Commun Nonlinear Sci Numer Simulat 43 (2017) 151–157

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

An integrable generalization of the super AKNS hierarchy and its bi-Hamiltonian formulation

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ARTICLE INFO

Article history:
Received 21 March 2016
Accepted 29 June 2016
Available online 1 July 2016

PACS:
02.30.Ik
05.45.Yv
11.30.Pb

Keywords:
A Lie super-algebra
Super Hamiltonian structure
Self-consistent sources

ABSTRACT

Based on a Lie super-algebra B(0, 1), an integrable generalization of the super AKNS isospectral problem is introduced and its corresponding generalized super AKNS hierarchy is generated. By making use of the super-trace identity (or the super variational identity), the resulting super soliton hierarchy can be put into a super bi-Hamiltonian form. A generalized super AKNS soliton hierarchy with self-consistent sources is also presented.

1. Introduction

It is interesting to search for new integrable equations in soliton theory and integrable systems. A traditional method is to use zero curvature equations associated with finite-dimensional Lie algebras and to make use of the trace identity (or more generally, the variational identity), to put the corresponding soliton equations into Hamiltonian forms [1–3]. This method was extended to the super integrable equations in Ref. [4–6]. Many important super integrable equations were constructed and written in super Hamiltonian forms by making use of the super trace identity (or the super variational identity [3]). For examples, the super AKNS hierarchy [6–10], the super Dirac hierarchy [6,11,12], the super Kaup-Newell (KN) hierarchy [13,14], and others [15–17]. Both of the odd variables and the even variables are involved in super integrable equations, which has attracted many researchers' attention [18–22].

In Ref. [23], the authors considered a hierarchy of generalized AKNS equations, where the spatial spectral problem is given by

$$\phi_x = U\phi, \quad U = \begin{pmatrix} -\lambda - \mu qr & q \\ r & \lambda + \mu qr \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{1}$$

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http://dx.doi.org/10.1016/j.cnsns.2016.06.033
1007-5704/

where q and r are both scalar potentials, λ is the spectral parameter, and μ is an arbitrary constant. The case of $\mu=0$ reduces to the spatial spectral problem of the standard AKNS hierarchy [24]. Another three versions of generalized AKNS equations were also discussed in refs. [25–27]. The same idea to generalize the KN hierarchy [28,29] and the Wadati-Konno-Ichikaw (WKI) hierarchy [30], whose bi-Hamiltonian structures were constructed. Inspired by those generalizations, we would, in this paper, like to construct a generalized super AKNS hierarchy.

The paper is organized as follows. In the next section, we shall construct a generalized super AKNS hierarchy. In Section 3, the super bi-Hamiltonian form will be presented for the obtained super AKNS hierarchy by making use of the super trace identity. Moreover, we propose a generalized super AKNS hierarchy with self-consistent sources generated from the variational derivative of spectra. Some conclusions and discussions will be listed in the last section.

2. A hierarchy of generalized super AKNS equations

In this section, we shall construct a generalized super AKNS hierarchy starting from a Lie super-algebra. First, let us choose a set of basis vectors for the Lie super-algebra $B(0,1)$:

$$e_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, e_2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, e_3=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, e_4=\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix}, e_5=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}, \tag{2}$$

which satisfy the following relationship

$$\begin{aligned}&[e_1,e_2\}=-[e_4,e_4\}=2e_2, [e_3,e_1\}=[e_5,e_5\}=2e_3, [e_1,e_4\}=[e_2,e_5\}=e_4,\\ &[e_5,e_1\}=[e_3,e_4\}=e_5, [e_2,e_3\}=[e_4,e_5\}=e_1, [e_2,e_4\}=[e_3,e_5\}=0,\end{aligned}$$

where $[a,b\}=ab-(-1)^{p(a)p(b)}ba$ is the super Lie bracket, and $p(f)$ denotes the parity of an arbitrary odd or even element f. Consider the following spatial spectral problem:

$$\phi_x=U\phi,\quad U=(\lambda+\omega)e_1+qe_2+re_3+\alpha e_4+\beta e_5=\begin{pmatrix}\lambda+\omega&q&\alpha\\r&-\lambda-\omega&\beta\\\beta&-\alpha&0\end{pmatrix}, \phi=\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\end{pmatrix}, \tag{3}$$

where $\omega=\mu(qr+2\alpha\beta)$ with μ is an arbitrary even constant, λ is the spectral parameter, q and r are even potentials, and α and β are odd potentials. Note that $u=\begin{pmatrix}q\\r\\\alpha\\\beta\end{pmatrix}$ is the potential. Obviously, the spatial spectral problem (3) with $\mu=0$ reduces to the standard super AKNS case [6–10].

