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N-fold Darboux Transformation for Integrable Couplings of AKNS Equations*

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Abstract *For the integrable couplings of Ablowitz-Kaup-Newell-Segur (ICAKNS) equations, N-fold Darboux transformation (DT) T_N, which is a 4×4 matrix, is constructed in this paper. Each element of this matrix is expressed by a ratio of the $(4N+1)$-order determinant and $4N$-order determinant of eigenfunctions. By making use of these formulae, the determinant expressions of N-transformed new solutions $p^{[N]}$, $q^{[N]}$, $r^{[N]}$ and $s^{[N]}$ are generated by this N-fold DT. Furthermore, when the reduced conditions $q=-p^*$ and $s=-r^*$ are chosen, we obtain determinant representations of N-fold DT and N-transformed solutions for the integrable couplings of nonlinear Schrödinger (ICNLS) equations. Starting from the zero seed solutions, one-soliton solutions are explicitly given as an example.*

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1 Introduction

Integrable couplings of soliton equations have been attracted much attention over the last few decades. Generally speaking, for a given integrable system of evolution equations $u_t=K(u)$, a new system consisting of the original system and its linearized system

$$u_t=K(u),\quad v_t=K'(u)[v]\,, \tag{1}$$

is still integrable. Here $K'(u)[v]$ denotes the Gateaux derivative of $K(u)\equiv K(u,D_xu,\ldots)$ with respect to u in a direction v, i.e.

$$K'(u)[v]=\frac{\partial}{\partial\varepsilon}K(u+\varepsilon v)|_{\varepsilon=0}\,.$$

The second part $v_t=K'(u)[v]$ in the above new system (1) is a special integrable couplings of the original system $u_t=K(u)$. So far, the method for constructing integrable couplings of soliton equations mainly included perturbations,[1−2] enlarging spectral problems,[3−6] creating new loop algebras,[7−8] and multi-integrable couplings.[9−11] The study of integrable couplings of soliton equations brought two major benefits. One is to generalize the symmetry problem,[12−13] the other is to provide clues towards complete classification of integrable systems.

It is well-known that DT[14−19] is always regarded as one of the most effective method to construct solutions of integrable equations. The main procedures are listed as follows. Suppose the integrable equations are associated with the following spectral problem

$$\psi_x=U\psi,\quad \psi_t=V\psi\,, \tag{2}$$

which is transformed into

$$\psi^{[1]}_x=U^{[1]}\psi^{[1]},\quad \psi^{[1]}_t=V^{[1]}\psi^{[1]}\,, \tag{3}$$

under a gauge transformation

$$\psi^{[1]}=T\psi\,. \tag{4}$$

By Refs. [19–20] , we know that T in Eq. (4) must satisfy following conditions

$$T_x+TU=U^{[1]}T,\quad T_t+TV=V^{[1]}T\,, \tag{5}$$

according to Eq. (3). It is crucial for us to search for T in Eq. (4) so that $U^{[1]}$ and $V^{[1]}$ have the same forms as U and V. Repeating N times, N-fold DT can be constructed. In 2006, the authors proposed determinant representation of DT for the integrable equations. Without iterating, N-fold DT and N-transformed solutions are derived for many integrable equations, such as AKNS equation,[21] derivative nonlinear Schrödinger

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(DNLS) equation,[22] Gerdjikov-Ivanov (GI) equation,[23] Chen-Lee-Liu (CLL) equation,[24] and so on.[25–28] Accordingly, many important solutions of these equations are constructed in these references. For example, soliton solutions, rogue wave solutions, breather solutions, and so forth. To this day, no researchers considered determinant representations of the N-fold DT and N-transformed new solutions for the integrable couplings of soliton equations. So we will solve this question in this paper.

The paper is organized as follows. In the next section, we recall the construction of ICAKNS equations. Then in Sec. 3, determinant representations of N-fold DT and N-transformed solutions are constructed for the ICAKNS equations. And in Sec. 4, under the constraint condition $q=-p^*$ and $s=-r^*$, we obtain ones of ICNLS equations. Some conclusions and discussions are listed in the last section.

2 Integrable Couplings of the AKNS Equations

In this section, we will briefly recall the construction of ICAKNS equations. Firstly, we know that the ICAKNS system is associated with the following the spatial spectral problem

$$\phi_x = U\phi, \quad U=\begin{pmatrix} U_0 & U_1 \\ 0 & U_0 \end{pmatrix},$$
$$U_0=\begin{pmatrix} -\mathrm{i}\lambda & p \\ q & \mathrm{i}\lambda \end{pmatrix}, \quad U_1=\begin{pmatrix} -\mathrm{i}\lambda & r \\ s & \mathrm{i}\lambda \end{pmatrix}, \tag{6}$$

where λ is a spectral parameter, p,q,r and s are potentials, $\phi=(\phi_1,\phi_2,\phi_3,\phi_4)^{\mathrm{T}}$ is an eigenfunction. Solving the stationary equation

$$V_x=[U,V], \tag{7}$$

where

$$V=\begin{pmatrix} V_0 & V_1 \\ 0 & V_0 \end{pmatrix}, \quad V_0=\begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$
$$V_1=\begin{pmatrix} E & F \\ G & -E \end{pmatrix},$$

we have

$$\begin{aligned} &A_x=pC-qB, \quad B_x=-2\mathrm{i}\lambda B-2pA, \\ &C_x=2\mathrm{i}\lambda C+2qA, \quad E_x=pG-qF+rC-sB, \\ &F_x=-2\mathrm{i}\lambda F-2\mathrm{i}\lambda B-2pE-2rA, \\ &G_x=2\mathrm{i}\lambda G+2\mathrm{i}\lambda C+2qE+2sA. \end{aligned} \tag{8}$$

