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The Bargmann Symmetry Constraint and Binary Nonlinearization of the Super Dirac Systems*****

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Abstract An explicit Bargmann symmetry constraint is computed and its associated binary nonlinearization of Lax pairs is carried out for the super Dirac systems. Under the obtained symmetry constraint, the n-th flow of the super Dirac hierarchy is decomposed into two super finite-dimensional integrable Hamiltonian systems, defined over the supersymmetry manifold $R^{4N|2N}$ with the corresponding dynamical variables x and t_n. The integrals of motion required for Liouville integrability are explicitly given.

Keywords Symmetry constraints, Binary nonlinearization, Super Dirac systems, Super finite-dimensional integrable Hamiltonian systems
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1 Introduction

For almost twenty years, much attention has been paid to the construction of finite-dimensional integrable systems from soliton equations by using symmetry constraints. Either $(2+1)$-dimensional soliton equations (see [1–3]) or $(1+1)$-dimensional soliton equations (see [4, 5]) can be decomposed into compatible finite-dimensional integrable systems. It is known that a crucial idea in carrying out symmetry constraints is the nonlinearization of Lax pairs for soliton hierarchies, and symmetry constraints give relations of potentials with eigenfunctions and adjoint eigenfunctions of Lax pairs so that solutions to soliton equations can be obtained by solving Jacobi inversion problems (see [6]). The nonlinearization of Lax pairs is classified into mono-nonlinearization (see [7–9]) and binary nonlinearization (see [10, 11]).

The technique of nonlinearization has been successfully applied to many well-known $(1+1)$-dimensional soliton equations, such as the AKNS system (see [4, 8]), the KdV equation (see [5]) and the Dirac system (see [12]). But there are few results on nonlinearization of super integrable systems, existing in the literature. Studies provide many examples of supersymmetry

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integrable systems, with super dependent variables and/or super independent variables (see [13–18]). Very recently, nonlinearization was made for the super AKNS system (see [19]) and the corresponding super finite-dimensional Hamiltonian systems were generated. In this paper, we would like to analyze binary nonlinearization for the super Dirac systems under a Bargmann symmetry constraint.

The paper is organized as follows. In the next section, we will recall the super Dirac soliton hierarchy and its super Hamiltonian structure. Then in Section 3, we compute a Bargmann symmetry constraint for the potential of the super Dirac hierarchy. In Section 4, we apply binary nonlinearization to the super Dirac hierarchy, and then obtain super finite-dimensional integrable Hamiltonian systems on the supersymmetry manifold $R^{4N|2N}$, whose integrals of motion are explicitly given. Some conclusions and remarks are listed in Section 5.

2 The Super Dirac Hierarchy

The super Dirac spectral problem associated with the Lie super-algebra $B(0,1)$ is given by

$$\phi_x = U\phi, \quad U = \begin{pmatrix} r & \lambda + s & \alpha \\ -\lambda + s & -r & \beta \\ \beta & -\alpha & 0 \end{pmatrix}, \quad u = \begin{pmatrix} r \\ s \\ \alpha \\ \beta \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.1}$$

where λ is a spectral parameter, r and s are even variables, and α and β are odd variables (see [20]). Taking

$$V = \begin{pmatrix} C & A + B & \rho \\ A - B & -C & \delta \\ \delta & -\rho & 0 \end{pmatrix},$$

the co-adjoint equation associated with (2.1) $V_x = [U, V]$ gives

$$\begin{cases} A_x = -2\lambda C + 2rB - \alpha\rho + \beta\delta, \\ B_x = 2rA - 2sC - \alpha\rho - \beta\delta, \\ C_x = 2\lambda A - 2sB + \alpha\delta + \beta\rho, \\ \rho_x = -\beta(A + B) - \alpha C + (\lambda + s)\delta + r\rho, \\ \delta_x = (-\lambda + s)\rho - r\delta - \alpha(A - B) + \beta C. \end{cases} \tag{2.2}$$

If we set

$$A = \sum_{i\geq 0} A_i\lambda^{-i}, \quad B = \sum_{i\geq 0} B_i\lambda^{-i}, \quad C = \sum_{i\geq 0} C_i\lambda^{-i}, \quad \rho = \sum_{i\geq 0} \rho_i\lambda^{-i}, \quad \delta = \sum_{i\geq 0} \delta_i\lambda^{-i}, \tag{2.3}$$

then equation (2.2) is equivalent to

$$\begin{cases} A_0 = C_0 = \rho_0 = \delta_0 = 0, & \\ A_{i+1} = \frac{1}{2}C_{i,x} + sB_i - \frac{1}{2}\alpha\delta_i - \frac{1}{2}\beta\rho_i, & i \geq 0, \\ C_{i+1} = -\frac{1}{2}A_{i,x} + rB_i - \frac{1}{2}\alpha\rho_i + \frac{1}{2}\beta\delta_i, & i \geq 0, \\ \rho_{i+1} = -\delta_{i,x} - r\delta_i + s\rho_i - \alpha(A_i - B_i) + \beta C_i, & i \geq 0, \\ \delta_{i+1} = \rho_{i,x} - r\rho_i - s\delta_i + \beta(A_i + B_i) + \alpha C_i, & i \geq 0, \\ B_{i+1,x} = 2rA_{i+1} - 2sC_{i+1} - \alpha\rho_{i+1} - \beta\delta_{i+1}, & i \geq 0, \end{cases} \tag{2.4}$$

