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Lump solutions of a new generalized Kadomtsev–Petviashvili equation

Jing Yu[*,†,§] and Wen-Xiu Ma[†,‡]

[*] *School of Science, Hangzhou Dianzi University,
Hangzhou, Zhejiang 310018, China*
[†] *Department of Mathematics and Statistics,
University of South Florida, Tampa, FL 33620-5700, USA*
[‡] *International Institute for Symmetry Analysis and Mathematical Modelling,
Department of Mathematical Sciences, North-West University, Mafikeng Campus,
Private Bag X2046, Mmabatho 2735, South Africa*
[§] *yujing615@hdu.edu.cn*

Shou-Ting Chen

*School of Mathematics and Physical Science,
Xuzhou Institute of Technology, Xuzhou, Jiangsu 221111, China*

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A new generalized Kadomtsev–Petviashvili (GKP) equation is derived from a bilinear differential equation by taking the transformation $u = 2(\ln f)_x$. By symbolic computation with Maple, lump solutions, rationally localized in all directions in the space, to the GKP equation are presented. The obtained lump solutions contain a set of six free parameters, four of which should satisfy a nonzero determinant condition. As special examples, six particular lump solutions are constructed and depicted with $t = 1$.

Keywords: Lump solution; GKP equation; Hirota bilinear form.

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1. Introduction

Solving nonlinear differential equations has been one of the interesting topics in the field of soliton and integrable system. Many methods have been developed to derive solutions of nonlinear differential equations, such as Darboux transformation,[1] Hirota bilinear method,[2] inverse scattering transformation[3] and so on. Among these methods, Hirota bilinear method is regarded as a powerful tool to obtain soliton solutions,[4–6] rational function solutions,[7–9] etc. Lump solutions, which are located

[§]Corresponding author.

in all directions in the space, are a kind of rational function solutions and they have been widely constructed by many researchers for many integrable equations, such as $(2+1)$-dimensional nonlinear Schrödinger equation (i.e. Davey–Stewartson equation),[10] Kadomtsev–Petviashvili (KP) equation,[11,12] BKP equation,[13] three-dimensional three-wave resonant interaction,[14] and so on.[15–22]

In Ref. 12, one of the authors gave a class of lump solutions to the $(2+1)$-dimensional KPI equation by Maple symbolic computation. Those lump solutions contain six free parameters, four of which satisfy a nonzero determinant condition guaranteeing analyticity and localization of the solutions. In fact, any lump solution of KPI equation has this form of the sum of squares. Inspired by this, we will construct a type of new GKP equation, and try to search for its lump solutions.

The plan of this paper is as follows. In Sec. 2, starting from Hirota bilinear operators, a new bilinear differential equation is derived. By the transformation $u = 2(\ln f)_x$, a GKP equation is constructed. Then we will search for positive quadratic function solutions of this bilinear differential equation by a Maple program. Accordingly, lump solutions of the GKP equation are obtained, and as its application, six particular cases with specific values of the involved parameters are plotted. Some conclusions and discussions are listed in Sec. 3.

2. Lump Solutions of the GKP Equation

Suppose that P is a polynomial in variables x, y, t:

$$P(x,y,t) = c_1x^4 + c_2x^2yt + c_3y^2 + c_4xy + c_5xt\,,$$

where coefficients c_i $(1 \le i \le 5)$ are arbitrary constants. And the corresponding $(2+1)$-dimensional bilinear differential equation read

$$\begin{aligned} P(D_x, D_y, D_t)f\cdot f &= (c_1D_x^4 + c_2D_x^2D_yD_t + c_3D_y^2 + c_4D_xD_y + c_5D_xD_t)f\cdot f \\ &= 2c_1(ff_{xxxx} - 4f_{xxx}f_x + 3f_{xx}^2) + 2c_2(ff_{xxyt} - f_{xxy}f_t - f_{xxt}f_y \\ &\quad + f_{xx}f_{yt} - 2f_{xyt}f_x + 2f_{xy}f_{xt}) + 2c_3(ff_{yy} - f_y^2) \\ &\quad + 2c_4(ff_{xy} - f_xf_y) + 2c_5(ff_{xt} - f_xf_t) = 0\,, \end{aligned} \tag{1}$$

where f is a function associated with variables x, y, t, and D_x, D_y, D_t are Hirota bilinear operators, defined by[2,23]

$$\begin{aligned} D_x^mD_y^nD_t^kf\cdot g &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k \\ &\quad \times f(x,y,t)g(x',y',t')|_{x'=x,y'=y,t'=t}\,, \end{aligned} \tag{2}$$

for non-negative integers m, n and k.