Taking

$$A=\sum_{j\ge 0}A_j\lambda^{-j}, \quad B=\sum_{j\ge 0}B_j\lambda^{-j}, \quad C=\sum_{j\ge 0}C_j\lambda^{-j},$$
$$E=\sum_{j\ge 0}E_j\lambda^{-j}, \quad F=\sum_{j\ge 0}F_j\lambda^{-j}, \quad G=\sum_{j\ge 0}G_j\lambda^{-j},$$

and comparing the coefficients of the same power of λ, we have

$$B_0=C_0=F_0=G_0=0,$$

$$\begin{aligned} &A_{j,x}=pC_j-qB_j, \\ &B_{j,x}=-2\mathrm{i}B_{j+1}-2pA_j, \\ &C_{j,x}=2\mathrm{i}C_{j+1}+2qA_j, \\ &E_{j,x}=pG_j-qF_j+rC_j-sB_j, \\ &F_{j,x}=-2\mathrm{i}F_{j+1}-2\mathrm{i}B_{j+1}-2pE_j-2rA_j, \\ &G_{j,x}=2\mathrm{i}G_{j+1}+2\mathrm{i}C_{j+1}+2qE_j+2sA_j, \end{aligned} \tag{9}$$

where $j\ge 0$. It is easy to find that $A_{0,x}=E_{0,x}=0$. Choosing $A_0=E_0=-\mathrm{i}$, and taking constants of integral to be zero, we obtain all terms A_j,B_j,C_j,E_j,F_j,G_j $(j\ge 1)$. The first three terms are listed as follows:

$$\begin{aligned} &A_1=E_1=0, \quad B_1=p, \quad C_1=q, \quad F_1=r, \quad G_1=s, \\ &A_2=-\frac{1}{2}\mathrm{i}pq, \quad B_2=\frac{1}{2}\mathrm{i}p_x, \quad C_2=-\frac{1}{2}\mathrm{i}q_x, \\ &E_2=\frac{1}{2}\mathrm{i}(pq-ps-qr), \quad F_2=\frac{1}{2}\mathrm{i}(r_x-p_x), \\ &G_2=\frac{1}{2}\mathrm{i}(q_x-s_x), \quad A_3=\frac{1}{4}(p_xq-pq_x), \\ &B_3=-\frac{1}{4}p_{xx}+\frac{1}{2}p^2q, \quad C_3=-\frac{1}{4}q_{xx}+\frac{1}{2}pq^2, \\ &E_3=\frac{1}{4}(p_xs-ps_x+2pq_x-2p_xq+qr_x-q_xr), \\ &F_3=-\frac{1}{4}r_{xx}+\frac{1}{2}p_{xx}+\frac{1}{2}p^2s-p^2q+pqr, \\ &G_3=-\frac{1}{4}s_{xx}+\frac{1}{2}q_{xx}+\frac{1}{2}q^2r-pq^2+pqs. \end{aligned}$$

Secondly, let us introduce the temporal parts of the spectral problem (6)

$$\phi_{t_n}=V^{(n)}\phi, \quad V^{(n)}=(\lambda^nV)_+, \quad n\ge 0, \tag{10}$$

where "+" means non-negative power. The compatible condition of (6) and (10) yields to

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0, \quad n\ge 0. \tag{11}$$

Substituting U and $V^{(n)}$ into Eq. (11), we have

$$\begin{aligned} &p_{t_n}=-2\mathrm{i}B_{n+1}, \quad q_{t_n}=2\mathrm{i}C_{n+1}, \\ &r_{t_n}=-2\mathrm{i}F_{n+1}-2\mathrm{i}B_{n+1}, \\ &s_{t_n}=2\mathrm{i}G_{n+1}+2\mathrm{i}C_{n+1}. \end{aligned} \tag{12}$$

Lastly, when $n=2$ in Eq. (12), we obtain the ICAKNS equations

$$\begin{aligned} &p_{t_2}=\frac{1}{2}\mathrm{i}p_{xx}-\mathrm{i}p^2q, \quad q_{t_2}=-\frac{1}{2}\mathrm{i}q_{xx}+\mathrm{i}pq^2, \\ &r_{t_2}=\frac{1}{2}\mathrm{i}r_{xx}-\frac{1}{2}\mathrm{i}p_{xx}+\mathrm{i}p^2q-\mathrm{i}p^2s-2\mathrm{i}pqr, \\ &s_{t_2}=-\frac{1}{2}\mathrm{i}s_{xx}+\frac{1}{2}\mathrm{i}q_{xx}-\mathrm{i}pq^2+\mathrm{i}q^2r+2\mathrm{i}pqs, \end{aligned} \tag{13}$$

whose Lax pairs are given by U and $V^{(2)}$ as follows:

$$V^{(2)}=\begin{pmatrix} V_0^{(2)} & V_1^{(2)} \\ 0 & V_0^{(2)} \end{pmatrix},$$

with

$$V_0^{(2)} = \begin{pmatrix} -\mathrm{i}\lambda^2 - \frac{1}{2}\mathrm{i}pq & p\lambda + \frac{1}{2}\mathrm{i}p_x \\ q\lambda - \frac{1}{2}\mathrm{i}q_x & \mathrm{i}\lambda^2 + \frac{1}{2}\mathrm{i}pq \end{pmatrix},$$
$$V_1^{(2)} = \begin{pmatrix} -\mathrm{i}\lambda^2 + \frac{1}{2}\mathrm{i}(pq - ps - qr) & r\lambda + \frac{1}{2}\mathrm{i}(r_x - p_x) \\ s\lambda + \frac{1}{2}\mathrm{i}(q_x - s_x) & \mathrm{i}\lambda^2 - \frac{1}{2}\mathrm{i}(pq - ps - qr) \end{pmatrix}.$$

