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ORIGINAL PAPER

Multiple-soliton solutions and lumps of a (3+1)-dimensional generalized KP equation

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Abstract In this paper, we study a (3+1)-dimensional generalized Kadomtsev–Petviashvili equation, which is physically meaningful. Applying the simplified Hirota's method, we derive multiple-soliton solutions and lumps for this new model, where the coefficients of spatial variables are not constrained by any conditions. But the phase and the new model are dependent on all these coefficients. Moreover, this new model passes the Painlevé integrability test.

Keywords Simplified Hirota's method · Multiple-soliton solution · Lumps · Painlevé test

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1 Introduction

For the research on nonlinear complex phenomena, we all know that nonlinear partial differential equations play an much more essential role. And they have got many applications, such as nonlinear optics, fluid mechanics and so forth. In order to know how to be in charge of nonlinear systems, a lot of experts have paid a little bit more attention to the multi-soliton solutions and lumps in the recent years. Among the efficient approachers are the Hirota bilinear method [1], the generalized bilinear method [2,3] the Bäcklund transformation method [4–6], the Darboux transformation [7–9], the inverse scattering method [10], the Painlevé analysis [11–13] and other methods.

Recently, More people are interested in the multiple solitons [14–16] and lump, which is a completely new locally nonlinear waves. For the soliton solutions, there is a good balance between dispersion effects and the nonlinearity. But the lump solution is a sort of rational solution and is localized in all of the directions of space [17–22].

The KP equation of (2+1)-dimension [23] can give perfect descriptions of the nonlinear and long waves with small amplitudes. In the past decades, some extended KP equations have been presented and investigated in the following references [24,25]

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} - u_{zz} = 0. \tag{1}$$

under the transformation $u = 2(\ln f)_x$, becomes

$$(D_x^3 D_y + D_t D_x + D_t D_y - D_z^2) f \times f = 0, \tag{2}$$

where D_t, D_x, D_y and D_z are Hirota's bilinear operators. Moreover, its resonant solitons have been constructed.

In this work, we will introduce a (3+1)-dimensional generalized KP equation:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tz} - u_{xx} - u_{zz} = 0, \tag{3}$$

where extra terms u_{tz}, u_{xx} are added to Eq. (1). Using $u = 2(\ln f)_x$, we obtain its Hirota bilinear form

$$(D_x^3 D_y + D_t D_x + D_t D_y + D_t D_z - D_x^2 - D_z^2) f \times f = 0, \tag{4}$$

Employing Hirota's method, we firstly establish multiple-soliton solutions and lumps for Eq. (3). To the best of our knowledge, there are few papers about lumps of (3+1)-dimensional generalized KP equations. Then, we prove that it is Painlevé integrable.

2 Multiple-soliton solutions of Eq. (3)

2.1 One-soliton solutions

In order to derive the dispersion relation of Eq. (3), we first substitute

$$u = e^{k_i x + r_i y + s_i z - c_i t} \tag{5}$$

into linear terms of (3) and obtain by direct computations

$$c_i = \frac{k_i^3 r_i - k_i^2 - s_i^2}{k_i + r_i + s_i}, \quad i = 1, 2, \ldots, N. \tag{6}$$

Therefore, the dispersion variables are

$$\theta_i = e^{k_i x + r_i y + s_i z - \frac{k_i^3 r_i - k_i^2 - s_i^2}{k_i + r_i + s_i} t}, \quad i = 1, 2, \ldots, N, \tag{7}$$

which implies that the one-soliton solution

$$u = 2(\ln f)_x, \tag{8}$$

where

$$f = 1 + e^{\theta_i}, \quad i = 1, 2 \ldots, N. \tag{9}$$

2.2 Two-soliton solutions

To construct two-soliton solutions, we plug

$$f = 1 + e^{\theta_i} + e^{\theta_j} + a_{ij} e^{\theta_i + \theta_j}, \quad 1 \le i < j \le N \tag{10}$$

into (4) and obtain the phase shift as follows:

$$a_{ij} = \frac{L_{ij}}{M_{ij}}, \quad 1 \le i < j \le N, \tag{11}$$

where

$$L_{ij} = L_{ij}(k_i, r_i, s_i, c_i; k_j, r_j, s_j, c_j),$$
$$M_{ij} = M_{ij}(k_i, r_i, s_i, c_i; k_j, r_j, s_j, c_j) \tag{12}$$

are two polynomials of degree 6. For simplicity, we will not give L_{ij}, Mij in detail since they are too complicated. But we noted that no constraints are on the coefficients of variables x, y, z, which means they are completely free. If $k_i = r_i = s_i$, the phase shift a_{ij} will be the Hirota's type. Otherwise, a_{ij} differs completely from the Hirota's type, which implies that some new two-soliton solutions have been constructed. Moreover, there is not any resonant phenomenon shown by Eq. (3) since $a_{ij} \neq 0$ or ∞ while $k_i = k_j, i \neq j, i, j = 1, 2, \ldots, N$.

