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Research Article

Diversity of Interaction Solutions of a Shallow Water Wave Equation

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In this paper, we study the diversity of interaction solutions of a shallow water wave equation, the generalized Hirota–Satsuma–Ito (gHSI) equation. Using the Hirota direct method, we establish a general theory for the diversity of interaction solutions, which can be applied to generate many important solutions, such as lumps and lump-soliton solutions. This is an interesting feature of this research. In addition, we prove this new model is integrable in Painlevé sense. Finally, the diversity of interactive wave solutions of the gHSI is graphically displayed by selecting specific parameters. All the obtained results can be applied to the research of fluid dynamics.

1. Introduction

The Hirota method played an important role in solving partial differential equations [1]. And, we can solve the corresponding Hirota bilinear equations using many efficient techniques, for example, applying the Wronskian technique [2, 3], we can get positons and complexitons [4]. And, if we take a long wave limit, the lumps, which are locally rationalized along all spacial directions, can be obtained [5–8]. Since the interaction solutions among different classes of solutions can describe more diverse nonlinear phenomena [3], studying interaction solutions is a hot topic for the researchers of mathematical physics [9–16]. Particularly, the interactions between the lumps and kinks [17, 18]. A lot of useful references can be found in [19–27]. Reference [1] presented a shallow water wave equation as follows:

$$\begin{aligned} u_t &= u_{xxt} + 3uu_t - 3u_x v_t - u_x, \\ v_x &= -u, \end{aligned} \tag{1}$$

of which the Hirota bilinear form is

$$\left(D_t D_x^3 - D_t D_x - D_x^2\right) f \cdot f = 0, \tag{2}$$

via the transformation $u = 2(\ln f)_{xx}$. This kind of transformations is an important part of Bell polynomial theory of partial differential equations [21].

In this study, we will investigate the diversity of a $(2+1)$-dimensional generalized HSI equation that reads

$$\begin{aligned}\text{gHSI} &:= u_{tt} + u_{xxxt} + 6u_x u_t + 3uu_{xt} - 3u_{xx}v_t \\ &\quad + \beta u_{xt} + u_{yt} + \alpha u_{xx} = 0, \\ v_x &= -u,\end{aligned} \tag{3}$$

which has the following Hirota bilinear form:

$$\left(D_t^2 + D_t D_x^3 + \beta D_t D_x + D_t D_y + \alpha D_x^2\right) f \cdot f = 0, \tag{4}$$

through the transformation $u = 2(\ln f)_{xx}$. The parameters $\alpha, \beta \neq 0$ are all real constant and D_x, D_t, and D_y are Hirota derivatives [1] which are

$$\left(D_x^m D_y^n D_t^k f \cdot g\right)(x, y, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k \Bigg|_{x=x', y=y', t=t'}, \tag{5}$$

with integers $m, n, k \geq 0$.

We will establish the general theory of interaction solutions of equation (3) so that we can build a general method to find the interaction solutions between lumps and other types of solutions of the $(2+1)$-dimensional gHSI equation by using the Hirota direct approach. Lump solutions and interaction solutions are presented to show diverse nonlinear phenomena. In Section 2, we derive the general approach for finding lumps and interaction solutions. Some applications are presented in Section 3 to illustrate obtained method in Section 2. In the meantime, the diversity of the interaction solutions of the gHSI equation is illustrated vividly by some graphs. In Section 4, the gHSI equation (3) showed that it is integrable in Painlevé sense. Finally, some remarks will be given in the conclusion part.

2. Diversity of Interaction Solutions

There are many ways to find solutions, for example, the symmetry method, the Hirota direct method, and the generalized bilinear method [21–26]. In this section, we will apply the Hirota direct method to establish the theory for the diversity of interaction solutions of the $(2+1)$-dimensional gHSI equation (3). Hence, the combined solutions of the HSI equation can be found efficiently.