Remark 1. Term of $D_x^2D_yD_tf\cdot f$ is useful for the following procedure of constructing rational function solutions. However, it has never appeared in the previous relevant references.

Taking the transformation

$$u = 2(\ln f)_x\,, \tag{3}$$

Eq. (1) can be written as the following nonlinear differential equation:

$$c_1(u_{xxx} + 3u_x^2) + c_2(u_{xyt} + u_x v_t + 2u_t u_y) + c_3 v_y + c_4 u_y + c_5 u_t = 0\,, \tag{4}$$

where $u_y = v_x$.

Remark 2. If f solves the bilinear differential equation (1), then u, defined by (3), solves the nonlinear differential equation (4).

The nonlinear differential equation (4) is identified as the famous KPI equation when $c_2 = c_4 = 0$, $c_1 = -c_3 = c_5 = 1$ and KPII equation when $c_2 = c_4 = 0$, $c_1 = c_3 = c_5 = 1$. Therefore, Eq. (4) is regarded as the GKP equation. Furthermore, we have known that lump solutions of the KP equation have been studied in Ref. 12. Here, we will search for lump solutions of Eq. (4). To this end, we firstly construct polynomial function solutions of Eq. (1). Suppose

$$f = g^2 + h^2 + a_9\,, \tag{5}$$

where

$$\begin{cases} g = a_1 x + a_2 y + a_3 t + a_4\,, \\ h = a_5 x + a_6 y + a_7 t + a_8\,, \end{cases} \tag{6}$$

and a_i $(1 \le i \le 9)$ are real parameters to be determined. Substituting (5) into Eq. (1), we obtain the constrained equation associated with parameters a_i $(1 \le i \le 9)$ and c_k $(1 \le k \le 5)$. After a direct Maple program, the parameters a_i $(1 \le i \le 9)$ can be solved as follows:

$$\begin{aligned} \Big\{ & a_1 = a_1, \quad a_2 = a_2, \quad a_3 = -\frac{(a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6)c_3}{(a_1^2 + a_5^2)c_5} - \frac{a_2 c_4}{c_5}\,, \\ & a_4 = a_4, \quad a_5 = a_5, \quad a_6 = a_6\,, \\ & a_7 = -\frac{(2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2)c_3}{(a_1^2 + a_5^2)c_5} - \frac{a_6 c_4}{c_5}, \quad a_8 = a_8\,, \\ & a_9 = \frac{1}{(a_1 a_6 - a_2 a_5)^2 c_3 c_5}[-3(a_1^2 + a_5^2)^3 c_1 c_5 + (a_1 a_2 + a_5 a_6) \\ & \quad \times (3a_1^2 a_2^2 - a_2^2 a_5^2 + 8a_1 a_2 a_5 a_6 + 3a_5^2 a_6^2 - a_1^2 a_6^2)c_2 c_3 \\ & \quad + (a_1^2 + a_5^2)(3a_1^2 a_2^2 + a_1^2 a_6^2 + 4a_1 a_2 a_5 a_6 + a_2^2 a_5^2 + 3a_5^2 a_6^2)c_2 c_4] \Big\}, \end{aligned} \tag{7}$$

where $a_1, a_2, a_4, a_5, a_6, a_8$ are free real constants, $c_3 \neq 0$, $c_5 \neq 0$, $a_1 a_6 - a_2 a_5 \neq 0$ and $a_1^2 + a_5^2 \neq 0$. In fact, if $a_1 a_6 - a_2 a_5 \neq 0$, then $a_1^2 + a_5^2 \neq 0$. But vice is not true. The set (7) leads to a class of quadratic function solutions of Eq. (1):