which results in the recurrence relations

$$\begin{cases}(C_{i+1}, A_{i+1}, \delta_{i+1}, -\rho_{i+1})^{\mathrm{T}} = \mathcal{L}(C_i, A_i, \delta_i, -\rho_i)^{\mathrm{T}}, & i \geq 0,\\ B_i = \partial^{-1}(2rA_i - 2sC_i - \alpha\rho_i - \beta\delta_i), & i \geq 0,\end{cases} \tag{2.5}$$

where

$$\mathcal{L} = \begin{pmatrix} -2r\partial^{-1}s & -\frac{1}{2}\partial + 2r\partial^{-1}r & \frac{1}{2}\beta - r\partial^{-1}\beta & \frac{1}{2}\alpha + r\partial^{-1}\alpha \\ \frac{1}{2}\partial - 2s\partial^{-1}s & 2s\partial^{-1}r & -\frac{1}{2}\alpha - s\partial^{-1}\beta & \frac{1}{2}\beta + s\partial^{-1}\alpha \\ \alpha - 2\beta\partial^{-1}s & \beta + 2\beta\partial^{-1}r & -s - \beta\partial^{-1}\beta & -\partial + r + \beta\partial^{-1}\alpha \\ -\beta + 2\alpha\partial^{-1}s & \alpha - 2\alpha\partial^{-1}r & \partial + r + \alpha\partial^{-1}\beta & s - \alpha\partial^{-1}\alpha \end{pmatrix}.$$

Upon choosing the initial conditions

$$A_0 = C_0 = \rho_0 = \delta_0 = 0, \quad B_0 = 1,$$

all other $A_i, B_i, C_i, \rho_i, \delta_i$ $(i \geq 1)$ can be worked out uniquely by the recurrence relations (2.5). The first few results are as follows:

$$\begin{aligned}
&A_1 = s, \quad B_1 = 0, \quad C_1 = r, \quad \rho_1 = \alpha, \quad \delta_1 = \beta,\\
&A_2 = \frac{1}{2}r_x, \quad B_2 = \frac{1}{2}(r^2+s^2) + \alpha\beta, \quad C_2 = -\frac{1}{2}s_x, \quad \rho_2 = -\beta_x, \quad \delta_2 = \alpha_x,\\
&A_3 = -\frac{1}{4}s_{xx} + \frac{1}{2}(r^2+s^2)s + s\alpha\beta - \frac{1}{2}\alpha\alpha_x + \frac{1}{2}\beta\beta_x,\\
&B_3 = -\frac{1}{2}(rs_x - r_xs) + \alpha\alpha_x + \beta\beta_x,\\
&C_3 = -\frac{1}{4}r_{xx} + \frac{1}{2}(r^2+s^2)r + r\alpha\beta + \frac{1}{2}\alpha\beta_x - \frac{1}{2}\alpha_x\beta,\\
&\rho_3 = -\alpha_{xx} + \frac{1}{2}(r^2+s^2)\alpha - \frac{1}{2}r_x\alpha - \frac{1}{2}s_x\beta - r\alpha_x - s\beta_x,\\
&\delta_3 = -\beta_{xx} + \frac{1}{2}(r^2+s^2)\beta + \frac{1}{2}r_x\beta - \frac{1}{2}s_x\alpha - s\alpha_x + r\beta_x.
\end{aligned}$$

Let us associate the spectral problem (2.1) with the following auxiliary spectral problem:

$$\phi_{t_n} = V^{(n)}\phi = (\lambda^n V)_+\phi \tag{2.6}$$

with

$$V^{(n)} = \sum_{i=0}^{n} \begin{pmatrix} C_i & A_i + B_i & \rho_i \\ A_i - B_i & -C_i & \delta_i \\ \delta_i & -\rho_i & 0 \end{pmatrix} \lambda^{n-i},$$

where the plus symbol "+" denotes taking the non-negative part in the power of λ.

The compatible conditions of the spectral problem (2.1) and the auxiliary spectral problem (2.6) are

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad n \geq 0, \tag{2.7}$$

which infer the super Dirac soliton hierarchy

$$u_{t_n} = K_n = (2A_{n+1}, -2C_{n+1}, \delta_{n+1}, -\rho_{n+1})^{\mathrm{T}}, \quad n \geq 0. \tag{2.8}$$

Here $u_{t_n} = K_n$ in (2.8) is called the n-th Dirac flow of the hierarchy.

Using the super trace identity

$$\frac{\delta}{\delta u}\int \mathrm{Str}\Big(V\frac{\partial U}{\partial \lambda}\Big)\mathrm{d}x = \Big(\lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\Big)\mathrm{Str}\Big(\frac{\partial U}{\partial u}V\Big), \tag{2.9}$$

where Str means the super trace (see [20, 21]), we have

$$\begin{pmatrix} C_{i+1} \\ A_{i+1} \\ \delta_{i+1} \\ -\rho_{i+1} \end{pmatrix} = \frac{\delta}{\delta u}H_i, \quad H_i = \int \frac{B_{i+2}}{i+1}\,\mathrm{d}x, \quad i \geq 0. \tag{2.10}$$

Therefore, the super Dirac soliton hierarchy (2.8) can be written as the following super Hamiltonian form:

$$u_{t_n} = J\frac{\delta H_n}{\delta u}, \tag{2.11}$$

where

$$J = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a supersymplectic operator, and H_n is given by (2.10).