Assume that the $(2+1)$-dimensional general bilinear equation be as follows:

$$P\left(D_x, D_y, D_t\right)(f \cdot f) = 0, \tag{6}$$

where $P(x, y, t)$ is a polynomial of even degree and satisfies $P(0, 0, 0) = 0$. Let

$$\begin{aligned} f &= G + \sum_{i=1}^{n} \left(d_i e^{\eta_i}\right), \\ \eta_i &= a_i x + b_i y + c_i t, \end{aligned} \tag{7}$$

where $G = G(x, y, t)$ is a function of x, y, and t and $d_i's$ are all real constant to be determined. Moreover, we assume

(1) $\eta_i, \eta_i + \eta_j \neq 0$ and η_i, η_{j+k} are all distinct for all $i, j, k = 1, \ldots, n$.

(2) G is a positive polynomial and $d_i \geq 0$ and $H = \sum_{i=1}^{n} \left(d_i e^{\eta_i}\right)$ with $\eta_i = a_i x + b_i y + c_i t$. According to the Hirota derivatives, we obtain

$$\begin{aligned} P\left(D_x, D_y, D_t\right)(f \cdot f) &= P\left(D_x, D_y, D_t\right)(G \cdot G) \\ &\quad + P\left(D_x, D_y, D_t\right)(G \cdot H) \\ &\quad + P\left(D_x, D_y, D_t\right)(H \cdot H), \\ P\left(D_x, D_y, D_t\right)\left(e^{\eta_i} \cdot e^{\eta_i}\right) &= 0, \\ P\left(D_x, D_y, D_t\right)\left(e^{\eta_j} \cdot e^{\eta_k}\right) &= P\left(a_j - a_k, b_j - b_k, c_j - c_k\right) e^{\eta_j + \eta_k}, \end{aligned} \tag{8}$$

which implies that (8) can be rewritten as follows:

$$\begin{aligned} P\left(D_x, D_y, D_t\right)(f \cdot f) &= P\left(D_x, D_y, D_t\right)(G \cdot G) \\ &\quad + 2\sum_{i=1}^{n} \left(d_i P\left(D_x, D_y, D_t\right)\left(G \cdot e^{\eta_i}\right)\right) \\ &\quad + 2\sum_{1 \leq j < k < n} \left(d_j d_k P\left(D_x, D_y, D_t\right)\left(e^{\eta_j + \eta_k}\right)\right). \end{aligned} \tag{9}$$

Hence, if

$$\begin{aligned} P\left(D_x, D_y, D_t\right)\left(G \cdot e^{\eta_i}\right) &= 0, \quad i = 1, \ldots, n, \\ P\left(D_x, D_y, D_t\right)\left(e^{\eta_j} \cdot e^{\eta_k}\right) &= 0, \end{aligned} \tag{10}$$

where $i, j, k = 1, \ldots, n$ and $j \neq k$, then f is a solution of equation (6) if and only if G is also a solution of equation (6). Therefore, using the transformations $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$, we can get the interact solutions: lump-soliion solutions of the gHSI equation (3).

Remark. (1) If we further let

$$f = g^2 + h^2 + d + ke^l, \tag{11}$$

where

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= b_1 x + b_2 y + b_3 t + b_4, \\ l &= c_1 x + c_2 y + c_3 t, \end{aligned} \tag{12}$$

and $d, k \geq 0$, then f is a solution of equation (6) if and only if $g^2 + h^2 + d$ is also a solution of equation (6) under the condition

$$P\left(D_x, D_y, D_t\right)\left(\left(g^2 + h^2 + d\right) \cdot e^l\right) = 0. \quad (13)$$