3 DT of ICAKNS Equations (13)

In this section, we will derive determinant representation of DT for the ICAKNS equations (13). To this end, we firstly suppose the Darboux matrix is

$$T = \begin{pmatrix} T_0 & T_1 \\ 0 & T_0 \end{pmatrix}. \tag{14}$$

The spectral problem $\phi_x = U\phi, \phi_{t_2} = V^{(2)}\phi$ is transformed into $\phi_x^{[1]} = U^{[1]}\phi^{[1]}$, $\phi_{t_2}^{[1]} = (V^{(2)})^{[1]}\phi^{[1]}$ under the DT (14). Here $U^{[1]}$, $(V^{(2)})^{[1]}$ share the same forms as U, $V^{(2)}$. After a direct calculation, we have

$$\begin{aligned} &T_x + TU = U^{[1]}T, \\ &T_{t_2} + TV^{(2)} = (V^{(2)})^{[1]}T, \end{aligned} \tag{15}$$

or

$$\begin{aligned} &T_{0,x} + T_0U_0 = U_0^{[1]}T_0, \\ &T_{1,x} + T_0U_1 + T_1U_0 = U_0^{[1]}T_1 + U_1^{[1]}T_0, \\ &T_{0,t_2} + T_0V_0^{(2)} = (V_0^{(2)})^{[1]}T_0, \\ &T_{1,t_2} + T_0V_1^{(2)} + T_1V_0^{(2)} \\ &\quad = (V_0^{(2)})^{[1]}T_1 + (V_1^{(2)})^{[1]}T_0. \end{aligned} \tag{16}$$

Secondly, without loss of generality of the DT, let us suppose the trial Darboux sub-matrices T_0 and T_1 are of the forms:

$$T_0 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \tag{17}$$

$$T_1 = \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}\lambda + \begin{pmatrix} e_0 & f_0 \\ g_0 & h_0 \end{pmatrix}. \tag{18}$$

Here, $a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k$ $(k = 0, 1)$, which are determined later, are all functions of eigenfunctions. Substituting (17) and (18) into the spatial part of Eq. (16), and comparing the coefficients of the same power of λ, we have

$$\begin{aligned} \lambda^2:\ & b_1 = c_1 = f_1 = g_1 = 0, \\ \lambda:\ & a_{1,x} = d_{1,x} = e_{1,x} = h_{1,x} = 0, \\ & pa_1 + 2\mathrm{i}b_0 = p^{[1]}d_1, \\ & qd_1 - 2\mathrm{i}c_0 = q^{[1]}a_1, \\ & ra_1 + pe_1 + 2\mathrm{i}b_0 + 2\mathrm{i}f_0 = p^{[1]}h_1 + r^{[1]}d_1, \\ & sd_1 + qh_1 - 2\mathrm{i}c_0 - 2\mathrm{i}g_0 = q^{[1]}e_1 + s^{[1]}a_1, \\ \lambda^0:\ & a_{0,x} + qb_0 = p^{[1]}c_0, \\ & b_{0,x} + pa_0 = p^{[1]}d_0, \\ & c_{0,x} + qd_0 = q^{[1]}a_0, \\ & d_{0,x} + pc_0 = q^{[1]}b_0, \\ & e_{0,x} + sb_0 + qf_0 = p^{[1]}g_0 + r^{[1]}c_0, \\ & f_{0,x} + ra_0 + pe_0 = p^{[1]}h_0 + r^{[1]}d_0, \\ & g_{0,x} + sd_0 + qh_0 = q^{[1]}e_0 + s^{[1]}a_0, \\ & h_{0,x} + rc_0 + pg_0 = q^{[1]}f_0 + s^{[1]}b_0. \end{aligned} \tag{19}$$

Meanwhile, if we apply the similar procedure to the temporal part of Eq. (16), we have