The first non-trivial nonlinear equation of hierarchy (2.11) is given by the second Dirac flow

$$\begin{cases} r_{t_2} = -\frac{1}{2}s_{xx} + (r^2+s^2)s + 2s\alpha\beta - \alpha\alpha_x + \beta\beta_x, \\ s_{t_2} = \frac{1}{2}r_{xx} - (r^2+s^2)r - 2r\alpha\beta - \alpha\beta_x + \alpha_x\beta, \\ \alpha_{t_2} = -\beta_{xx} + \frac{1}{2}(r^2+s^2)\beta + r\beta_x - s\alpha_x + \frac{1}{2}r_x\beta - \frac{1}{2}s_x\alpha, \\ \beta_{t_2} = \alpha_{xx} - \frac{1}{2}(r^2+s^2)\alpha + r\alpha_x + s\beta_x + \frac{1}{2}r_x\alpha + \frac{1}{2}s_x\beta, \end{cases} \tag{2.12}$$

which possesses a Lax pair of U defined in (2.1) and $V^{(2)}$ defined by

$$V^{(2)} = \begin{pmatrix} r\lambda - \frac{1}{2}s_x & \lambda^2 + s\lambda + \frac{1}{2}r_x + \frac{1}{2}(r^2+s^2) + \alpha\beta & \alpha\lambda - \beta_x \\ -\lambda^2 + s\lambda + \frac{1}{2}r_x - \frac{1}{2}(r^2+s^2) - \alpha\beta & -r\lambda + \frac{1}{2}s_x & \beta\lambda + \alpha_x \\ \beta\lambda + \alpha_x & -\alpha\lambda + \beta_x & 0 \end{pmatrix}.$$

Remark 2.1 We consider all differential equations in the real field and explore the Liouville integrability on real symplectic manifolds. We do not see any equivalence between the real Dirac soliton hierarchy and the real AKNS soliton hierarchy. For example, it is clear that the AKNS system of nonlinear Schrödinger equations and the Dirac system of nonlinear Schrödinger equations can not be transformed into each other by any real linear transformations. There is a similar situation between the super Dirac soliton hierarchy and the super AKNS soliton hierarchy, and between the Liouville integrable constrained flows associated with the two super soliton hierarchies.

3 The Bargmann Symmetry Constraint

In order to compute a Bargmann symmetry constraint, we consider the following adjoint spectral problem of the spectral problem (2.1):

$$\psi_x = -U^{\mathrm{St}}\psi = \begin{pmatrix} -r & \lambda - s & \beta \\ -\lambda - s & r & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix}\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{3.1}$$

where St means the super transposition. The following result is a general formula for the variational derivative with respect to the potential u (see [4] for the classical case).

Lemma 3.1 *Let $U(u,\lambda)$ be an even matrix of order $m+n$ depending on $u, u_x, u_{xx}, \cdots$ and a parameter λ. Suppose that $\phi = (\phi_{\mathrm{e}}, \phi_{\mathrm{o}})^{\mathrm{T}}$ and $\psi = (\psi_{\mathrm{e}}, \psi_{\mathrm{o}})^{\mathrm{T}}$ satisfy the spectral problem and the adjoint spectral problem*

$$\phi_x = U(u,\lambda)\phi, \quad \psi_x = -U^{\mathrm{St}}(u,\lambda)\psi,$$

where $\phi_{\mathrm{e}} = (\phi_1, \cdots, \phi_m)$ and $\psi_{\mathrm{e}} = (\psi_1, \cdots, \psi_m)$ are even eigenfunctions, and $\phi_{\mathrm{o}} = (\phi_{m+1}, \cdots, \phi_{m+n})$ and $\psi_{\mathrm{o}} = (\psi_{m+1}, \cdots, \psi_{m+n})$ are odd eigenfunctions. Then, the variational derivative of the spectral parameter λ with respect to the potential u is given by

$$\frac{\delta\lambda}{\delta u} = \frac{(\psi_{\mathrm{e}}, (-1)^{p(u)}\psi_{\mathrm{o}})(\frac{\partial U}{\partial u})\phi}{-\int \psi^{\mathrm{T}}(\frac{\partial U}{\partial \lambda})\phi \,\mathrm{d}x}, \tag{3.2}$$

where we denote

$$p(v) = \begin{cases} 0, & v \text{ is an even variable}, \\ 1, & v \text{ is an odd variable}. \end{cases} \tag{3.3}$$

By Lemma 3.1, it is not difficult to find that

$$\frac{\delta\lambda}{\delta u} = \begin{pmatrix} \psi_1\phi_1 - \psi_2\phi_2 \\ \psi_1\phi_2 + \psi_2\phi_1 \\ \psi_1\phi_3 + \psi_3\phi_2 \\ \psi_2\phi_3 - \psi_3\phi_1 \end{pmatrix}. \tag{3.4}$$

When zero boundary conditions $\lim\limits_{|x|\to\infty}\phi = \lim\limits_{|x|\to\infty}\psi = 0$ are imposed, we can obtain a characteristic property — a recurrence relation for the variational derivative of λ:

$$\mathcal{L}\frac{\delta\lambda}{\delta u} = \lambda\frac{\delta\lambda}{\delta u}, \tag{3.5}$$

where $\mathcal{L}$ and $\frac{\delta\lambda}{\delta u}$ are given by (2.5) and (3.4), respectively.