Second, to derive a soliton hierarchy associated with the spatial spectral problem (3), we solve the stationary zero curvature equation

$$V_x=[U,V], \tag{4}$$

where

$$V=\begin{pmatrix}A&B&\rho\\C&-A&\delta\\\delta&-\rho&0\end{pmatrix}. \tag{5}$$

Substituting U in (3) and V in (5) into the above Eq. (4), we have

$$\begin{cases}A_x=qC-rB+\alpha\delta+\beta\rho,\\ B_x=2(\lambda+\omega)B-2qA-2\alpha\rho,\\ C_x=-2(\lambda+\omega)C+2rA+2\beta\delta,\\ \rho_x=(\lambda+\omega)\rho-\alpha A-\beta B+q\delta,\\ \delta_x=-(\lambda+\omega)\delta+\beta A-\alpha C+r\rho.\end{cases} \tag{6}$$

Choosing

$$A=\sum_{j\geq0}a_j\lambda^{-j},\quad B=\sum_{j\geq0}b_j\lambda^{-j},\quad C=\sum_{j\geq0}c_j\lambda^{-j},\quad \rho=\sum_{j\geq0}\rho_j\lambda^{-j},\quad \delta=\sum_{j\geq0}\delta_j\lambda^{-j}, \tag{7}$$

and comparing the coefficients of the same powers of λ in Eq. (6), we obtain

$$\begin{cases}b_0=c_0=\rho_0=\delta_0=0,\\ a_{j,x}=qc_j-rb_j+\alpha\delta_j+\beta\rho_j,\quad j\geq0,\\ b_{j,x}=2b_{j+1}-2qa_j-2\alpha\rho_j+2\omega b_j,\quad j\geq0,\\ c_{j,x}=-2c_{j+1}+2ra_j+2\beta\delta_j-2\omega c_j,\quad j\geq0,\\ \rho_{j,x}=\rho_{j+1}-\alpha a_j-\beta b_j+q\delta_j+\omega\rho_j,\quad j\geq0,\\ \delta_{j,x}=-\delta_{j+1}+\beta a_j-\alpha c_j+r\rho_j-\omega\delta_j,\quad j\geq0,\end{cases} \tag{8}$$

which leads to a recursive relationship

$$\begin{pmatrix} c_{j+1} \\ b_{j+1} \\ \delta_{j+1} \\ \rho_{j+1} \end{pmatrix} = L_1 \begin{pmatrix} c_j \\ b_j \\ \delta_j \\ \rho_j \end{pmatrix}, \quad j \geq 0, \tag{9}$$

where the recursive operator L_1 is given as follows:

$$L_1 = \begin{pmatrix} -\frac{1}{2}\partial + r\partial^{-1}q - \omega & -r\partial^{-1}r & r\partial^{-1}\alpha + \beta & r\partial^{-1}\beta \\ q\partial^{-1}q & \frac{1}{2}\partial - q\partial^{-1}r - \omega & q\partial^{-1}\alpha & q\partial^{-1}\beta + \alpha \\ \beta\partial^{-1}q - \alpha & -\beta\partial^{-1}r & -\partial + \beta\partial^{-1}\alpha - \omega & \beta\partial^{-1}\beta + r \\ \alpha\partial^{-1}q & -\alpha\partial^{-1}r + \beta & \alpha\partial^{-1}\alpha - q & \partial + \alpha\partial^{-1}\beta - \omega \end{pmatrix}.$$