$$\begin{aligned} \lambda^3:\ & b_1 = c_1 = f_1 = g_1 = 0, \\ \lambda^2:\ & pa_1 + 2\mathrm{i}b_0 = p^{[1]}d_1, \\ & qd_1 - 2\mathrm{i}c_0 = q^{[1]}a_1, \\ & ra_1 + pe_1 + 2\mathrm{i}b_0 + 2\mathrm{i}f_0 = p^{[1]}h_1 + r^{[1]}d_1, \\ & sd_1 + qh_1 - 2\mathrm{i}c_0 - 2\mathrm{i}g_0 = q^{[1]}e_1 + s^{[1]}a_1, \\ \lambda:\ & a_{1,t_2} = d_{1,t_2} = e_{1,t_2} = h_{1,t_2} = 0, \\ & pa_0 + \frac{1}{2}\mathrm{i}p_xa_1 = p^{[1]}d_0 + \frac{1}{2}\mathrm{i}p_x^{[1]}d_1, \\ & qd_0 - \frac{1}{2}\mathrm{i}q_xd_1 = q^{[1]}a_0 - \frac{1}{2}\mathrm{i}q_x^{[1]}a_1, \\ & pe_0 + \frac{1}{2}\mathrm{i}p_xe_1 + ra_0 + \frac{1}{2}(r_x - p_x)a_1 \\ & \quad = r^{[1]}d_0 + \frac{1}{2}(r_x^{[1]} - p_x^{[1]})d_1 + p^{[1]}h_0 + \frac{1}{2}\mathrm{i}p_x^{[1]}h_1, \\ & qh_0 - \frac{1}{2}\mathrm{i}q_xh_1 + sd_0 + \frac{1}{2}\mathrm{i}(q_x - s_x)d_1 + s^{[1]}a_0 \\ & \quad + \frac{1}{2}\mathrm{i}(q_x^{[1]} - s_x^{[1]})a_1 + q^{[1]}e_0 - \frac{1}{2}\mathrm{i}q_x^{[1]}e_1, \\ \lambda^0:\ & a_{0,t_2} - \frac{1}{2}\mathrm{i}pqa_0 - \frac{1}{2}\mathrm{i}q_xb_0 = -\frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}a_0 + \frac{1}{2}\mathrm{i}p_x^{[1]}c_0, \\ & b_{0,t_2} + \frac{1}{2}\mathrm{i}p_xa_0 + \frac{1}{2}\mathrm{i}pqb_0 = -\frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}b_0 + \frac{1}{2}\mathrm{i}p_x^{[1]}d_0, \\ & c_{0,t_2} - \frac{1}{2}\mathrm{i}pqc_0 - \frac{1}{2}\mathrm{i}q_xd_0 = -\frac{1}{2}\mathrm{i}q_x^{[1]}a_0 + \frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}c_0, \\ & d_{0,t_2} + \frac{1}{2}\mathrm{i}p_xc_0 + \frac{1}{2}\mathrm{i}pqd_0 = -\frac{1}{2}\mathrm{i}p_x^{[1]}b_0 + \frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}d_0, \\ & e_{0,t_2} - \frac{1}{2}\mathrm{i}pqe_0 - \frac{1}{2}\mathrm{i}p_xf_0 + \frac{1}{2}\mathrm{i}(pq - ps - qr)a_0 \\ & \quad + \frac{1}{2}\mathrm{i}(q_x - s_x)b_0 \\ & \quad = \frac{1}{2}\mathrm{i}(p^{[1]}q^{[1]} - p^{[1]}s^{[1]} - q^{[1]}r^{[1]})a_0 \\ & \quad + \frac{1}{2}\mathrm{i}(r_x^{[1]} - p_x^{[1]})c_0 - \frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}e_0 + \frac{1}{2}p_x^{[1]}g_0, \\ & f_{0,t_2} + \frac{1}{2}\mathrm{i}p_xe_0 + \frac{1}{2}\mathrm{i}pqf_0 \\ & \quad + \frac{1}{2}\mathrm{i}(r_x - p_x)a_0 - \frac{1}{2}\mathrm{i}(pq - ps - qr)b_0 \\ & \quad = \frac{1}{2}\mathrm{i}(p^{[1]}q^{[1]} - p^{[1]}s^{[1]} - q^{[1]}r^{[1]})b_0 \end{aligned}$$

$$+\frac{1}{2}\mathrm{i}(r_x^{[1]}-p_x^{[1]})d_0-\frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}f_0+\frac{1}{2}p_x^{[1]}h_0\,,$$

$$\begin{aligned} &g_{0,t_2}-\frac{1}{2}\mathrm{i}pqg_0-\frac{1}{2}\mathrm{i}p_xh_0\\ &\quad+\frac{1}{2}\mathrm{i}(pq-ps-qr)c_0+\frac{1}{2}\mathrm{i}(q_x-s_x)d_0\\ &=\frac{1}{2}\mathrm{i}(q_x^{[1]}-s_x^{[1]})a_0\\ &\quad-\frac{1}{2}\mathrm{i}(p^{[1]}q^{[1]}-p^{[1]}s^{[1]}-q^{[1]}r^{[1]})c_0\\ &\quad-\frac{1}{2}\mathrm{i}p_x^{[1]}e_0+\frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}g_0\,,\\ &h_{0,t_2}+\frac{1}{2}\mathrm{i}p_xg_0+\frac{1}{2}\mathrm{i}pqh_0+\frac{1}{2}\mathrm{i}(r_x-p_x)c_0\\ &\quad-\frac{1}{2}\mathrm{i}(pq-ps-qr)d_0\\ &=\frac{1}{2}\mathrm{i}(q_x^{[1]}-s_x^{[1]})b_0\\ &\quad-\frac{1}{2}\mathrm{i}(p^{[1]}q^{[1]}-p^{[1]}s^{[1]}-q^{[1]}r^{[1]})d_0\\ &\quad-\frac{1}{2}\mathrm{i}p_x^{[1]}f_0+\frac{1}{2}\mathrm{i}p^{[1]}q^{[1]}h_0\,. \end{aligned} \tag{20}$$

It is easy to know that a_1, d_1, e_1 and h_1 are all independent of variables x and t_2. So, a_1, d_1, e_1, and h_1 are all constants. In order to obtain the non-trivial new solutions and without losing any generality, we choose $a_1 = d_1 = e_1 = h_1 = 1$. Thus, the Darboux sub-matrices T_0 and T_1 are of the forms

$$T_0=\begin{pmatrix}\lambda+a_0 & b_0\\ c_0 & \lambda+d_0\end{pmatrix},\qquad T_1=\begin{pmatrix}\lambda+e_0 & f_0\\ g_0 & \lambda+h_0\end{pmatrix}, \tag{21}$$

where $a_0, b_0, c_0, d_0, e_0, f_0, g_0, h_0$ are undetermined functions of (x, t). And the transformed new solutions $p^{[1]}, q^{[1]}, r^{[1]}, s^{[1]}$ are given by

$$p^{[1]}=p+2\mathrm{i}b_0,\quad q^{[1]}=q-2\mathrm{i}c_0\,,\qquad r^{[1]}=r+2\mathrm{i}f_0,\quad s^{[1]}=s-2\mathrm{i}g_0\,. \tag{22}$$

Denote that $2N$ eigenfunctions

$$f_k=(f_{k1},f_{k2},f_{k3},f_{k4})^{\mathrm{T}}, \tag{23}$$

are basic solutions of systems (6) and (10) ($n = 2$) with $\lambda = \lambda_k$ ($1 \le k \le 2N$).