Let us now discuss the two spatial and temporal systems:

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(u,\lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = \begin{pmatrix} r & \lambda_j + s & \alpha \\ -\lambda_j + s & -r & \beta \\ \beta & -\alpha & 0 \end{pmatrix}\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_x = -U^{\mathrm{St}}(u,\lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix} = \begin{pmatrix} -r & \lambda_j - s & \beta \\ -\lambda_j - s & r & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix}\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix} \end{cases} \tag{3.6}$$

and

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = V^{(n)}(u,\lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} \\ \qquad = \begin{pmatrix} \sum\limits_{i=0}^{n} C_i\lambda_j^{n-i} & \sum\limits_{i=0}^{n} (A_i+B_i)\lambda_j^{n-i} & \sum\limits_{i=0}^{n} \rho_i\lambda_j^{n-i} \\ \sum\limits_{i=0}^{n} (A_i-B_i)\lambda_j^{n-i} & -\sum\limits_{i=0}^{n} C_i\lambda_j^{n-i} & \sum\limits_{i=0}^{n} \delta_i\lambda_j^{n-i} \\ \sum\limits_{i=0}^{n} \delta_i\lambda_j^{n-i} & -\sum\limits_{i=0}^{n} \rho_i\lambda_j^{n-i} & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_{t_n} = -(V^{(n)})^{\mathrm{St}}(u,\lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix} \\ \qquad = \begin{pmatrix} -\sum\limits_{i=0}^{n} C_i\lambda_j^{n-i} & -\sum\limits_{i=0}^{n} (A_i-B_i)\lambda_j^{n-i} & \sum\limits_{i=0}^{n} \delta_i\lambda_j^{n-i} \\ -\sum\limits_{i=0}^{n} (A_i+B_i)\lambda_j^{n-i} & \sum\limits_{i=0}^{n} C_i\lambda_j^{n-i} & -\sum\limits_{i=0}^{n} \rho_i\lambda_j^{n-i} \\ -\sum\limits_{i=0}^{n} \rho_i\lambda_j^{n-i} & -\sum\limits_{i=0}^{n} \delta_i\lambda_j^{n-i} & 0 \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \end{cases} \tag{3.7}$$

where $1 \le j \le N$ and $\lambda_1, \cdots, \lambda_N$ are N distinct spectral parameters. Now for the systems (3.6) and (3.7), we have the following symmetry constraints:

$$\frac{\delta}{\delta u} H_k = \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u}, \quad k \ge 0. \tag{3.8}$$

The symmetry constraint in the case of $k = 0$ is called a Bargmann symmetry constraint (see [11]). It leads to an explicit expression for the potential u, i.e.,

$$\begin{cases} r = \langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle, \\ s = \langle \Psi_1, \Phi_2 \rangle + \langle \Psi_2, \Phi_1 \rangle, \\ \alpha = -\langle \Psi_2, \Phi_3 \rangle + \langle \Psi_3, \Phi_1 \rangle, \\ \beta = \langle \Psi_1, \Phi_3 \rangle + \langle \Psi_3, \Phi_2 \rangle, \end{cases} \tag{3.9}$$

where we use the following notation:

$$\Phi_i = (\phi_{i1}, \cdots, \phi_{iN})^{\mathrm{T}}, \quad \Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^{\mathrm{T}}, \quad i = 1, 2, 3,$$

and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of the Euclidian space R^N.

4 Binary Nonlinearization

In this section, we want to perform binary nonlinearization for the Lax pairs and adjoint Lax pairs of the super Dirac hierarchy (2.11). To this end, let us substitute (3.9) into the Lax pairs and adjoint Lax pairs (3.6) and (3.7), and then we obtain the following nonlinearized Lax pairs and adjoint Lax pairs:

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(\widetilde{u}, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = \begin{pmatrix} \widetilde{r} & \lambda_j + \widetilde{s} & \widetilde{\alpha} \\ -\lambda_j + \widetilde{s} & -\widetilde{r} & \widetilde{\beta} \\ \widetilde{\beta} & -\widetilde{\alpha} & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_x = -U^{\mathrm{St}}(\widetilde{u}, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix} = \begin{pmatrix} -\widetilde{r} & \lambda_j - \widetilde{s} & \widetilde{\beta} \\ -\lambda_j - \widetilde{s} & \widetilde{r} & -\widetilde{\alpha} \\ -\widetilde{\alpha} & -\widetilde{\beta} & 0 \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix} \end{cases} \tag{4.1}$$