$$f = \left[a_1 x + a_2 y - \left(\frac{(a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6)c_3}{(a_1^2 + a_5^2)c_5} + \frac{a_2 c_4}{c_5}\right) t + a_4\right]^2$$

$$+\left[a_5x+a_6y-\left(\frac{(a_5a_6^2-a_2^2a_5+2a_1a_2a_6)c_3}{(a_1^2+a_5^2)c_5}+\frac{a_6c_4}{c_5}\right)t+a_8\right]^2$$
$$+\frac{1}{(a_1a_6-a_2a_5)^2c_3c_5}[-3(a_1^2+a_5^2)^3c_1c_5+(a_1a_2+a_5a_6)(3(a_1a_2+a_5a_6)^2$$
$$-(a_1a_6-a_2a_5)^2)c_2c_3+(a_1^2+a_5^2)(3(a_1a_2+a_5a_6)^2+(a_1a_6-a_2a_5)^2)c_2c_4]\,. \tag{8}$$

Moreover, under the transformation (3), lump solutions of the nonlinear differential equation (4) can be derived in the following form:

$$u=4\frac{a_1g_1+a_5h_1}{f}\,, \tag{9}$$

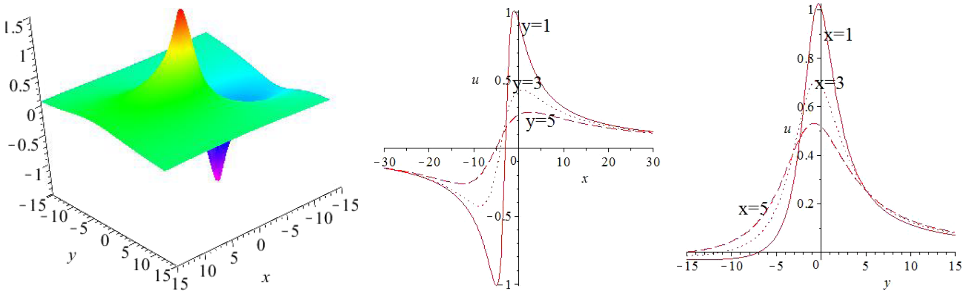
where

$$\begin{cases} g_1=a_1x+a_2y-\left(\dfrac{(a_1a_2^2-a_1a_6^2+2a_2a_5a_6)c_3}{(a_1^2+a_5^2)c_5}+\dfrac{a_2c_4}{c_5}\right)t+a_4\,,\\ h_1=a_5x+a_6y-\left(\dfrac{(a_5a_6^2-a_2^2a_5+2a_1a_2a_6)c_3}{(a_1^2+a_5^2)c_5}+\dfrac{a_6c_4}{c_5}\right)t+a_8\,,\end{cases}$$

and f are given by (8). Note that lump solutions (9) of the nonlinear equation (4) are analytic in the xy-plane if and only if the parameter $a_9>0$ in (7). Furthermore, we know that lump solutions (9) contain six free parameters $(a_1,a_2,a_4,a_5,a_6,a_8)$, two of which (a_4,a_8) are due to the translation invariance of the GKP equation (4) and the other four of which satisfy a nonzero determinant condition $(a_1a_6-a_2a_5\neq 0)$, which indicates that g_1 and h_1 are independent.