After a direct calculation, we get $a_{0,x} = 0$. For the sake of simplicity, we take $a_0 = 1$. Moreover, all constants of integration are chosen as zero. That is to say, $a_j|_{u=0} = b_j|_{u=0} = c_j|_{u=0} = \rho_j|_{u=0} = \delta_j|_{u=0} = 0$ $(j \geq 1)$. Thus, the first three sets can be computed as follows:

$$\begin{aligned}
&b_1 = q, c_1 = r, \rho_1 = \alpha, \delta_1 = \beta, a_1 = 0, \\
&b_2 = \frac{1}{2}q_x - \omega q, c_2 = -\frac{1}{2}r_x - \omega r, \rho_2 = \alpha_x - \omega\alpha, \delta_2 = -\beta_x - \omega\beta, a_2 = -\frac{1}{2}(qr + 2\alpha\beta), \\
&b_3 = \frac{1}{4}q_{xx} - \frac{1}{2}q^2r - q\alpha\beta + \alpha\alpha_x - \frac{1}{2}\omega_x q - \omega q_x + \omega^2 q, \\
&c_3 = \frac{1}{4}r_{xx} - \frac{1}{2}qr^2 - r\alpha\beta - \beta\beta_x + \frac{1}{2}\omega_x r + \omega r_x + \omega^2 r, \\
&\rho_3 = \alpha_{xx} - \frac{1}{2}qr\alpha + \frac{1}{2}q_x\beta + q\beta_x - \mu[(qr\alpha)_x + qr\alpha_x + 2\alpha_x\alpha\beta] + \mu^2q^2r^2\alpha, \\
&\delta_3 = \beta_{xx} - \frac{1}{2}qr\beta + \frac{1}{2}r_x\alpha + r\alpha_x + \mu[(qr\beta)_x + qr\beta_x + 2\alpha\beta\beta_x] + \mu^2q^2r^2\beta, \\
&a_3 = \frac{1}{4}(qr_x - q_xr) + \alpha\beta_x - \alpha_x\beta + \mu qr(qr + 4\alpha\beta).
\end{aligned}$$

Last, let us introduce the temporal spectral problems;

$$\phi_{t_n} = V^{[n]}\phi, \tag{10}$$

where

$$V^{[n]} = \sum_{j=0}^{n} \begin{pmatrix} a_j & b_j & \rho_j \\ c_j & -a_j & \delta_j \\ \delta_j & -\rho_j & 0 \end{pmatrix} \lambda^{n-j} + \Delta_n, \quad n \geq 0,$$

with Δ_n being the modification term, which doesn't appear in the standard super AKNS case. Upon supposing $\Delta_n = \begin{pmatrix} a & b & e \\ c & -a & f \\ f & -e & 0 \end{pmatrix}$, the compatibility condition of the spectral problems (3) and (10) yields the following zero curvature equation

$$U_{t_n} - V_x^{[n]} + [U, V^{[n]}] = 0, \tag{11}$$

where $n \geq 0$. Making use of (8), we have

$$\begin{cases} \omega_{t_n} = a_x, \quad b = c = e = f = 0, \\ q_{t_n} = b_{n,x} - 2\omega b_n + 2qa_n + 2\alpha\rho_n + 2qa = 2b_{n+1} + 2qa, \\ r_{t_n} = c_{n,x} + 2\omega c_n - 2ra_n - 2\beta\delta_n - 2ra = -2c_{n+1} - 2ra, \\ \alpha_{t_n} = \rho_{n,x} - \omega\rho_n + \alpha a_n + \beta b_n - q\delta_n + \alpha a = \rho_{n+1} + \alpha a, \\ \beta_{t_n} = \delta_{n,x} + \omega\delta_n - \beta a_n + \alpha c_n - r\rho_n - \beta a = -\delta_{n+1} - \beta a, \end{cases} \tag{12}$$

which guarantees the following identity:

$$(qr + 2\alpha\beta)_{t_n} = -2(qc_{n+1} - rb_{n+1} + \alpha\delta_{n+1} + \beta\rho_{n+1}) = -2a_{n+1,x}.$$

Choosing $a = -2\mu a_{n+1}$, we arrive at the following hierarchy:

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \\ \alpha_{t_n} \\ \beta_{t_n} \end{pmatrix} = \begin{pmatrix} 2b_{n+1} - 4\mu q a_{n+1} \\ -2c_{n+1} + 4\mu r a_{n+1} \\ \rho_{n+1} - 2\mu\alpha a_{n+1} \\ -\delta_{n+1} + 2\mu\beta a_{n+1} \end{pmatrix}, \tag{13}$$

where $n \geq 0$. The case of Eq. (13) with $\mu = 0$ is exactly the standard super AKNS hierarchy [5]. Therefore, Eq. (13) is called the generalized super AKNS hierarchy.