and

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_{t_n} = V^{(n)}(\widetilde{u}, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} \\ \qquad = \begin{pmatrix} \sum\limits_{i=0}^{n} \widetilde{C}_i \lambda_j^{n-i} & \sum\limits_{i=0}^{n} (\widetilde{A}_i + \widetilde{B}_i) \lambda_j^{n-i} & \sum\limits_{i=0}^{n} \widetilde{\rho}_i \lambda_j^{n-i} \\ \sum\limits_{i=0}^{n} (\widetilde{A}_i - \widetilde{B}_i) \lambda_j^{n-i} & -\sum\limits_{i=0}^{n} \widetilde{C}_i \lambda_j^{n-i} & \sum\limits_{i=0}^{n} \widetilde{\delta}_i \lambda_j^{n-i} \\ \sum\limits_{i=0}^{n} \widetilde{\delta}_i \lambda_j^{n-i} & -\sum\limits_{i=0}^{n} \widetilde{\rho}_i \lambda_j^{n-i} & 0 \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_{t_n} = -(V^{(n)})^{\mathrm{St}}(\widetilde{u}, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix} \\ \qquad = \begin{pmatrix} -\sum\limits_{i=0}^{n} \widetilde{C}_i \lambda_j^{n-i} & -\sum\limits_{i=0}^{n} (\widetilde{A}_i - \widetilde{B}_i) \lambda_j^{n-i} & \sum\limits_{i=0}^{n} \widetilde{\delta}_i \lambda_j^{n-i} \\ -\sum\limits_{i=0}^{n} (\widetilde{A}_i + \widetilde{B}_i) \lambda_j^{n-i} & \sum\limits_{i=0}^{n} \widetilde{C}_i \lambda_j^{n-i} & -\sum\limits_{i=0}^{n} \widetilde{\rho}_i \lambda_j^{n-i} \\ -\sum\limits_{i=0}^{n} \widetilde{\rho}_i \lambda_j^{n-i} & -\sum\limits_{i=0}^{n} \widetilde{\delta}_i \lambda_j^{n-i} & 0 \end{pmatrix} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \end{cases} \tag{4.2}$$

where $1 \leq j \leq N$ and $\widetilde{P}$ means an expression of $P(u)$ under the explicit constraint (3.9). Note that the spatial part of the nonlinearized system (4.1) is a system of ordinary differential equations with an independent variable x, but for a given n ($n \geq 2$), the t_n-part of the nonlinearized system (4.2) is a system of ordinary differential equations. Obviously, the system (4.1) can be written as

$$\begin{cases} \Phi_{1,x} = (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)\Phi_1 + (\Lambda + \langle \Psi_1, \Phi_2 \rangle + \langle \Psi_2, \Phi_1 \rangle)\Phi_2 \\ \qquad + (-\langle \Psi_2, \Phi_3 \rangle + \langle \Psi_3, \Phi_1 \rangle)\Phi_3, \\ \Phi_{2,x} = (-\Lambda + \langle \Psi_1, \Phi_2 \rangle + \langle \Psi_2, \Phi_1 \rangle)\Phi_1 - (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)\Phi_2 \\ \qquad + (\langle \Psi_1, \Phi_3 \rangle + \langle \Psi_3, \Phi_2 \rangle)\Phi_3, \\ \Phi_{3,x} = (\langle \Psi_1, \Phi_3 \rangle + \langle \Psi_3, \Phi_2 \rangle)\Phi_1 - (-\langle \Psi_2, \Phi_3 \rangle + \langle \Psi_3, \Phi_1 \rangle)\Phi_2, \\ \Psi_{1,x} = -(\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)\Psi_1 + (\Lambda - \langle \Psi_1, \Phi_2 \rangle - \langle \Psi_2, \Phi_1 \rangle)\Psi_1 \\ \qquad + (\langle \Psi_1, \Phi_3 \rangle + \langle \Psi_3, \Phi_2 \rangle)\Psi_3, \\ \Psi_{2,x} = -(\Lambda + \langle \Psi_1, \Phi_2 \rangle + \langle \Psi_2, \Phi_1 \rangle)\Psi_1 + (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle)\Psi_2 \\ \qquad - (-\langle \Psi_2, \Phi_3 \rangle + \langle \Psi_3, \Phi_1 \rangle)\Psi_3, \\ \Psi_{3,x} = -(-\langle \Psi_2, \Phi_3 \rangle + \langle \Psi_3, \Phi_1 \rangle)\Psi_1 - (\langle \Psi_1, \Phi_3 \rangle + \langle \Psi_3, \Phi_2 \rangle)\Psi_2, \end{cases} \tag{4.3}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. When $n = 1$, the system (4.2) is exactly the system (4.1) with $t_1 = x$. When $n = 2$, the system (4.2) is