From the above procedure, we find that free parameters a_1,a_2,a_4,a_5,a_6,a_8 should satisfy the condition $a_1a_6-a_2a_5\neq 0$, coefficients c_k $(1\leq k\leq 5)$ should satisfy the conditions $c_3\neq 0, c_5\neq 0$. Moreover, the condition $a_9>0$ should be satisfied. After direct calculation of a_9 in (7), we know that when $\frac{c_1}{c_3}\leq 0, \frac{c_2c_4}{c_3c_5}\geq 0, (a_1a_2+a_5a_6)(3(a_1a_2+a_5a_6)^2-(a_1a_6-a_2a_5)^2)\frac{c_2}{c_5}\geq 0$ while $c_1^2+c_2^2\neq 0$, the condition $a_9>0$ is satisfied. In order to display lump solutions of the GKP equation (1) more specifically, we choose the following five kinds of cases of $c_1=0$, $c_2=0$, $c_4=0$, $c_1=c_4=0$, $c_2=c_4=0$ and the others are nonzero. And all of c_k $(1\leq k\leq 5)$ are nonzero as the sixth case. Here, we point out that the chosen a_i $(1\leq i\leq 9)$ and c_k $(1\leq k\leq 5)$ must comply with the above conditions. In the following, we will display lump solutions of the GKP equation (4) under these six cases. Note that the same free parameters a_1,a_2,a_4,a_5,a_6, a_8 are chosen here.

Case 1. If we choose $a_1=1, a_2=2, a_5=1, a_6=-1, c_1=0, c_2=1, c_3=-1, c_4=1, c_5=-1, a_4=0, a_8=0$, then $a_9=\frac{10}{3}>0$. The positive quadratic function



(a) (b) (c)

Fig. 1. (Color online) The plots of (10) at (a) $t = 1$, (b) x-curves and (c) y-curves.

solutions and lump solutions are given by

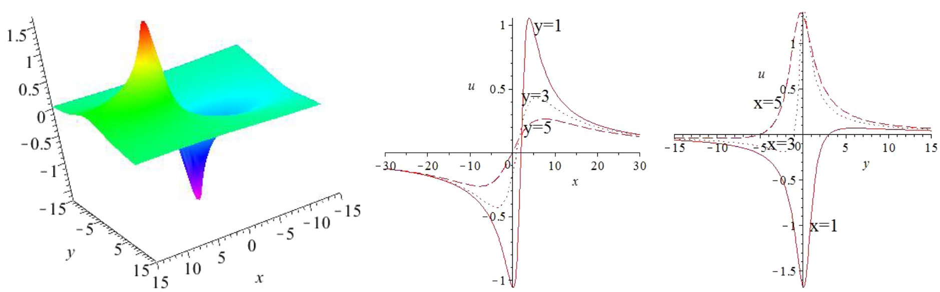
$$\begin{cases} f = \left(x + 2y + \dfrac{5}{2}t\right)^2 + \left(x - y + \dfrac{5}{2}t\right)^2 + \dfrac{10}{3}\,, \\ u = \dfrac{4(2x + y + 5t)}{(x + 2y + \frac{5}{2}t)^2 + (x - y + \frac{5}{2}t)^2 + \frac{10}{3}}\,. \end{cases} \tag{10}$$

The plots of u in (10) with $t = 1$ are depicted by Fig. 1.

Case 2. If we choose $a_1 = 1, a_2 = 2, a_5 = 1, a_6 = -1, c_1 = 1, c_2 = 0, c_3 = -1, c_4 = 1, c_5 = 1, a_4 = 0, a_8 = 0$, then $a_9 = \frac{8}{3} > 0$. The positive quadratic function solutions and lump solutions are given by

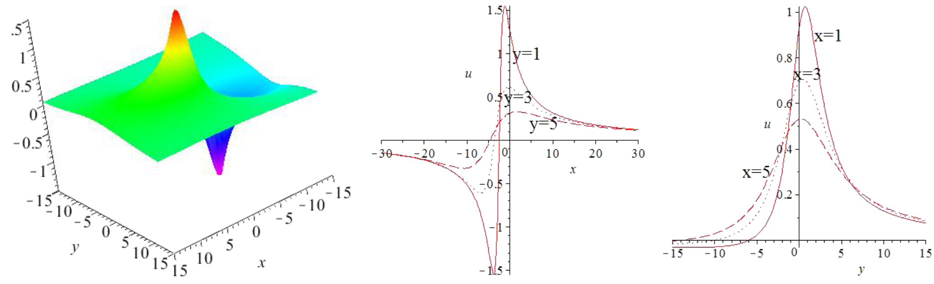
$$\begin{cases} f = \left(x + 2y - \dfrac{5}{2}t\right)^2 + \left(x - y - \dfrac{5}{2}t\right)^2 + \dfrac{8}{3}\,, \\ u = \dfrac{4(2x + y - 5t)}{(x + 2y - \frac{5}{2}t)^2 + (x - y - \frac{5}{2}t)^2 + \frac{8}{3}}\,. \end{cases} \tag{11}$$

The plots of u in (11) with $t = 1$ are depicted by Fig. 2.