When $n=1$ in Eq. (13), the flow is trivial. When $n=2$ in Eq. (13), we obtain the first non-trivial flow:

$$\begin{cases} q_{t_2} = \frac{1}{2}q_{xx} - q^2r - 2q\alpha\beta + 2\alpha\alpha_x - 2\mu(qq_xr + q^2r_x - q\alpha_x\beta + 3q\alpha\beta_x + 2q_x\alpha\beta) - 2\mu^2q^2r(qr+4\alpha\beta), \\ r_{t_2} = -\frac{1}{2}r_{xx} + qr^2 + 2r\alpha\beta + 2\beta\beta_x - 2\mu(q_xr^2 + qrr_x - r\alpha\beta_x + 3r\alpha_x\beta + 2r_x\alpha\beta) + 2\mu^2qr^2(qr+4\alpha\beta), \\ \alpha_{t_2} = \alpha_{xx} + \frac{1}{2}q_x\beta + q\beta_x - \frac{1}{2}qr\alpha - \frac{1}{2}\mu(q_xr\alpha + 3qr_x\alpha + 4qr\alpha_x - 8\alpha\alpha_x\beta) - \mu^2q^2r^2\alpha, \\ \beta_{t_2} = -\beta_{xx} - \frac{1}{2}r_x\alpha - r\alpha_x + \frac{1}{2}qr\beta - \frac{1}{2}\mu(qr_x\beta + 3q_xr\beta + 4qr\beta_x + 8\alpha\beta\beta_x) + \mu^2q^2r^2\beta, \end{cases} \tag{14}$$

whose Lax pairs are determined by U in (3) and $V^{[2]}$, given by

$$V^{[2]} = \begin{pmatrix} V_{11}^{[2]} & V_{12}^{[2]} & V_{13}^{[2]} \\ V_{21}^{[2]} & -V_{11}^{[2]} & V_{23}^{[2]} \\ V_{23}^{[2]} & -V_{13}^{[2]} & 0 \end{pmatrix},$$

with

$$\begin{cases} V_{11}^{[2]} = \lambda^2 - \frac{1}{2}(qr+2\alpha\beta) - \frac{\mu}{2}(qr_x - q_xr) - 2\mu(\alpha\beta_x - \alpha_x\beta) - 2\mu^2qr(qr+4\alpha\beta), \\ V_{12}^{[2]} = q\lambda + \frac{1}{2}q_x - q\omega, \\ V_{13}^{[2]} = \alpha\lambda + \alpha_x - \omega\alpha, \\ V_{21}^{[2]} = r\lambda - \frac{1}{2}r_x - r\omega, \\ V_{23}^{[2]} = \beta\lambda - \beta_x - \omega\beta. \end{cases}$$

3. Super bi-Hamiltonian structures

In this section, we shall find super bi-Hamiltonian structures of the generalized super AKNS hierarchy (13). To this end, we shall use the super trace identity, which was discussed in [4,5] and rigorously proved by Ma et al. in ref. [6]:

$$\frac{\delta}{\delta u}\int Str\left(V\frac{\partial U}{\partial\lambda}\right)dx = \left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)Str\left(\frac{\partial U}{\partial u}V\right), \tag{15}$$

where Str denotes the super trace. It is easy to find that

$$\begin{aligned} &Str\left(V\frac{\partial U}{\partial\lambda}\right) = 2A, \quad Str\left(\frac{\partial U}{\partial q}V\right) = C + 2\mu rA, \quad Str\left(\frac{\partial U}{\partial r}V\right) = B + 2\mu qA, \\ &Str\left(\frac{\partial U}{\partial\alpha}V\right) = 2\delta + 4\mu\beta A, \quad Str\left(\frac{\partial U}{\partial\beta}V\right) = -2\rho - 4\mu\alpha A. \end{aligned} \tag{16}$$

Substituting (16) into (15), and balancing the coefficient of λ^{-n-2}, we have

$$\frac{\delta}{\delta u}\int 2a_{n+2}dx = (\gamma - n - 1)\begin{pmatrix} c_{n+1} + 2\mu ra_{n+1} \\ b_{n+1} + 2\mu qa_{n+1} \\ 2\delta_{n+1} + 4\mu\beta a_{n+1} \\ -2\rho_{n+1} - 4\mu\alpha a_{n+1} \end{pmatrix}, \quad n \geq 0.$$