2.3 N-soliton solutions

In the same way, N-soliton solutions of Eq. (3) can be written in the following form

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \theta_i + \sum_{1 \le i < j \le N}^{N} \mu_i \mu_j \ln(A_{ij}) \right), \tag{13}$$

where $\sum_{\mu=0,1}$ means the summation over all possible combinations of $\mu_1 = 0, 1, \ldots, \mu_N = 0, 1$. For example, three-soliton solutions are constructed by letting

$$f = 1 + e^{\theta_i} + e^{\theta_j} + e^{\theta_k} + a_{ij} e^{\theta_i + \theta_j}$$
$$+ a_{ik} e^{\theta_i + \theta_k} + a_{jk} e^{\theta_j + \theta_k}$$
$$+ a_{ijk} e^{\theta_i + \theta_j + \theta_k}, 1 \le i < j < k \le N. \tag{14}$$

Substituting (14) into (4), we obtain $a_{ijk} = a_{ij}a_{ik}a_{jk}$. Thus, we can get the three-soliton solutions through $u = 2(\ln f)_x$.

3 Lumps of Eq. (3)

The lumps of Eq. (3) are constructed in the form of the sum of positive quadratic functions in this section.

3.1 Two quadratic function solutions

In order to get the lump solutions described by the sum of two quadratic functions, we assume

$$f = g^2 + h^2 + \omega, \tag{15}$$

with

$$\begin{aligned} g &= a_1 x + b_1 y + c_1 z + d_1 t + r_1, \\ h &= a_2 x + b_2 y + c_2 z + d_2 t + r_2, \end{aligned} \tag{16}$$

where a_i's are real constants to be determined. Substituting (15) and (16) into (4) , we can get all the values of a_i's as follows:

$$\begin{aligned} b_1 &= \frac{a_1^2 d_1 + 2a_1 a_2 d_2 - a_2^2 d_1 + c_1^2 d_1 + 2c_1 c_2 d_2 - c_2^2 d_1}{d_1^2 + d_2^2} \\ &\quad - a_1 - c_1, \\ b_2 &= \frac{-(a_1^2 d_2 - 2a_1 a_2 d_1 - a_2^2 d_2 + c_1^2 d_2 - 2c_1 c_2 d_1 - c_2^2 d_2)}{d_1^2 + d_2^2} \\ &\quad + a_2 + c_2, \\ \omega &= \frac{p}{q}, \end{aligned} \tag{17}$$

with

$$\begin{aligned} p &= p(a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2), \\ q &= q(a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2) \end{aligned} \tag{18}$$

are two complicated polynomials of degree 6. For simplicity, we will not provide their detailed forms. It is observed that this resulting lump solutions contain eleven parameters, of which there are eight free parameters. The central points, which play key roles in studying lumps about velocity of waveform and so on, can be got via computing extremum points. We also noted that the lump $u = 2\ln(f)_x$ is analytic if and only if $\omega > 0$, and it is not degenerate if any two column vectors of matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ are linearly independent. Moreover, It is easily to find that the aforementioned lump solution $u \to 0$ if and only if $g^2 + h^2 \to \infty$, which is equivalent to $x^2 + y^2 + z^2 \to \infty$ while evolving. Hence, the analyticity and localization of the aforementioned lump can be completely guaranteed. The totally free parameters also determine the expansion and the deflection angle of lump, the smaller the absolute values of which, the greater the expansion of lump. Now, we choose some specific values of the parameters: $a_1 = 1$, $a_2 = 3$, $b_1 = -9/2$, $b_2 = 39/2$, $c_1 = 2$, $c_2 = 5$, $d_1 = 1$, $d_2 = 1$, $r_1 = 3$, $r_2 = 3$, $\omega = 3240/13$, , and the graphs of the lump wave are shown vividly in Fig.1.