$$\begin{cases} \Phi_{1,t_2} = \Big(\widetilde{r}\Lambda - \frac{1}{2}\widetilde{s}_x\Big)\Phi_1 + \Big(\Lambda^2 + \widetilde{s}\Lambda + \frac{1}{2}\widetilde{r}_x + \frac{1}{2}(\widetilde{r}^2 + \widetilde{s}^2) + \widetilde{\alpha}\widetilde{\beta}\Big)\Phi_2 + (\widetilde{\alpha}\Lambda - \widetilde{\beta}_x)\Phi_3, \\ \Phi_{2,t_2} = \Big(-\Lambda^2 + \widetilde{s}\Lambda + \frac{1}{2}\widetilde{r}_x - \frac{1}{2}(\widetilde{r}^2 + \widetilde{s}^2) - \widetilde{\alpha}\widetilde{\beta}\Big)\Phi_1 - \Big(\widetilde{r}\Lambda - \frac{1}{2}\widetilde{s}_x\Big)\Phi_2 \\ \qquad + (\widetilde{\beta}\Lambda + \widetilde{\alpha}_x)\Phi_3, \\ \Phi_{3,t_2} = (\widetilde{\beta}\Lambda + \widetilde{\alpha}_x)\Phi_1 - (\widetilde{\alpha}\Lambda - \widetilde{\beta}_x)\Phi_2, \\ \Psi_{1,t_2} = -\Big(\widetilde{r}\Lambda - \frac{1}{2}\widetilde{s}_x\Big)\Psi_1 + \Big(\Lambda^2 - \widetilde{s}\Lambda - \frac{1}{2}\widetilde{r}_x + \frac{1}{2}(\widetilde{r}^2 + \widetilde{s}^2) + \widetilde{\alpha}\widetilde{\beta}\Big)\Psi_2 \\ \qquad + (\widetilde{\beta}\Lambda + \widetilde{\alpha}_x)\Psi_3, \\ \Psi_{2,t_2} = -\Big(\Lambda^2 + \widetilde{s}\Lambda + \frac{1}{2}\widetilde{r}_x + \frac{1}{2}(\widetilde{r}^2 + \widetilde{s}^2) + \widetilde{\alpha}\widetilde{\beta}\Big)\Psi_1 + \Big(\widetilde{r}\Lambda - \frac{1}{2}\widetilde{s}_x\Big)\Psi_2 \\ \qquad - (\widetilde{\alpha}\Lambda - \widetilde{\beta}_x)\Psi_3, \\ \Psi_{3,t_2} = -(\widetilde{\alpha}\Lambda - \widetilde{\beta}_x)\Psi_1 - (\widetilde{\beta}\Lambda + \widetilde{\alpha}_x)\Psi_2, \end{cases} \tag{4.4}$$

where $\widetilde{r}, \widetilde{s}, \widetilde{\alpha}, \widetilde{\beta}$ denote the functions r, s, α, β defined by the explicit constraint (3.9), and $\widetilde{r}_x, \widetilde{s}_x, \widetilde{\alpha}_x, \widetilde{\beta}_x$ are given by

$$\begin{cases} \widetilde{r}_x = 2\langle\Lambda\Psi_1, \Phi_2\rangle + 2\langle\Lambda\Psi_2, \Phi_1\rangle + 2\langle\Psi_1, \Phi_2\rangle^2 - 2\langle\Psi_2, \Phi_1\rangle^2, \\ \widetilde{s}_x = -2\langle\Lambda\Psi_1, \Phi_1\rangle + 2\langle\Lambda\Psi_2, \Phi_2\rangle + 2(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle)(\langle\Psi_2, \Phi_1\rangle - \langle\Psi_1, \Phi_2\rangle), \\ \widetilde{\alpha}_x = \langle\Lambda\Psi_1, \Phi_3\rangle + \langle\Lambda\Psi_3, \Phi_2\rangle + (\langle\Psi_1, \Phi_2\rangle - \langle\Psi_2, \Phi_1\rangle)(\langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle), \\ \widetilde{\beta}_x = \langle\Lambda\Psi_2, \Phi_3\rangle - \langle\Lambda\Psi_3, \Phi_1\rangle - (\langle\Psi_1, \Phi_2\rangle - \langle\Psi_2, \Phi_1\rangle)(-\langle\Psi_2, \Phi_3\rangle + \langle\Psi_3, \Phi_1\rangle), \end{cases}$$

which are computed through using the spatial constrained flow (4.3).

In what follows, we want to prove that system (4.1) is a completely integrable Hamiltonian system in the Liouville sense. Furthermore, we shall prove that system (4.2) is also completely integrable under the control of system (4.1).

On the one hand, the system (4.1) or (4.3) can be represented as the following super Hamiltonian form:

$$\begin{cases} \Phi_{1,x} = \dfrac{\partial H_1}{\partial \Psi_1}, & \Phi_{2,x} = \dfrac{\partial H_1}{\partial \Psi_2}, & \Phi_{3,x} = \dfrac{\partial H_1}{\partial \Psi_3}, \\ \Psi_{1,x} = -\dfrac{\partial H_1}{\partial \Phi_1}, & \Psi_{2,x} = -\dfrac{\partial H_1}{\partial \Phi_2}, & \Psi_{3,x} = \dfrac{\partial H_1}{\partial \Phi_3}, \end{cases} \tag{4.5}$$

where

$$H_1 = \langle\Lambda\Psi_1, \Phi_2\rangle - \langle\Lambda\Psi_2, \Phi_1\rangle + \frac{1}{2}(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle)^2 + \frac{1}{2}(\langle\Psi_1, \Phi_2\rangle + \langle\Psi_2, \Phi_1\rangle)^2 \\ + (-\langle\Psi_2, \Phi_3\rangle + \langle\Psi_3, \Phi_1\rangle)(\langle\Psi_1, \Phi_3\rangle + \langle\Psi_3, \Phi_2\rangle).$$