The identity with $n=0$ tells $\gamma = 0$. Thus, we have

$$\begin{pmatrix} c_{n+1} + 2\mu ra_{n+1} \\ b_{n+1} + 2\mu qa_{n+1} \\ 2\delta_{n+1} + 4\mu\beta a_{n+1} \\ -2\rho_{n+1} - 4\mu\alpha a_{n+1} \end{pmatrix} = \frac{\delta\mathcal{H}_{n+1}}{\delta u}, \quad n \geq 0, \tag{17}$$

where $\mathcal{H}_{n+1} = -\frac{2}{n+1}\int a_{n+2}dx$. Moreover, it is easy to know that

$$\begin{pmatrix} c_{n+1} \\ b_{n+1} \\ \delta_{n+1} \\ \rho_{n+1} \end{pmatrix} = R_2\begin{pmatrix} c_{n+1} + 2\mu ra_{n+1} \\ b_{n+1} + 2\mu qa_{n+1} \\ 2\delta_{n+1} + 4\mu\beta a_{n+1} \\ -2\rho_{n+1} - 4\mu\alpha a_{n+1} \end{pmatrix}, \quad n \geq 0, \tag{18}$$

where R_2 is given by

$$R_2 = \begin{pmatrix} 1-2\mu r\partial^{-1}q & 2\mu r\partial^{-1}r & -\mu r\partial^{-1}\alpha & \mu r\partial^{-1}\beta \\ -2\mu q\partial^{-1}q & 1+2\mu q\partial^{-1}r & -\mu q\partial^{-1}\alpha & \mu q\partial^{-1}\beta \\ -2\mu\beta\partial^{-1}q & 2\mu\beta\partial^{-1}r & \frac{1}{2}-\mu\beta\partial^{-1}\alpha & \mu\beta\partial^{-1}\beta \\ -2\mu\alpha\partial^{-1}q & 2\mu\alpha\partial^{-1}r & -\mu\alpha\partial^{-1}\alpha & -\frac{1}{2}+\mu\alpha\partial^{-1}\beta \end{pmatrix}.$$

Thus, on the one hand, the hierarchy of generalized super AKNS Eq. (13) can be written as

$$u_{t_n} = R_1 \begin{pmatrix} c_{n+1} \\ b_{n+1} \\ \delta_{n+1} \\ \rho_{n+1} \end{pmatrix} = R_1R_2 \begin{pmatrix} c_{n+1}+2\mu r a_{n+1} \\ b_{n+1}+2\mu q a_{n+1} \\ 2\delta_{n+1}+4\mu\beta a_{n+1} \\ -2\rho_{n+1}-4\mu\alpha a_{n+1} \end{pmatrix} = J\frac{\delta\mathcal{H}_{n+1}}{\delta u}, \quad n \geq 0, \tag{19}$$

where

$$R_1 = \begin{pmatrix} -4\mu q\partial^{-1}q & 2+4\mu q\partial^{-1}r & -4\mu q\partial^{-1}\alpha & -4\mu q\partial^{-1}\beta \\ -2+4\mu r\partial^{-1}q & -4\mu r\partial^{-1}r & 4\mu r\partial^{-1}\alpha & 4\mu r\partial^{-1}\beta \\ -2\mu\alpha\partial^{-1}q & 2\mu\alpha\partial^{-1}r & -2\mu\alpha\partial^{-1}\alpha & 1-2\mu\alpha\partial^{-1}\beta \\ 2\mu\beta\partial^{-1}q & -2\mu\beta\partial^{-1}r & -1+2\mu\beta\partial^{-1}\alpha & 2\mu\beta\partial^{-1}\beta \end{pmatrix},$$

and

$$J = R_1R_2 = \begin{pmatrix} -8\mu q\partial^{-1}q & 2+8\mu q\partial^{-1}r & -4\mu q\partial^{-1}\alpha & 4\mu q\partial^{-1}\beta \\ -2+8\mu r\partial^{-1}q & -8\mu r\partial^{-1}r & 4\mu r\partial^{-1}\alpha & -4\mu r\partial^{-1}\beta \\ -4\mu\alpha\partial^{-1}q & 4\mu\alpha\partial^{-1}r & -2\mu\alpha\partial^{-1}\alpha & -\frac{1}{2}+2\mu\alpha\partial^{-1}\beta \\ 4\mu\beta\partial^{-1}q & -4\mu\beta\partial^{-1}r & -\frac{1}{2}+2\mu\beta\partial^{-1}\alpha & -2\mu\beta\partial^{-1}\beta \end{pmatrix},$$

which is a super Hamiltonian operator.