(2) If $G = g^2 + h^2 + d$ is a solution of equation (6), then we have

(i) $u = 2(\ln G)_x$ or $u = 2(\ln G)_{xx}$ is a lump solution

(ii) Moreover, if $k > 0$, then $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$ is a lump-soliton solution if and only if

$$P\left(D_x, D_y, D_t\right)\left(G \cdot e^l\right) = 0. \quad (14)$$

3. Application to Shallow Water Wave Equation

3.1. Lump Solution of the gHSI Equation. Firstly, we consider the lump solutions of equation (4). We suppose that

$$G = g^2 + h^2 + d = \left(a_1 x + a_2 y + a_3 t + a_4\right)^2 + \left(b_1 x + b_2 y + b_3 t + b_4\right)^2 + d, \quad (15)$$

where g and h are linearly independent and $d > 0$. The parameters $a_i' s$ are obtained via the direct computation as follows:

$$\begin{aligned} d &= -\frac{3a_1\left(a_1^2 + b_1^2\right)^2 a_3}{b_1^2\left(a_1^2 - b_1^2\right)\alpha}, \\ a_2 &= \frac{-\alpha a_1^2 + \alpha b_1^2 - \beta a_1 a_3 - a_3^2}{a_3}, \\ b_2 &= \frac{-b_1\left(\beta a_1^2 - \beta b_1^2 + 2a_1 a_3\right)}{a_1^2 - b_1^2}, \\ b_3 &= \frac{2a_3 b_1 a_1}{a_1^2 - b_1^2}, \end{aligned} \quad (16)$$

where $\alpha, \beta \neq 0$. Then, we can get the lump solution of equation (3) as

$$u = \frac{4\left(\left(a_1^2 + b_1^2\right) f - 2\left(a_1 g + b_1 h\right)^2\right)}{f^2}, \quad (17)$$

with $\alpha a_1 a_3 < 0$ and $a_1^2 - b_1^2 \neq 0$. It is observed that, at any given time t, the extremum points can be obtained by direct computation, from which the traveling speeds, along x-direction and y-direction, and the changes of waveform can be obtained. The amplitude of u is also attained. We also noted that the lump wave is analytic in the XY-plane if and only if $d > 0$. Moreover, it is easy to find the aforementioned lump solution $u \longrightarrow 0$ if and only if the sum of squares $g^2 + h^2 \longrightarrow \infty$, or equivalently, $x^2 + y^2 \longrightarrow \infty$ at any given time. The evolution profile, density plot, and contour plots of solution (15) with specific parameters are shown in Figure 1, from which we can see that the waveforms of (15) change only a little bit at different time.

3.2. Interaction Solutions of the gHSI Equation. In this part, we will find some lump-soliton solutions of the gHSI equation (3). Assume $f = g + h + d + ke^l$ with g, h, d, and k defined as in equation (11). By the logarithm transformation $u = 2(\ln f)_{xx}$, we get the lump-soliton solution as

$$u = 2\frac{f_{xx} f - f_x^2}{f^2}. \quad (18)$$

By theories in Section 2, we can find the solution of all the parameters as follows:

$$\begin{aligned} a_1 &= -\frac{3b_3 c_1^2}{2\alpha}, \\ a_2 &= -\frac{-9\alpha b_1 c_1^4 - 9\beta b_3 c_1^4 + 4\alpha^2 b_1}{6\alpha c_1^2}, \\ a_3 &= \frac{2\alpha b_1}{3c_1^2}, \\ c_3 &= -\frac{2\alpha}{3c_1}, \\ b_2 &= -\frac{-9b_3 c_1^4 + 4\alpha\beta b_1 + 4\alpha b_3}{4\alpha}, \\ d &= 0, \\ b_4 &= \frac{3a_4 b_3 c_1^2}{2\alpha b_1}, \\ c_2 &= \frac{3c_1^4 - 6\beta c_1^2 + 4\alpha}{6c_1}, \end{aligned} \quad (19)$$

which yields the following functions:

$$\begin{aligned} g &= -\frac{3b_3 c_1^2}{2\alpha} x - \frac{-9\alpha b_1 c_1^4 - 9\beta b_3 c_1^4 + 4\alpha^2 b_1}{6\alpha c_1^2} y + \frac{2\alpha b_1}{3c_1^2} t + a_4, \\ h &= b_1 x - \frac{-9b_3 c_1^4 + 4\alpha\beta b_1 + 4\alpha b_3}{4\alpha} y + b_3 t + b_4, \\ ke^l &= c_1 x + \frac{3c_1^4 - 6\beta c_1^2 + 4\alpha}{6c_1} y - \frac{2\alpha}{3c_1} t. \end{aligned} \quad (20)$$