In addition, the characteristic property (3.5) and the recurrence relations (2.5) ensure that

$$\begin{cases} \widetilde{A}_{i+1} = \langle\Lambda^i\Psi_1, \Phi_2\rangle + \langle\Lambda^i\Psi_2, \Phi_1\rangle, & i \geq 0, \\ \widetilde{B}_{i+1} = \langle\Lambda^i\Psi_2, \Phi_1\rangle - \langle\Lambda^i\Psi_1, \Phi_2\rangle, & i \geq 0, \\ \widetilde{C}_{i+1} = \langle\Lambda^i\Psi_1, \Phi_1\rangle - \langle\Lambda^i\Psi_2, \Phi_2\rangle, & i \geq 0, \\ \widetilde{\delta}_{i+1} = \langle\Lambda^i\Psi_1, \Phi_3\rangle + \langle\Lambda^i\Psi_3, \Phi_2\rangle, & i \geq 0, \\ \widetilde{\rho}_{i+1} = -\langle\Lambda^i\Psi_2, \Phi_3\rangle + \langle\Lambda^i\Psi_3, \Phi_1\rangle, & i \geq 0. \end{cases} \tag{4.6}$$

Then the co-adjoint representation equation $\widetilde{V}_x = [\widetilde{U}, \widetilde{V}]$ remains true. Furthermore, we know that the equality $\widetilde{V}_x^2 = [\widetilde{U}, \widetilde{V}^2]$ is also true. Let

$$F = \frac{1}{4}\,\mathrm{Str}\,\widetilde{V}^2. \tag{4.7}$$

Then it is easy to find that $F_x = 0$. That is to say, F is a generating function of integrals of motion for the system (4.1) or (4.3). Due to $F = \sum_{n\geq 0} F_n\lambda^{-n}$, we obtain the following formulas of integrals of motion:

$$\begin{aligned} &F_0 = -\frac{1}{2}\widetilde{B}_0^2, \quad F_1 = -\widetilde{B}_0\widetilde{B}_1, \\ &F_n = -\widetilde{B}_0\widetilde{B}_n + \frac{1}{2}\sum_{i=1}^{n-1}(\widetilde{A}_i\widetilde{A}_{n-i} - \widetilde{B}_i\widetilde{B}_{n-i} + \widetilde{C}_i\widetilde{C}_{n-i} + 2\widetilde{\rho}_i\widetilde{\delta}_{n-i}), \quad n\geq 2. \end{aligned} \tag{4.8}$$

Substituting (4.6) into the above formulas of integrals of motion, we obtain the following expressions of F_m $(m\geq 0)$:

$$\begin{aligned} F_0 &= -\frac{1}{2}, \quad F_1 = \langle\Psi_1,\Phi_2\rangle - \langle\Psi_2,\Phi_1\rangle, \\ F_n &= \langle\Lambda^{n-1}\Psi_1,\Phi_2\rangle - \langle\Lambda^{n-1}\Psi_2,\Phi_1\rangle + \sum_{i=1}^{n-1}\Big[2(\langle\Lambda^{i-1}\Psi_1,\Phi_2\rangle\langle\Lambda^{n-i-1}\Psi_2,\Phi_1\rangle) \\ &\quad + \frac{1}{2}(\langle\Lambda^{i-1}\Psi_1,\Phi_1\rangle - \langle\Lambda^{i-1}\Psi_2,\Phi_2\rangle)(\langle\Lambda^{n-i-1}\Psi_1,\Phi_1\rangle - \langle\Lambda^{n-i-1}\Psi_2,\Phi_2\rangle) \\ &\quad + (-\langle\Lambda^{i-1}\Psi_2,\Phi_3\rangle + \langle\Lambda^{i-1}\Psi_3,\Phi_1\rangle)(\langle\Lambda^{n-i-1}\Psi_1,\Phi_3\rangle + \langle\Lambda^{n-i-1}\Psi_3,\Phi_2\rangle)\Big], \quad n\geq 2. \end{aligned} \tag{4.9}$$

On the other hand, let us consider the temporal part of the nonlinearized system (4.2). Making use of (4.6) and (4.9), the system (4.2) can be represented as the following super Hamiltonian form:

$$\begin{cases} \Phi_{1,t_n} = \dfrac{\partial F_{n+1}}{\partial\Psi_1}, & \Phi_{2,t_n} = \dfrac{\partial F_{n+1}}{\partial\Psi_2}, \quad \Phi_{3,t_n} = \dfrac{\partial F_{n+1}}{\partial\Psi_3}, \\ \Psi_{1,t_n} = -\dfrac{\partial F_{n+1}}{\partial\Phi_1}, & \Psi_{2,t_n} = -\dfrac{\partial F_{n+1}}{\partial\Phi_2}, \quad \Psi_{3,t_n} = \dfrac{\partial F_{n+1}}{\partial\Phi_3}. \end{cases} \tag{4.10}$$