(a) (b) (c)

Fig. 2. (Color online) The plots of (11) at (a) $t = 1$, (b) x-curves and (c) y-curves.



(a) (b) (c)

Fig. 3. (Color online) The plots of (12) at (a) $t = 1$, (b) x-curves and (c) y-curves.

Case 3. If we choose $a_1 = 1, a_2 = 2, a_5 = 1, a_6 = -1, c_1 = 1, c_2 = 1, c_3 = -1, c_4 = 0, c_5 = -1, a_4 = 0, a_8 = 0$, then $a_9 = \frac{10}{3} > 0$. The positive quadratic function solutions and lump solutions are given by

$$\begin{cases} f = \left(x + 2y + \dfrac{1}{2}t\right)^2 + \left(x - y + \dfrac{7}{2}t\right)^2 + \dfrac{10}{3}, \\ u = \dfrac{4(2x + y + 4t)}{(x + 2y + \frac{1}{2}t)^2 + (x - y + \frac{7}{2}t)^2 + 10/3}. \end{cases} \tag{12}$$

The plots of u in (12) with $t = 1$ are depicted by Fig. 3.

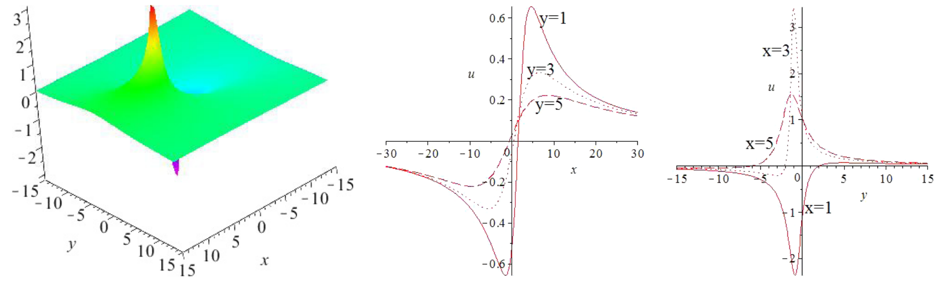
Case 4. If we choose $a_1 = 1, a_2 = 2, a_5 = 1, a_6 = -1, c_1 = 0, c_2 = 1, c_3 = 1, c_4 = 0, c_5 = -1, a_4 = 0, a_8 = 0$, then $a_9 = \frac{2}{3} > 0$. The positive quadratic function solutions and lump solutions are given by

$$\begin{cases} f = \left(x + 2y - \dfrac{1}{2}t\right)^2 + \left(x - y - \dfrac{7}{2}t\right)^2 + \dfrac{2}{3}, \\ u = \dfrac{4(2x + y - 4t)}{(x + 2y - \frac{1}{2}t)^2 + (x - y - \frac{7}{2}t)^2 + \frac{2}{3}}. \end{cases} \tag{13}$$

The plots of u in (13) with $t = 1$ are depicted by Fig. 4.