Theorem 1 The elements of one-fold DT for the ICAKNS equations (13) are determined by the eigenfunctions f_1, f_2 associated with the parameters λ_1, λ_2 as

$$a_0=\frac{1}{|W_4|}\begin{vmatrix}-\lambda_1f_{11}-\lambda_1f_{13} & f_{12} & f_{13} & f_{14}\\ -\lambda_2f_{21}-\lambda_2f_{23} & f_{22} & f_{23} & f_{24}\\ -\lambda_1f_{13} & f_{14} & 0 & 0\\ -\lambda_2f_{23} & f_{24} & 0 & 0\end{vmatrix},$$

$$b_0=\frac{1}{|W_4|}\begin{vmatrix}f_{11} & -\lambda_1f_{11}-\lambda_1f_{13} & f_{13} & f_{14}\\ f_{21} & -\lambda_2f_{21}-\lambda_2f_{23} & f_{23} & f_{24}\\ f_{13} & -\lambda_1f_{13} & 0 & 0\\ f_{23} & -\lambda_2f_{23} & 0 & 0\end{vmatrix},$$

$$c_0=\frac{1}{|W_4|}\begin{vmatrix}-\lambda_1f_{12}-\lambda_1f_{14} & f_{12} & f_{13} & f_{14}\\ -\lambda_2f_{22}-\lambda_2f_{24} & f_{22} & f_{23} & f_{24}\\ -\lambda_1f_{14} & f_{14} & 0 & 0\\ -\lambda_2f_{24} & f_{24} & 0 & 0\end{vmatrix},$$

$$d_0=\frac{1}{|W_4|}\begin{vmatrix}f_{11} & -\lambda_1f_{12}-\lambda_1f_{14} & f_{13} & f_{14}\\ f_{21} & -\lambda_2f_{22}-\lambda_2f_{24} & f_{23} & f_{24}\\ f_{13} & -\lambda_1f_{14} & 0 & 0\\ f_{23} & -\lambda_2f_{24} & 0 & 0\end{vmatrix},$$

$$e_0=\frac{1}{|W_4|}\begin{vmatrix}f_{11} & f_{12} & -\lambda_1f_{11}-\lambda_1f_{13} & f_{14}\\ f_{21} & f_{22} & -\lambda_2f_{21}-\lambda_2f_{23} & f_{24}\\ f_{13} & f_{14} & -\lambda_1f_{13} & 0\\ f_{23} & f_{24} & -\lambda_2f_{23} & 0\end{vmatrix},$$

$$f_0=\frac{1}{|W_4|}\begin{vmatrix}f_{11} & f_{12} & f_{13} & -\lambda_1f_{11}-\lambda_1f_{13}\\ f_{21} & f_{22} & f_{23} & -\lambda_2f_{21}-\lambda_2f_{23}\\ f_{13} & f_{14} & 0 & -\lambda_1f_{13}\\ f_{23} & f_{24} & 0 & -\lambda_2f_{23}\end{vmatrix},$$

$$g_0=\frac{1}{|W_4|}\begin{vmatrix}f_{11} & f_{12} & -\lambda_1f_{12}-\lambda_1f_{14} & f_{14}\\ f_{21} & f_{22} & -\lambda_2f_{22}-\lambda_2f_{24} & f_{24}\\ f_{13} & f_{14} & -\lambda_1f_{14} & 0\\ f_{23} & f_{24} & -\lambda_2f_{24} & 0\end{vmatrix},$$

$$h_0=\frac{1}{|W_4|}\begin{vmatrix}f_{11} & f_{12} & f_{13} & -\lambda_1f_{12}-\lambda_1f_{14}\\ f_{21} & f_{22} & f_{23} & -\lambda_2f_{22}-\lambda_2f_{24}\\ f_{13} & f_{14} & 0 & -\lambda_1f_{14}\\ f_{23} & f_{24} & 0 & -\lambda_2f_{24}\end{vmatrix}, \tag{24}$$

$$\Leftrightarrow T_1=T_1(\lambda;\lambda_1,\lambda_2,f_1,f_2)=\begin{pmatrix}T_{1,0} & T_{1,1}\\ 0 & T_{1,0}\end{pmatrix}, \tag{25}$$

where the Darboux sub-matrices $T_{1,0}$ and $T_{1,1}$ can be written as the determinant forms:

$$T_{1,0}=\frac{1}{|W_4|}\begin{pmatrix}\begin{vmatrix}p_1 & \lambda\\ W_4 & \xi_1\end{vmatrix} & \begin{vmatrix}p_2 & 0\\ W_4 & \xi_1\end{vmatrix}\\ \begin{vmatrix}p_1 & 0\\ W_4 & \xi_2\end{vmatrix} & \begin{vmatrix}p_2 & \lambda\\ W_4 & \xi_2\end{vmatrix}\end{pmatrix},\qquad T_{1,1}=\frac{1}{|W_4|}\begin{pmatrix}\begin{vmatrix}p_3 & \lambda\\ W_4 & \xi_1\end{vmatrix} & \begin{vmatrix}p_4 & 0\\ W_4 & \xi_1\end{vmatrix}\\ \begin{vmatrix}p_3 & 0\\ W_4 & \xi_2\end{vmatrix} & \begin{vmatrix}p_4 & \lambda\\ W_4 & \xi_2\end{vmatrix}\end{pmatrix}, \tag{26}$$

with

$$W_4=\begin{pmatrix}f_{11} & f_{12} & f_{13} & f_{14}\\ f_{21} & f_{22} & f_{23} & f_{24}\\ f_{13} & f_{14} & 0 & 0\\ f_{23} & f_{24} & 0 & 0\end{pmatrix},$$

$$p_1=(1,0,0,0),\quad p_2=(0,1,0,0)\,,$$
$$p_3=(0,0,1,0),\quad p_4=(0,0,0,1)\,,$$
$$\xi_1=(\lambda_1f_{11}+\lambda_1f_{13},\lambda_2f_{21}+\lambda_2f_{23},\lambda_1f_{13},\lambda_2f_{23})^{\mathrm{T}},$$
$$\xi_2=(\lambda_1f_{12}+\lambda_1f_{14},\lambda_2f_{22}+\lambda_2f_{24},\lambda_1f_{14},\lambda_2f_{24})^{\mathrm{T}}.$$