This can be checked pretty easily. For example, we can show the last but one equality in the above system as follows:

$$\begin{aligned} \Psi_{2,t_n} &= -\sum_{i=0}^{n}(\widetilde{A}_i + \widetilde{B}_i)\Lambda^{n-i}\Psi_1 + \sum_{i=0}^{n}\widetilde{C}_i\Lambda^{n-i}\Psi_2 - \sum_{i=0}^{n}\widetilde{\rho}_i\Lambda^{n-i}\Psi_3 \\ &= -\Lambda^n\Psi_1 - 2\sum_{i=1}^{n}\langle\Lambda^{i-1}\Psi_2,\Phi_1\rangle\Lambda^{n-i}\Psi_1 + \sum_{i=1}^{n}(\langle\Lambda^{i-1}\Psi_1,\Phi_1\rangle - \langle\Lambda^{i-1}\Psi_2,\Phi_2\rangle)\Lambda^{n-i}\Psi_2 \\ &\quad + \sum_{i=1}^{n}(\langle\Lambda^{i-1}\Psi_2,\Phi_3\rangle - \langle\Lambda^{i-1}\Psi_3,\Phi_1\rangle)\Lambda^{n-i}\Psi_3 \\ &= -\frac{\partial F_{n+1}}{\partial\Phi_2}. \end{aligned}$$

In order to further show the Liouville integrability for the constrained flows (4.1) and (4.2), we need to prove the commutative property of the integrals of motion $\{F_m\}_{m\geq 0}$, under the corresponding Poisson bracket:

$$\{F, G\} = \sum_{i=1}^{3}\sum_{j=1}^{N}\Big(\frac{\partial F}{\partial \phi_{ij}}\frac{\partial G}{\partial \psi_{ij}} - (-1)^{p(\phi_{ij})p(\psi_{ij})}\frac{\partial F}{\partial \psi_{ij}}\frac{\partial G}{\partial \phi_{ij}}\Big). \tag{4.11}$$

At this time, we still have an equality $\widetilde{V}_{t_n} = [\widetilde{V}^{(n)}, \widetilde{V}]$, and after a similar discussion, we know that F is also a generating function of integrals of motion for (4.2). Hence F_m $(m \geq 0)$ are integrals of motion for the system (4.2) or (4.10), which implies

$$\{F_{m+1}, F_{n+1}\} = \frac{\partial}{\partial t_n}F_{m+1} = 0, \quad m, n \geq 0. \tag{4.12}$$

The above equality (4.12) shows that $\{F_m\}_{m\geq 0}$ are in involution in pair under the Poisson bracket (4.11).

In addition, similarly to [22], we know that

$$f_k = \psi_{1k}\phi_{1k} + \psi_{2k}\phi_{2k} + \psi_{3k}\phi_{3k}, \quad 1 \leq k \leq N \tag{4.13}$$

are integrals of motion for (4.1) and (4.2). It is not difficult to verify that the $3N$ functions $\{f_k\}_{k=1}^{N}$ and $\{F_m\}_{m=1}^{2N}$ are in involution in pair. To show the functional independence of the $3N$ functions $\{f_k\}_{k=1}^{N}$ and $\{F_m\}_{m=1}^{2N}$, we can use, as in [19], the technique developed by Ma et al [22, 23]. Therefore, the $3N$ functions $\{f_k\}_{k=1}^{N}$ and $\{F_m\}_{m=1}^{2N}$ are functionally independent over some region of the supersymmetry manifold $R^{4N|2N}$. Now, all of the above analysis gives the following theorem.

Theorem 4.1 *Both the spatial and temporal constrained flows* (4.1) *and* (4.2) *are Liouville integrable super Hamiltonian systems defined on the supersymmetry manifold* $R^{4N|2N}$*, which possess* $3N$ *functionally independent and involutive integrals of motion* $\{f_k\}_{k=1}^{N}$ *and* $\{F_m\}_{m=1}^{2N}$ *defined by* (4.13) *and* (4.9)*. Moreover, formula* (3.9) *provides a Bäcklund transform from the constrained flows* (4.1) *and* (4.2) *to the Dirac systems* (2.11)*.*

Remark 4.1 The super-system on supermanifolds $R^{2M|2N}$ is Liouville integrable (see [24]) if it possesses M even valued conserved quantities and N odd valued conserved quantities that are independent and are also in involution. Furthermore, similar to the classical case, there exists a super analogue of Liouville's theorem (see [24, 25]). Let θ be an odd variable in superspace. Then $\widetilde{f}_k = \theta f_k$ $(k = 1, 2, \cdots, N)$ are N odd valued conserved quantities for super-finite dimensional in Theorem 4.1 because this system is involved only with even flow t_n.

5 Conclusions and Remarks

In this paper, we have applied the binary nonlinearization method to the super Dirac systems by the Bargmann symmetry constraint (3.9). We have also shown in Theorem 4.1 that the nonlinearized systems (4.1) and (4.2) are two super finite-dimensional integrable Hamiltonian systems, whose super Hamiltonian forms and integrals of motion have been presented explicitly. We would also like to emphasize that the new formula (3.2) is a general result for calculating the variational derivative of the spectral parameter λ with respect to the potential u. The crucial difference between the nonlinearziation processes of the super AKNS system and the super Dirac system is due to the variational derivatives of λ defined by formula (21) in reference [19] and formula (3.4) in this paper.

We remark that the super Dirac systems (2.8) or (2.11), e.g., (2.12), only possess super (odd and even) independent variables. The fully supersymmetric Dirac systems possessing both super dependent variables and super independent variables seem to be a very interesting object for our future research. For more detailed discussions on the supersymmetry theory and supersymmetric analysis, we would like to refer readers to [26].

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