Therefore, we can get the function $f = g^2 + h^2 + d + ke^l$ which implies that the lump-soliton solution of the gHSI equation is also obtained by equation (20). We can also get the extremum points by direct computation in Maple, which play an important role in studying the wave equations, for example, the velocities, along x-direction and y-direction, the amplitude of u, and the changes of waveform can be obtained via the extremum points. We also found that the lump wave is analytic in the XY-plane if and only if $c_1 \neq 0$ and $b_1 \neq 0$. The aforementioned lump-soliton solution is an interactive solution; hence, during the collision, they interact like fusion and fission phenomenon in physics. At first, the energy of the lump wave is stronger than the stripe wave

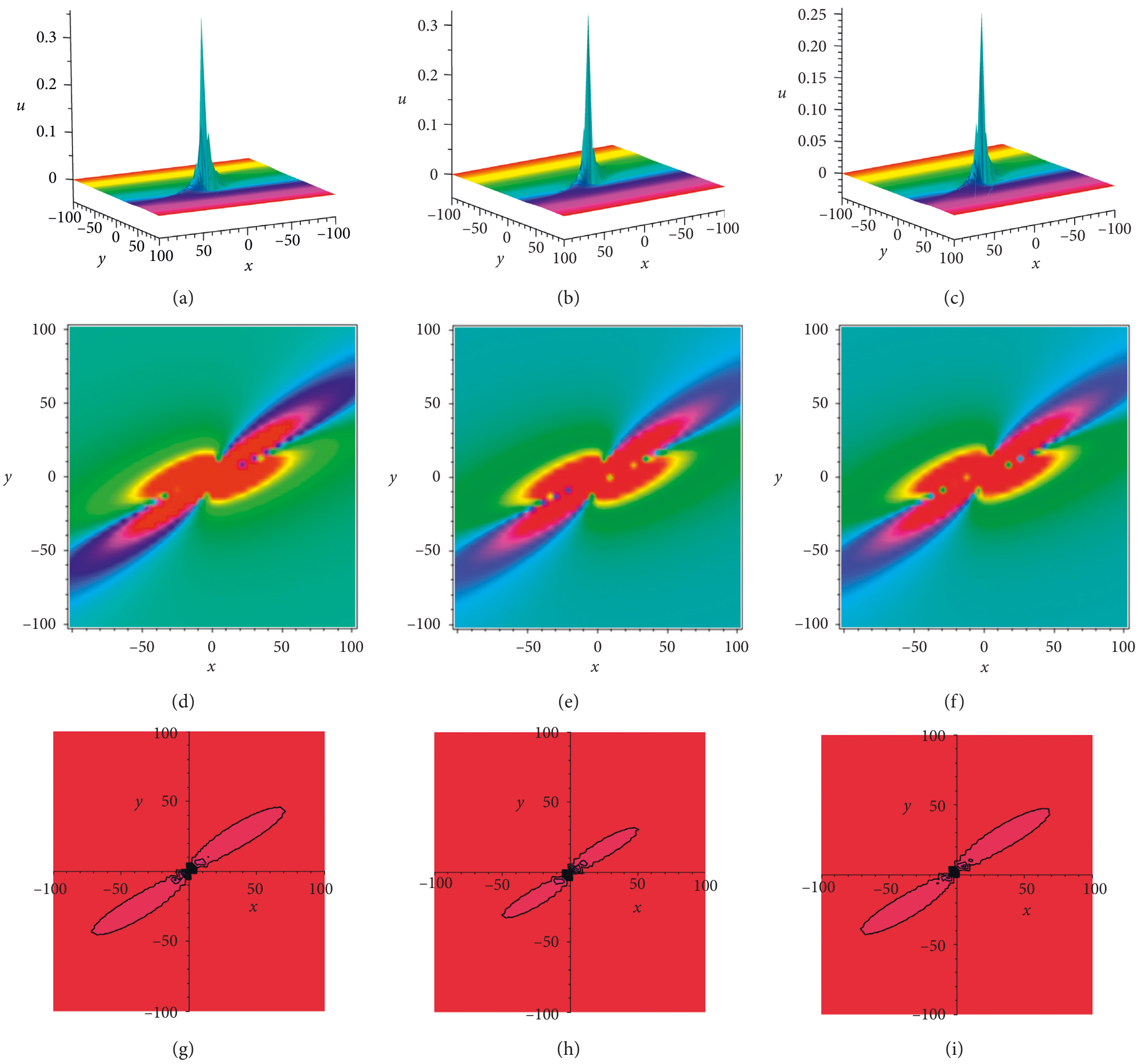

Figure 1: 3D plots, density plots, and contour plots of lump wave solution (19) with the specific parameters $a_1 = 2, a_3 = 1, a_4 = 1, b_1 = 1, b_3 = 1, b_4 = 1, \alpha = -1$, and $\beta = 2$. (a), (d), and (g) are for $t = -2$; (b), (e), and (h) are for $t = 0$; (c), (f), and (i) are for $t = 6$.

described by the exponential function, but finally the lump wave are gradually swallowed by the stripe soliton, which implies that its energy is also transferred to the stripe soliton completely. They become one soliton. The evolution profiles and contour plots of solution (20) with specific parameters are shown in Figure 2, from which we observed that the intersect solution (20) of the gHSI equation change greatly at different time.

4. Painlevé Analysis

It is well known that Painlevé analysis is a very powerful tool for finding the integrable model from given nonlinear equations [27]. Using the WTC-Kruskal approach, we firstly analyze the leading order to the negative integer α, then determine the resonant points, and finally obtain the compatibility conditions, which must be completely satisfied for all the positive resonant points. Baldwin et al. presented two packages in Mathematica based on the WTC approach and Kruskal's simplification.

Applying the aforementioned packages in Mathematica to test the integrability of the $(2+1)$-dimensional gHSI equation (3), we find five resonant points $j = -1, 1, 4, 5, 6$. In all the cases, equation (3) does pass the Painlevé test. It is noted that the presence of soliton solutions can indicate the integrability of the tested equation. But, this is not enough since it should be supported by the Painlevé test, or the Lax pair of the examined equation or other approaches. In this study, we formally obtained lump solutions and lump-soliton solutions of the gHSI equation (3) and showed that it passed the Painlevé test, which implies that it is an integrable equation in Painlevé sense.