Then, the new solutions $p^{[1]}, q^{[1]}, r^{[1]}, s^{[1]}$ are given by

$$p^{[1]}=p+2\mathrm{i}\frac{1}{|W_4|}\begin{vmatrix} f_{11} & -\lambda_1 f_{11}-\lambda_1 f_{13} & f_{13} & f_{14}\\ f_{21} & -\lambda_2 f_{21}-\lambda_2 f_{23} & f_{23} & f_{24}\\ f_{13} & -\lambda_1 f_{13} & 0 & 0\\ f_{23} & -\lambda_2 f_{23} & 0 & 0\end{vmatrix},$$

$$q^{[1]}=q-2\mathrm{i}\frac{1}{|W_4|}\begin{vmatrix} -\lambda_1 f_{12}-\lambda_1 f_{14} & f_{12} & f_{13} & f_{14}\\ -\lambda_2 f_{22}-\lambda_2 f_{24} & f_{22} & f_{23} & f_{24}\\ -\lambda_1 f_{14} & f_{14} & 0 & 0\\ -\lambda_2 f_{24} & f_{24} & 0 & 0\end{vmatrix},$$

$$r^{[1]}=r+2\mathrm{i}\frac{1}{|W_4|}\begin{vmatrix} f_{11} & f_{12} & f_{13} & -\lambda_1 f_{11}-\lambda_1 f_{13}\\ f_{21} & f_{22} & f_{23} & -\lambda_2 f_{21}-\lambda_2 f_{23}\\ f_{13} & f_{14} & 0 & -\lambda_1 f_{13}\\ f_{23} & f_{24} & 0 & -\lambda_2 f_{23}\end{vmatrix},$$

$$s^{[1]}=s-2\mathrm{i}\frac{1}{|W_4|}\begin{vmatrix} f_{11} & f_{12} & -\lambda_1 f_{12}-\lambda_1 f_{14} & f_{14}\\ f_{21} & f_{22} & -\lambda_2 f_{22}-\lambda_2 f_{24} & f_{24}\\ f_{13} & f_{14} & -\lambda_1 f_{14} & 0\\ f_{23} & f_{24} & -\lambda_2 f_{24} & 0\end{vmatrix}. \tag{27}$$

Proof 1 By making use of the general fact of the DT, i.e. $T_1(\lambda;\lambda_k)|_{\lambda=\lambda_k} f_k=0$ $(k=1,2)$, all $a_0,b_0,c_0,d_0,e_0,f_0,g_0,h_0$ are expressed by the eigenfunctions f_1, f_2 associated with λ_1,λ_2. Substituting b_0,c_0,f_0,g_0 given in Eq. (24) into Eq. (22), new solutions are given as in Eq. (27). After a direct and tedious calculation, we show that T_1 in Eq. (25) and new solutions in Eq. (27) indeed satisfy the temporal part (20). So the ICAKNS equations (13) are covariant under the transformation T_1 in Eq. (25). Thus, T_1 in Eq. (25) is the DT of ICAKNS equations (13).

In what follows, our key task is to establish the determinant representation of the N-fold DT for the ICAKNS equations (13). To this end, we arrive at the following conclusion.

Theorem 2 N-fold DT for the ICAKNS equations (13) can be expressed by

$$T_N=T_N(\lambda;\lambda_1,\ldots,\lambda_{2N},f_1,\ldots,f_{2N})=\begin{pmatrix} T_{N,0} & T_{N,1}\\ 0 & T_{N,0}\end{pmatrix}, \tag{28}$$

where the Darboux sub-matrices $T_{N,0}$, $T_{N,1}$ are given by the following determinant forms:

$$T_{N,0}=\frac{1}{|W_{4N}|}\begin{pmatrix}\begin{vmatrix} p_{4N-3} & \lambda^N\\ W_{4N} & \xi_{4N-3}\end{vmatrix} & \begin{vmatrix} p_{4N-2} & 0\\ W_{4N} & \xi_{4N-3}\end{vmatrix}\\ \begin{vmatrix} p_{4N-3} & 0\\ W_{4N} & \xi_{4N-2}\end{vmatrix} & \begin{vmatrix} p_{4N-2} & \lambda^N\\ W_{4N} & \xi_{4N-2}\end{vmatrix}\end{pmatrix},$$

$$T_{N,1}=\frac{1}{|W_{4N}|}\begin{pmatrix}\begin{vmatrix} p_{4N-1} & N\lambda^N\\ W_{4N} & \xi_{4N-3}\end{vmatrix} & \begin{vmatrix} p_{4N} & 0\\ W_{4N} & \xi_{4N-3}\end{vmatrix}\\ \begin{vmatrix} p_{4N-1} & 0\\ W_{4N} & \xi_{4N-2}\end{vmatrix} & \begin{vmatrix} p_{4N} & N\lambda^N\\ W_{4N} & \xi_{4N-2}\end{vmatrix}\end{pmatrix}, \tag{29}$$