3.2 Multiple quadratic function solutions

Similarly, we firstly set

$$f = \sum_{i=1}^{N} (a_i x + b_i y + c_i z + d_i t + r_i)^2 + \omega, \tag{19}$$

where $N \geq 2$ is an integer and $a_i, b_i, c_i, d_i, r_i, \omega$ are real constants to be determined. In order to explain the proposed method, we consider the three quadratic function solutions by letting

$$f = \sum_{i=1}^{3} (a_i x + b_i y + c_i z + d_i t + r_i)^2 + \omega. \tag{20}$$

Substituting (20) into (4) and solving for $a_i, b_i, c_i, d_i, r_i, \omega$, we can get the following results

$$b_1 = \frac{(5a_1^2 + a_1 d_1 - 2a_2^2 - 2a_3^2)}{d_1}, \quad b_2 = \frac{6a_1 a_2}{d_1},$$

$$b_3 = \frac{6a_1 a_3}{d_1}, \quad d_2 = d_3 = 0, \quad \omega = \frac{p}{q}, \tag{21}$$

where

$$p = p(a_1, d_1, a_2, r_2, a_3, r_3), \quad q = q(d_1, a_2, a_3), \tag{22}$$

which are two polynomials of degree 5 and degree 3, respectively. For the simplicity, they will not be presented here. Thus, we obtained the three quadratic function solutions explicitly via the formula $u = 2(\ln f)_x$.

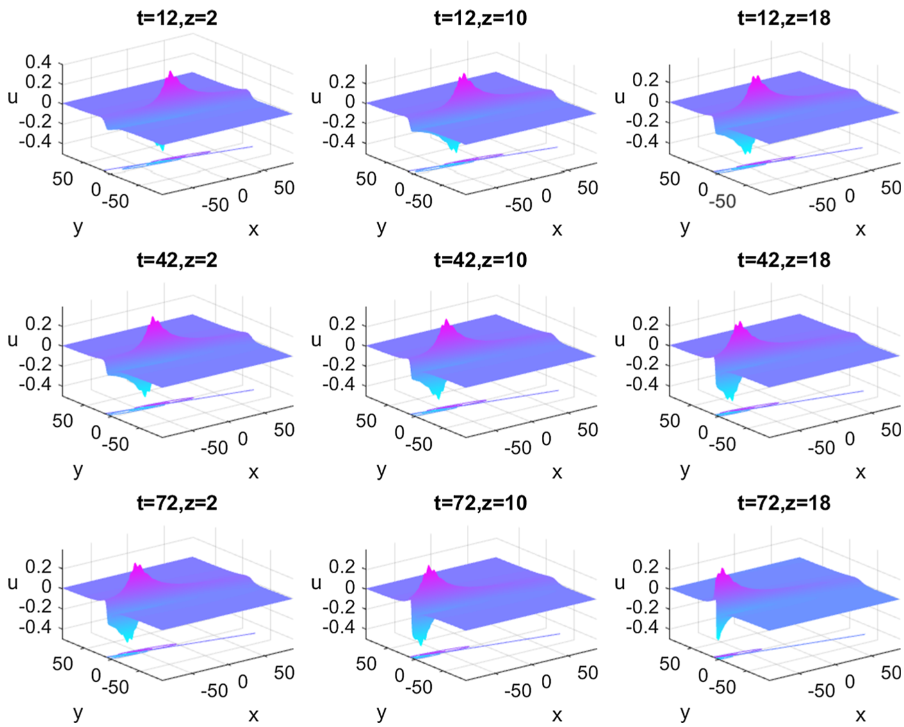

Fig. 1 Evolution profiles of lump wave with the specific parameters: $a_1 = 1,\ a_2 = 3,\ b_1 = -9/2,\ b_2 = 39/2,\ c_1 = 2,\ c_2 = 5,\ d_1 = 1,\ d_2 = 1,\ r_1 = 3,\ r_2 = 3,\ \omega = 3240/13$

It is observed that this resulting lump wave contains twelve parameters, of which six are free parameters. The the extremum points can help us get the central points. We also noted that the lump wave $u = 2\ln(f)_x$ is analytic if and only if $\omega > 0$, and it is not degenerate if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ holds. Moreover, It is not difficult to find that the aforementioned lump solution $u \to 0$ if and only if the sum of squares $x^2 + y^2 + z^2 \to \infty$ at any given time. Hence, the analyticity and localization of the obtained lump are guaranteed. Moreover, the deflection and the expansion of lump are determined by the totally free parameters, Now, we choose some particular parameters: $a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ d_1 = 1,\ r_1 = 1,\ r_2 = 2,\ r_3 = 1,\ c_1 = -2,\ c_2 = -2,\ c_3 = -3,\ b_1 = -20,\ b_2 = 12,\ b_3 = 18,\ d_2 = 0,\ d_3 = 0,\ \omega = 1202/13$, and the graphs of the lump wave are shown vividly in Fig. 2.