On the other hand, by making use of the recursive relationship (9), the generalized super AKNS hierarchy (13) can be written as follows:

$$u_{t_n} = R_1L_1 \begin{pmatrix} c_n \\ b_n \\ \delta_n \\ \rho_n \end{pmatrix} = R_1L_1R_2 \begin{pmatrix} c_n+2\mu r a_n \\ b_n+2\mu q a_n \\ 2\delta_n+4\mu\beta a_n \\ -2\rho_n-4\mu\alpha a_n \end{pmatrix} = M\frac{\delta\mathcal{H}_n}{\delta u}, \quad n \geq 0, \tag{20}$$

where $M = R_1L_1R_2 = (M_{ij})_{4\times 4}$, presented by as follows:

$$\begin{aligned}
M_{11} &= 2q\partial^{-1}q - 2\mu\partial q\partial^{-1}q + 4\mu q(\partial^{-1}q\omega + \omega\partial^{-1}q) - 4\mu^2 q\Omega\partial^{-1}q, \\
M_{12} &= -2q\partial^{-1}r - 2\omega + 2\mu\partial q\partial^{-1}r - 4\mu q(\partial^{-1}r\omega + \omega\partial^{-1}r) + 4\mu^2 q\Omega\partial^{-1}r, \\
M_{13} &= q\partial^{-1}\alpha - \mu\partial q\partial^{-1}\alpha + 2\mu q(\partial^{-1}\alpha\omega + \omega\partial^{-1}\alpha) - 2\mu^2 q\Omega\partial^{-1}\alpha, \\
M_{14} &= -\alpha - q\partial^{-1}\beta + \mu\partial q\partial^{-1}\beta - 2\mu q(\partial^{-1}\beta\omega + \omega\partial^{-1}\beta) + 2\mu^2 q\Omega\partial^{-1}\beta, \\
M_{21} &= -2r\partial^{-1}q + 2\omega - 2\mu\partial r\partial^{-1}q - 4\mu r(\partial^{-1}q\omega + \omega\partial^{-1}q) + 4\mu^2 r\Omega\partial^{-1}q, \\
M_{22} &= 2r\partial^{-1}r + 2\mu\partial r\partial^{-1}r + 4\mu r(\partial^{-1}r\omega + \omega\partial^{-1}r) - 4\mu^2 r\Omega\partial^{-1}r, \\
M_{23} &= -\beta - r\partial^{-1}\alpha - \mu\partial r\partial^{-1}\alpha - 2\mu r(\partial^{-1}\alpha\omega + \omega\partial^{-1}\alpha) + 2\mu^2 r\Omega\partial^{-1}\alpha, \\
M_{24} &= r\partial^{-1}\beta + \mu\partial r\partial^{-1}\beta + 2\mu r(\partial^{-1}\beta\omega + \omega\partial^{-1}\beta) - 2\mu^2 r\Omega\partial^{-1}\beta, \\
M_{31} &= \alpha\partial^{-1}q - 2\mu\partial\alpha\partial^{-1}q + 2\mu\alpha(\partial^{-1}q\omega + \omega\partial^{-1}q) - 2\mu^2\alpha\Omega\partial^{-1}q, \\
M_{32} &= \beta - \alpha\partial^{-1}r + 2\mu\partial\alpha\partial^{-1}r - 2\mu\alpha(\partial^{-1}r\omega + \omega\partial^{-1}r) + 2\mu^2\alpha\Omega\partial^{-1}\beta, \\
M_{33} &= -q + \frac{1}{2}\alpha\partial^{-1}\alpha - \mu\partial\alpha\partial^{-1}\alpha + \mu\alpha(\partial^{-1}\alpha\omega + \omega\partial^{-1}\alpha) - \mu^2\alpha\Omega\partial^{-1}\alpha, \\
M_{34} &= -\frac{1}{2}\alpha\partial^{-1}\beta + \frac{1}{2}\omega + \mu\partial\alpha\partial^{-1}\beta - \mu\alpha(\partial^{-1}\beta\omega + \omega\partial^{-1}\beta) + \mu^2\alpha\Omega\partial^{-1}\beta, \\
M_{41} &= \alpha - \beta\partial^{-1}q - 2\mu\partial\beta\partial^{-1}q - 2\mu\beta(\partial^{-1}q\omega + \omega\partial^{-1}q) + 2\mu^2\beta\Omega\partial^{-1}q, \\
M_{42} &= \beta\partial^{-1}r + 2\mu\partial\beta\partial^{-1}r + 2\mu\beta(\partial^{-1}r\omega + \omega\partial^{-1}r) - 2\mu^2\beta\Omega\partial^{-1}r, \\
M_{43} &= -\frac{1}{2}\beta\partial^{-1}\alpha + \frac{1}{2}\omega - \mu\partial\beta\partial^{-1}\alpha - \mu\beta(\partial^{-1}\alpha\omega + \omega\partial^{-1}\alpha) + \mu^2\beta\Omega\partial^{-1}\alpha, \\
M_{44} &= \frac{1}{2}r + \frac{1}{2}\beta\partial^{-1}\beta + \mu\partial\beta\partial^{-1}\beta + \mu\beta(\partial^{-1}\beta\omega + \omega\partial^{-1}\beta) - \mu^2\beta\Omega\partial^{-1}\beta,
\end{aligned}$$