with

$$\begin{aligned}
&W_{4N}=(\eta_1,\eta_2,\eta_3,\eta_4,\ldots,\eta_{4N-3},\eta_{4N-2},\eta_{4N-1},\eta_{4N}),\\
&\eta_{4k-3}=(\lambda_1^{k-1}f_{11},\ldots,\lambda_{2N}^{k-1}f_{2N,1},\lambda_1^{k-1}f_{13},\ldots,\lambda_{2N}^{k-1}f_{2N,3})^{\mathrm{T}},\quad 1\le k\le N,\\
&\eta_{4k-2}=(\lambda_1^{k-1}f_{12},\ldots,\lambda_{2N}^{k-1}f_{2N,2},\lambda_1^{k-1}f_{14},\ldots,\lambda_{2N}^{k-1}f_{2N,4})^{\mathrm{T}},\quad 1\le k\le N,\\
&\eta_{4k-1}=(\lambda_1^{k-1}f_{13},\ldots,\lambda_{2N}^{k-1}f_{2N,3},0,\ldots,0)^{\mathrm{T}},\quad 1\le k\le N,\\
&\eta_{4k}=(\lambda_1^{k-1}f_{14},\ldots,\lambda_{2N}^{k-1}f_{2N,4},0,\ldots,0)^{\mathrm{T}},\quad 1\le k\le N,\\
&p_{4N-3}=(1,0,0,0,\lambda,0,0,0,\ldots,\lambda^{N-1},0,0,0),\\
&p_{4N-2}=(0,1,0,0,0,\lambda,0,0,\ldots,0,\lambda^{N-1},0,0),\\
&p_{4N-1}=(0,0,1,0,0,0,\lambda,0,\ldots,0,0,\lambda^{N-1},0),\\
&p_{4N}=(0,0,0,1,0,0,0,\lambda,\ldots,0,0,0,\lambda^{N-1}),\\
&\xi_{4N-3}=(\lambda_1^N f_{11}+N\lambda_1^N f_{13},\ldots,\lambda_{2N}^N f_{2N,1}+N\lambda_{2N}^N f_{2N,3},\lambda_1^N f_{13},\ldots,\lambda_{2N}^N f_{2N,3})^{\mathrm{T}},
\end{aligned}$$

$$\xi_{4N-2} = (\lambda_1^N f_{12} + N\lambda_1^N f_{14}, \ldots, \lambda_{2N}^N f_{2N,2} + N\lambda_{2N}^N f_{2N,4}, \lambda_1^N f_{14}, \ldots, \lambda_{2N}^N f_{2N,4})^{\mathrm{T}}.$$

Correspondingly, N-transformed solutions of ICAKNS equations (13) become

$$p^{[N]} = p+2\mathrm{i}\tilde{b}_{N-1}, \quad q^{[N]} = q-2\mathrm{i}\tilde{c}_{N-1}, \quad r^{[N]} = r+2\mathrm{i}(1-N)\tilde{b}_{N-1}+2\mathrm{i}\tilde{f}_{N-1}, \quad s^{[N]} = s-2\mathrm{i}(1-N)\tilde{c}_{N-1}-2\mathrm{i}\tilde{g}_{N-1}\,, \tag{30}$$

where

$$\tilde{b}_{N-1} = \frac{-1}{|W_{4N}|}\det(\eta_1, \eta_2, \eta_3, \eta_4, \ldots, \eta_{4N-3}, \xi_{4N-3}, \eta_{4N-1}, \eta_{4N})\,,$$
$$\tilde{c}_{N-1} = \frac{-1}{|W_{4N}|}\det(\eta_1, \eta_2, \eta_3, \eta_4, \ldots, \xi_{4N-2}, \eta_{4N-2}, \eta_{4N-1}, \eta_{4N})\,,$$
$$\tilde{f}_{N-1} = \frac{-1}{|W_{4N}|}\det(\eta_1, \eta_2, \eta_3, \eta_4, \ldots, \eta_{4N-3}, \eta_{4N-2}, \eta_{4N-1}, \xi_{4N-3})\,,$$
$$\tilde{g}_{N-1} = \frac{-1}{|W_{4N}|}\det(\eta_1, \eta_2, \eta_3, \eta_4, \ldots, \eta_{4N-3}, \eta_{4N-2}, \xi_{4N-2}, \eta_{4N})\,.$$

Proof 2 According to the form of T_1 in Eq. (25), the N-fold DT should be of the form (28), where

$$T_{N,0} = \begin{pmatrix} \lambda^N + \tilde{a}_{N-1}\lambda^{N-1} + \cdots + \tilde{a}_1\lambda + \tilde{a}_0 & \tilde{b}_{N-1}\lambda^{N-1} + \cdots + \tilde{b}_1\lambda + \tilde{b}_0 \\ \tilde{c}_{N-1}\lambda^{N-1} + \cdots + \tilde{c}_1\lambda + \tilde{c}_0 & \lambda^N + \tilde{d}_{N-1}\lambda^{N-1} + \cdots + \tilde{d}_1\lambda + \tilde{d}_0 \end{pmatrix},$$
$$T_{N,1} = \begin{pmatrix} N\lambda^N + \tilde{e}_{N-1}\lambda^{N-1} + \cdots + \tilde{e}_1\lambda + \tilde{e}_0 & \tilde{f}_{N-1}\lambda^{N-1} + \cdots + \tilde{f}_1\lambda + \tilde{f}_0 \\ \tilde{g}_{N-1}\lambda^{N-1} + \cdots + \tilde{g}_1\lambda + \tilde{g}_0 & N\lambda^N + \tilde{h}_{N-1}\lambda^{N-1} + \cdots + \tilde{h}_1\lambda + \tilde{h}_0 \end{pmatrix},$$

with $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j, \tilde{e}_j, \tilde{f}_j, \tilde{g}_j, \tilde{h}_j$ $(0 \le j \le N-1)$ are the functions of x and t. We know that the kernel of T_N is zero, i.e.

$$T_N(\lambda; \lambda_1, \ldots, \lambda_{2N}, f_1, \ldots, f_{2N})|_{\lambda=\lambda_k} f_k = 0\,, \quad 1 \le k \le 2N\,. \tag{31}$$

Thus, all coefficients $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j, \tilde{e}_j, \tilde{f}_j, \tilde{g}_j, \tilde{h}_j$ $(0 \le j \le N-1)$, which can be written as the determinant forms (29), are uniquely solved by the Cramer's rule.

