,41]. Thus, all members in the soliton hierarchy (3.16) are bi-Hamiltonian

$$\bar{u}_{t_m}=\overline{K}_m=\bar{J}\frac{\delta\overline{\mathcal{H}}_m}{\delta\bar{u}}=\overline{M}\frac{\delta\overline{\mathcal{H}}_{m-1}}{\delta\bar{u}},\quad m\geqslant 1. \tag{3.34}$$

Therefore, the soliton hierarchy of bi-integrable couplings (3.16) is Liouville integrable. Every member in (3.16) possesses infinitely many independent commuting conserved functionals

$$\{\overline{\mathcal{H}}_k,\overline{\mathcal{H}}_l\}_{\bar{J}}:=\int\left(\frac{\delta\overline{\mathcal{H}}_k}{\delta\bar{u}}\right)^T\bar{J}\frac{\delta\overline{\mathcal{H}}_l}{\delta\bar{u}}\,dx=0,\quad k,l\geqslant 0,$$

$$\{\overline{\mathcal{H}}_k,\overline{\mathcal{H}}_l\}_{\overline{M}}:=\int\left(\frac{\delta\overline{\mathcal{H}}_k}{\delta\bar{u}}\right)^T\overline{M}\frac{\delta\overline{\mathcal{H}}_l}{\delta\bar{u}}\,dx=0,\quad k,l\geqslant 0,$$

and symmetries

$$[\overline{K}_k,\overline{K}_l]:=\overline{K}'_k(\bar{u})[\overline{K}_l]-\overline{K}'_l(\bar{u})[\overline{K}_k]=0,\quad k,l\geqslant 0.$$

4. Conclusions and discussions

There are many soliton hierarchies obtained by using matrix spectral problems based on certain real Lie algebras such as $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3,\mathbb{R})$. Recently, seeking for integrable couplings associated with non-semisimple Lie algebras has aroused more and more interest. Various examples of bi- and tri-integrable couplings bring us ideas to classify multi-component integrable systems. In this paper, we have introduced a new asymmetric 2×2 matrix spectral problem associated with $\mathrm{sl}(2,\mathbb{R})$ and derived a new KN type soliton hierarchy (2.14), together with bi-Hamiltonian structures firstly. Then based on a class of non-semisimple matrix Lie algebras consisting of 3×3 block matrices, we have derived bi-integrable couplings (3.16) of the obtained soliton hierarchy. Corresponding bi-Hamiltonian structures are constructed by the variational identity, which imply the system has infinitely many commuting symmetries and conserved functionals. In addition, we must point out that the mathematical software Maple is used to deal with some complicated computations in this paper.

In fact, we can construct other kinds of integrable couplings by means of more complex non-semisimple Lie algebras such as

$$\bar{\mathfrak{g}} = \{M(A_1,A_2,A_3)\},$$
$$M(A_1,A_2,A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1+\alpha A_2 & \beta A_2+\alpha A_3 \\ 0 & 0 & A_1+\alpha A_2 \end{bmatrix}$$

and

$$\bar{\mathfrak{g}} = \{M(A_1,A_2,A_3,A_4)\},$$
$$M(A_1,A_2,A_3,A_4) = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & A_1+\alpha A_2 & \alpha A_3 & \beta A_2+\alpha A_4 \\ 0 & 0 & A_1+\alpha A_2+\mu A_3 & \nu A_3 \\ 0 & 0 & 0 & A_1+\alpha A_2 \end{bmatrix}$$

presented in Refs. [40,41]. Considering the calculation is similar, we omit the lengthy process for the sake of convenience.

The algebra $\mathrm{sl}(2,\mathbb{R})$ is one of the only two three-dimensional real Lie algebras with a three-dimensional derived algebra, and the other one is the special orthogonal Lie algebra $\mathrm{so}(3,\mathbb{R})$. In a subsequent study, we will consider the construction of integrable couplings for the KN type soliton hierarchy associated with $\mathrm{so}(3,\mathbb{R})$ and $\mathrm{so}(n+1,\mathbb{R})$.

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