with $\Omega = \partial^{-1}q\partial r + \partial^{-1}r\partial q + 2\partial^{-1}\alpha\partial\beta - 2\partial^{-1}\beta\partial\alpha$. Here M is the second super Hamiltonian operator.

For m distinct spectral parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$, the spatial parts by (3) and the temporal parts by (10) read

$$\begin{cases} \begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_x = U(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}, \\ \begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_{t_n} = V^{[n]}(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}, \quad n \geq 0, \end{cases} \tag{21}$$

where $1 \leq j \leq m$. Referring to the refs. [17,31,32], we obtain the variational derivative of the spectral parameter λ_j with respect to the potential u:

$$\frac{\delta\lambda_j}{\delta u} = \begin{pmatrix}\frac{\delta\lambda_j}{\delta q}\\ \frac{\delta\lambda_j}{\delta r}\\ \frac{\delta\lambda_j}{\delta\alpha}\\ \frac{\delta\lambda_j}{\delta\beta}\end{pmatrix} = \frac{1}{E_j}\begin{pmatrix} -\phi_{2j}^2 - 2\mu r\phi_{1j}\phi_{2j}\\ \phi_{1j}^2 - 2\mu q\phi_{1j}\phi_{2j}\\ -2\phi_{2j}\phi_{3j} - 4\mu\beta\phi_{1j}\phi_{2j}\\ 2\phi_{1j}\phi_{3j} + 4\mu\alpha\phi_{1j}\phi_{2j}\end{pmatrix}, \tag{22}$$

where $E_j = 2\int \phi_{1j}\phi_{2j}dx$ and $1 \leq j \leq m$. Therefore, the generalized super AKNS hierarchy with self-consistent sources is given by:

$$u_{t_n} = J\frac{\delta\mathcal{H}_{n+1}}{\delta u} + J\sum_{j=1}^{m}\frac{\delta\lambda_j}{\delta u}, \quad n \geq 0, \tag{23}$$

where $m \geq 1$.

4. Conclusions and discussions

In this paper, starting from the Lie super-algebra B(0, 1), we constructed a generalized super AKNS hierarchy (13), which can be written as the super bi-Hamiltonian forms (19) and (20) by making use of the super trace identity (15). Choosing m distinct spectral parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ in the spatial spectral problem (3) and the temporal spectral problem (10), we calculated the variational derivative of λ_j with respect to u. Thus, we proposed a generalized super AKNS hierarchy with self-consistent sources (23). For other super integrable systems, can we construct their extensions by the similar method? Moreover, in our previous papers, we have successfully applied binary nonlinearization to the super AKNS hierarchy [9,10]. Can we do the nonlinearization of Lax pairs for the generalized super AKNS hierarchy (13)? These two questions will be discussed in a future paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under grant Nos. 61273077, 11371326, 11371361, 11271008, 11301454 and 11271168.

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