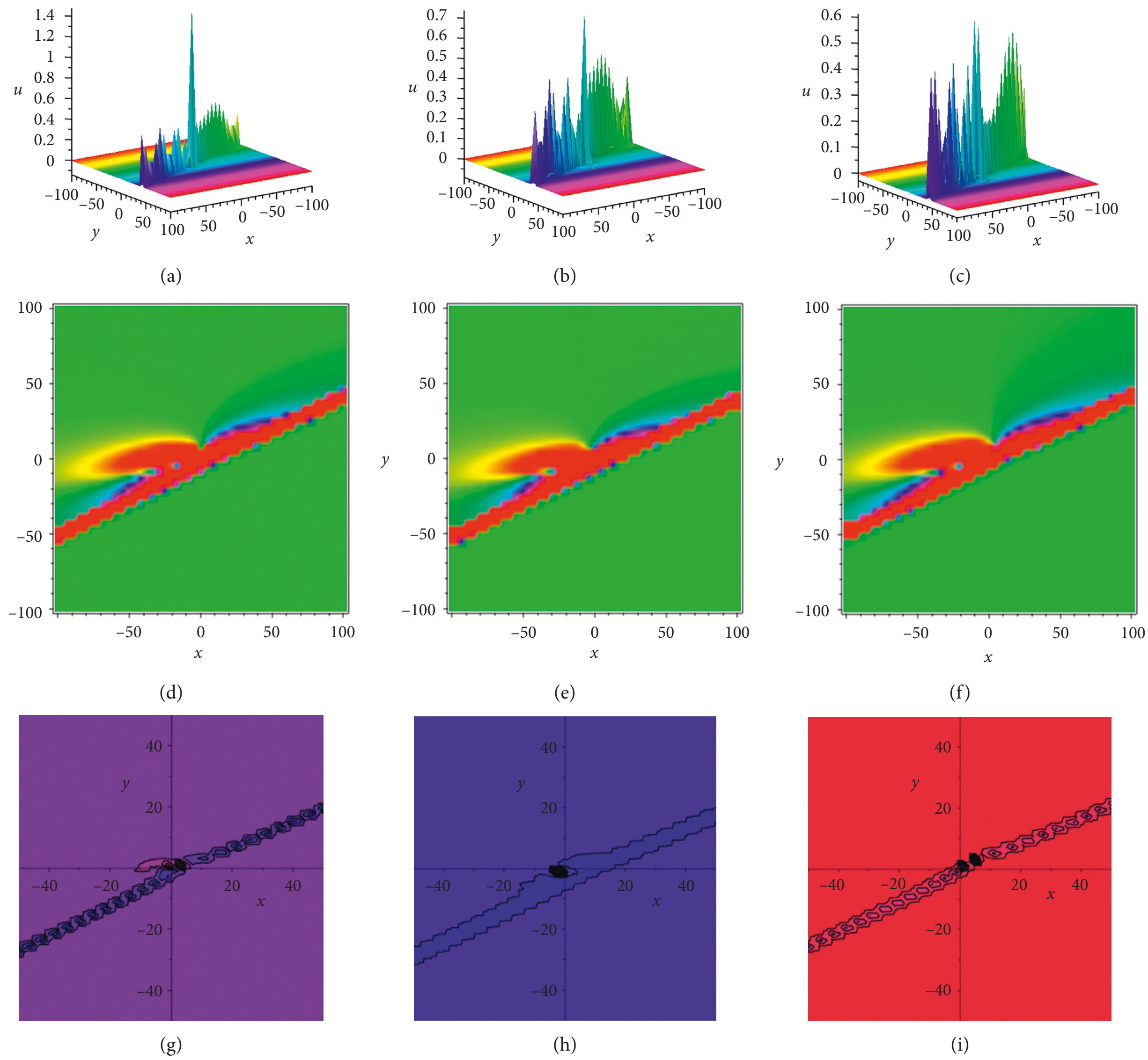

FIGURE 2: 3D plots, density plots, and contour plots of lump wave solution (20) with the specific parameters $k = 1, a_4 = 1, b_1 = 1, b_3 = 1, b_4 = 1, c[1] = 1, \alpha = 1$, and $\beta = 2$. (a), (d), and (g) are for $t = -5$; (b), (e), and (h) are for $t = 0$; (c), (f), and (i) are for $t = 6$.

5. Conclusions

In this research, we introduced a shallow water wave equation, the gHSI equation (3), and established the theory of its diversity of interactions, the lump solution, and lump-soliton solutions. All the computations are performed in Maple using the Hirota bilinear equations. Moreover, we proved that this gHSI equation (3) is Painlevé integrable. During the study, we found that the waveforms of (20) are completely different if we select different values of α and β. For example, if we choose $\alpha = -2$, the waveform has a unique peak at the maximum point.

The research of the diversity of interaction actions is an interesting and hot topic in mathematical physics since we can get a lot of useful solutions for the physical research. Hence, we will continue studying other interaction solutions, such as the interactions between the periodic function solutions and the hyperbolic function solutions. In addition, we hope that we can find whether equation (3) is integrable in Liouville sense or not.

In the meantime, this introduced shallow water equation has some applications in physics research. For example, it can be used to describe the flow under a pressure surface (sometimes a free surface) in a fluid, which implies that it can be applied to the research on the fluid dynamics.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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