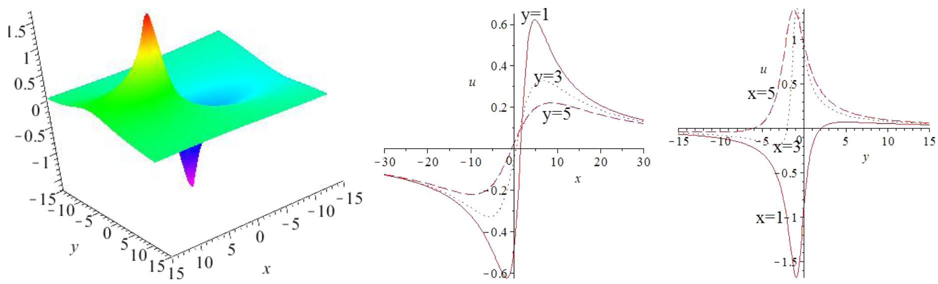
Case 5. If we choose $a_1 = 1, a_2 = 2, a_5 = 1, a_6 = -1, c_1 = 1, c_2 = 0, c_3 = -1, c_4 = 0, c_5 = 1, a_4 = 0, a_8 = 0$, then $a_9 = \frac{8}{3} > 0$. The positive quadratic function solutions and lump solutions are given by

$$\begin{cases} f = \left(x + 2y - \dfrac{1}{2}t\right)^2 + \left(x - y - \dfrac{7}{2}t\right)^2 + \dfrac{8}{3}, \\ u = \dfrac{4(2x + y - 4t)}{(x + 2y - \frac{1}{2}t)^2 + (x - y - \frac{7}{2}t)^2 + \frac{8}{3}}. \end{cases} \tag{14}$$



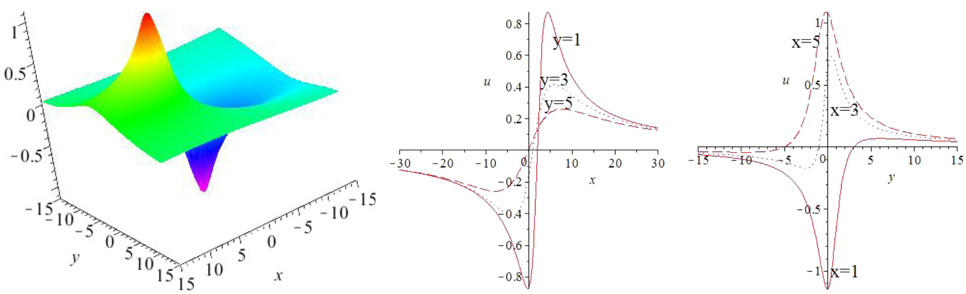
(a) (b) (c)

Fig. 4. (Color online) The plots of (13) at (a) $t = 1$, (b) x-curves and (c) y-curves.



(a) (b) (c)

Fig. 5. (Color online) The plots of (14) at (a) $t = 1$, (b) x-curves and (c) y-curves.



(a) (b) (c)

Fig. 6. (Color online) The plots of (15) at (a) $t = 1$, (b) x-curves and (c) y-curves.

The plots of u in (14) with $t = 1$ are depicted by Fig. 5.

Case 6. If we choose $a_1 = 1, a_2 = 2, a_5 = 1, a_6 = -1, c_1 = -1, c_2 = 1, c_3 = 1, c_4 = -1, c_5 = -1, a_4 = 0, a_8 = 0$, then $a_9 = 6 > 0$. The positive quadratic function solutions and lump solutions are given by

$$\begin{cases} f = \left(x + 2y - \frac{5}{2}t\right)^2 + \left(x - y - \frac{5}{2}t\right)^2 + 6\,, \\ u = \frac{4(2x + y - 5t)}{(x + 2y - \frac{5}{2}t)^2 + (x - y - \frac{5}{2}t)^2 + 6}\,. \end{cases} \tag{15}$$

The plots of u in (15) with $t = 1$ are depicted by Fig. 6.

From above six cases, we notice that lump solutions $u \to 0$ if and only if the corresponding sum of squares $g_1^2 + h_1^2 \to \infty$. In fact, this condition is useful for any cases as long as the chosen a_i $(1 \leq i \leq 9)$ and c_k $(1 \leq k \leq 5)$ satisfy the constrained conditions.