4 Painlevé analysis

Applying the WTC–Kruskal approach [11], we first analyze the leading order of Eq. (3) by setting $u = \sum_{j=0}^{\infty} u_j \phi^{j-\rho}$, where $\rho > 0$ is an integer. Then, we got that $\rho = 1$ and $u_0 = 2\phi_x$. And four resonance points for Eq. (3) are also found, which are $j = -1, 1, 4, 6$. Since the max resonance point occurs at $j = 6$, to get the compatibility conditions, we need compute the u_j up to $j = 6$.

For $j = 1$, we have

$$u_1 = u_1, \tag{23}$$

which means the parameter u_1 is arbitrary.

For $j = 2$, we obtain

$$u_2 = -\frac{3\phi_x^2 + 3\phi_x\phi_y u_{1,x} - \phi_x^2}{6\phi_x^2\phi_y} + \frac{3\phi_x\phi_{xxy} + \phi_y\phi_x xx - 3\phi_{xx}\phi_{xy} - \phi_z^2 + \phi_z\phi_t}{6\phi_x^2\phi_y}. \tag{24}$$

Similarly, we can get $u_j,\ j = 3, 4, 5, 6$, and find u_4 and u_6 are also arbitrary. Therefore, the new (3+1)-dimensional model passes the Painlevé test. Hence, it is integrable in the sense of Painlevé.

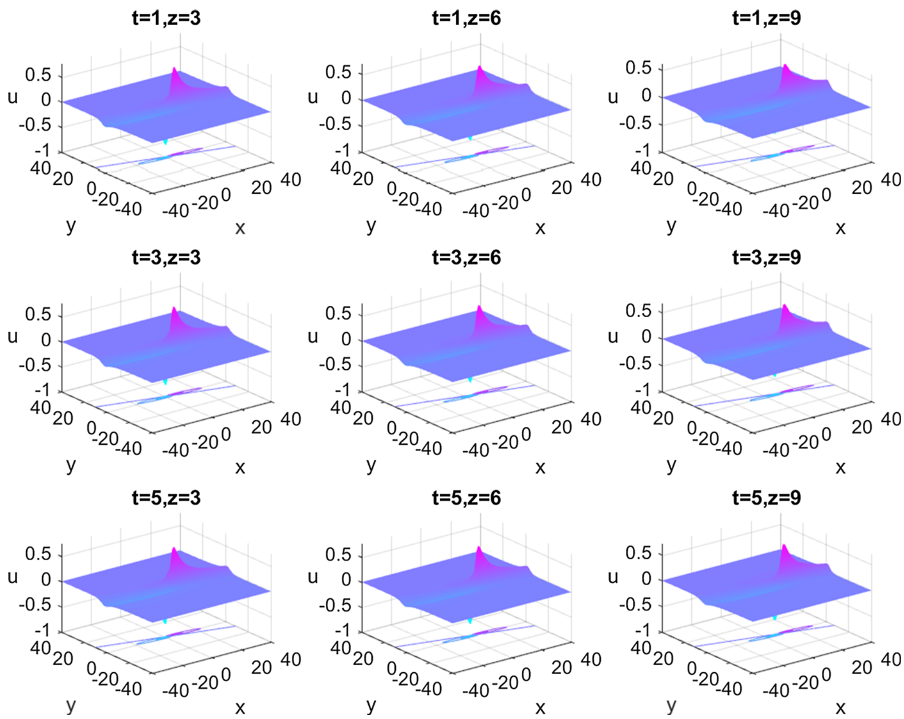

Fig. 2 Evolution profiles of lump wave with the specific parameters:$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ d_1 = 1,\ r_1 = 1,\ r_2 = 2,\ r_3 = 1,\ c_1 = -2,\ c_2 = -2,\ c_3 = -3,\ b_1 = -20,\ b_2 = 12,\ b_3 = 18,\ d_2 = 0,\ d_3 = 0,\ \omega = 1202/13$

5 Conclusions

In this study, the (3+1)-dimensional Eq. (3) is introduced and its multiple-soliton solutions and lump solutions are derived by using the Hirota bilinear approach. During the investigation, we remarked that there are not any constraints for the coefficients of spatial variables. In the meantime, the conditions which can guarantee the positiveness, the analyticity and the localization of lumps are obtained. The Painlevé analysis is presented to prove Eq. (3) is integrable in the sense of Painlevé. Actually, this introduced model Eq.(3) is very much physically meaningful since it can be applied to investigate some nonlinear phenomena in physics as well. For instance, it can be applied to study long wavelength water waves, which are associated with frequency dispersion and weakly nonlinear restoring forces, and it can also be employed to model three-dimensional matter-wave pulses in Bose–Einstein condensates and waves in ferromagnetic media.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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