3. Conclusions and Discussions

As for the bilinear GKP equation (1) and its corresponding nonlinear differential equation (4), positive quadratic functions solutions and lump solutions are constructed by symbolic computations with Maple, respectively. Comparing Ref. 12 and our paper, we find that Eq. (1) with $c_1 = -c_3 = c_5 = 1, c_2 = c_4 = 0$ is exactly Eq. (2.2) with $\sigma = 1$ in Ref. 12. Moreover, substituting $c_1 = -c_3 = c_5 = 1$, $c_2 = c_4 = 0$ into (8), positive quadratic function solutions of the bilinear GKP equation (1) are defined by

$$f = f_1^2 + f_2^2 + \frac{3(a_1^2 + a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2}\,, \tag{16}$$

where $f_1 = a_1x + a_2y + \frac{a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6}{a_1^2 + a_2^2}t + a_4$, $f_2 = a_5x + a_6y + \frac{2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2}{a_1^2 + a_5^2}t + a_8$, which is in complete accord with Eq. (2.7) in Ref. 12. Because the transformation (3) is different from (2.1) in Ref. 12, the corresponding lump solutions (Eq. (9) in our paper and Eq. (2.8) in Ref. 12) are also different. For example, lump solutions of the nonlinear GKP equation (4) with $c_1 = -c_3 = c_5 = 1, c_2 = c_4 = 0$ are given by

$$u = 4\frac{a_1 f_1 + a_5 f_2}{f_1^2 + f_2^2 + \frac{3(a_1^2 + a_5^2)^3}{(a_1a_6 - a_2a_5)^2}}\,, \tag{17}$$

which are solutions of potential KP equation. In our future paper, we will search for lump solutions of other nonlinear integrable equations.

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References

1. V. B. Matveev and M. A. Salle, *Darboux Transformation and Solitons* (Springer-Verlag, Berlin, 1991).
2. R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004).
3. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
4. X. Lü and M. S. Peng, *Chaos* **23**(1) (2013) 013122.
5. J. C. Chen, Y. Chen, B. F. Feng and H. M. Zhu, *Appl. Math. Lett.* **37** (2014) 15.
6. X. Lü, F. H. Lin and F. H. Qi, *Appl. Math. Model.* **39**(12) (2015) 3221.
7. Y. Zhang and W. X. Ma, *Appl. Math. Comput.* **256** (2015) 252.
8. C. G. Shi, B. Z. Zhao and W. X. Ma, *Appl. Math. Lett.* **48** (2015) 170.
9. J. C. Chen, Y. Chen, B. F. Feng and K. Maruno, *Phys. Lett. A* **379**(24–25) (2015) 1510.
10. J. Satsuma and M. J. Ablowitz, *J. Math. Phys.* **20**(7) (1979) 1496.
11. S. V. Manakov and V. E. Zakharov, *Phys. Lett. A* **63**(3) (1977) 205.
12. W. X. Ma, *Phys. Lett. A* **379**(36) (2015) 1975.
13. C. R. Gilson and J. J. C. Nimmo, *Phys. Lett. A* **147**(8–9) (1990) 472.
14. D. J. Kaup, *J. Math. Phys.* **22**(6) (1981) 1176.
15. J. Satsuma and M. J. Ablowitz, *J. Math. Phys.* **20**(7) (1979) 1496.
16. P. G. Estévez, J. Prada and J. Villarroel, *J. Phys. A: Math. Theor.* **40**(26) (2007) 7213.
17. P. G. Estévez and J. Prada, *J. Nonlinear Math. Phys.* **15**(3) (2008) 166.
18. X. M. Zhu, D. J. Zhang and D. Y. Chen, *Commun. Theor. Phys.* **55**(1) (2011) 13.
19. W. X. Ma, Z. Y. Qin and X. Lü, *Nonlinear Dynam.* **84**(2) (2015) 923.
20. W. X. Ma, Y. Zhou and R. Dougherty, *Int. Mod. Phys. B* **30**(28–29) (2016) 1640018.
21. W. X. Ma and Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, preprint in 2015.
22. Z. S. Lü and Y. N. Chen, *Eur. Phys. J. B* **88**(7) (2015) 187.
23. R. Hirota, *Progr. Theor. Phys.* **52**(5) (1974) 1498.