Under a covariant requirement of the spectral problem of the ICAKNS equations (13), the transformed spatial spectral problem should be

$$\phi_x^{[N]} = U^{[N]}\phi^{[N]}, \tag{32}$$

where

$$U^{[N]} = \begin{pmatrix} U_0^{[N]} & U_1^{[N]} \\ 0 & U_0^{[N]} \end{pmatrix}, \quad U_0^{[N]} = \begin{pmatrix} -\mathrm{i}\lambda & p^{[N]} \\ q^{[N]} & \mathrm{i}\lambda \end{pmatrix},$$
$$U_1^{[N]} = \begin{pmatrix} -\mathrm{i}\lambda & r^{[N]} \\ s^{[N]} & \mathrm{i}\lambda \end{pmatrix},$$

and then

$$T_{N,x} + T_N U = U^{[N]} T\,. \tag{33}$$

Substituting T_N given by Eq. (28) into the above equation (33), and comparing the coefficients of λ^N, we get the N-transformed solutions (30). As for the temporal part (t_2) spectral problem of the ICAKNS equations (13), we arrive at the same conclusion after a similar discussion.

4 Solutions of Integrable Couplings of the NLS Equations

When the reduction conditions $q = -p^*$ and $s = -r^*$ are imposed on the ICAKNS equations (13), we have the ICNLS equations

$$p_{t_2} = \frac{1}{2}\mathrm{i}p_{xx} + \mathrm{i}|p|^2 p\,,$$
$$r_{t_2} = \frac{1}{2}\mathrm{i}(r_{xx} - p_{xx}) - \mathrm{i}|p|^2 p + \mathrm{i}p^2 r^* + 2\mathrm{i}|p|^2 r\,, \tag{34}$$

whose Lax pairs are given by

$$\phi_x = \begin{pmatrix} -\mathrm{i}\lambda & p & -\mathrm{i}\lambda & r \\ -p^* & \mathrm{i}\lambda & -r^* & \mathrm{i}\lambda \\ 0 & 0 & -\mathrm{i}\lambda & p \\ 0 & 0 & -p^* & \mathrm{i}\lambda \end{pmatrix}\phi\,,$$

$$\phi_{t_2} = \begin{pmatrix} -\mathrm{i}\lambda^2 + \frac{1}{2}\mathrm{i}|p|^2 & p\lambda + \frac{1}{2}\mathrm{i}p_x & -\mathrm{i}\lambda^2 + \frac{1}{2}\mathrm{i}(pr^* + p^*r - |p|^2) & r\lambda + \frac{1}{2}\mathrm{i}(r_x - p_x) \\ -p^*\lambda + \frac{1}{2}\mathrm{i}p_x^* & \mathrm{i}\lambda^2 - \frac{1}{2}\mathrm{i}|p|^2 & -r^*\lambda - \frac{1}{2}\mathrm{i}(p_x^* - r_x^*) & \mathrm{i}\lambda^2 - \frac{1}{2}\mathrm{i}(pr^* + p^*r - |p|^2) \\ 0 & 0 & -\mathrm{i}\lambda^2 + \frac{1}{2}\mathrm{i}|p|^2 & p\lambda + \frac{1}{2}\mathrm{i}p_x \\ 0 & 0 & -p^*\lambda + \frac{1}{2}\mathrm{i}p_x^* & \mathrm{i}\lambda^2 - \frac{1}{2}\mathrm{i}|p|^2 \end{pmatrix}\phi\,. \tag{35}$$

Lemma 1 If $(\phi_1,\phi_2,\phi_3,\phi_4)^{\mathrm{T}}$ is a solution of Eq. (35) with $\lambda=\lambda_1$, then $(-\phi_2^*,\phi_1^*,-\phi_4^*,\phi_3^*)^{\mathrm{T}}$ is a solution of Eq. (35) with $\lambda=\lambda_1^*$.

By making use of Lemma 1, if we choose $\lambda_{2k}=\lambda_{2k-1}^*$, $f_{2k,1}=-f_{2k-1,2}^*$, $f_{2k,2}=f_{2k-1,1}^*$, $f_{2k,3}=-f_{2k-1,4}^*$, $f_{2k,4}=f_{2k-1,3}^*$ $(1\leq k\leq N)$, then N-fold DT and N-transformed solutions of the ICNLS equations are obtained. In what follows, as a special example, we choose the seed solution $p=r=0$ in Eq. (35), we have

$$\begin{aligned}
&\phi_1=(-\mathrm{i}\lambda x-\mathrm{i}\lambda^2t+c_1)\,\mathrm{e}^{-\mathrm{i}\lambda x-\mathrm{i}\lambda^2t+c_3}\,,\\
&\phi_2=(\mathrm{i}\lambda x+\mathrm{i}\lambda^2t+c_1)\,\mathrm{e}^{\mathrm{i}\lambda x+\mathrm{i}\lambda^2t+c_4}\,,\\
&\phi_3=\mathrm{e}^{-\mathrm{i}\lambda x-\mathrm{i}\lambda^2t+c_3}\,,\\
&\phi_4=\mathrm{e}^{\mathrm{i}\lambda x+\mathrm{i}\lambda^2t+c_4}\,.
\end{aligned}\tag{36}$$

Furthermore, let $c_1=c_2=c_3=c_4=0$ in Eq. (36), and then the corresponding eigenfunction $f_1=(f_{11},f_{12},f_{13},f_{14})^{\mathrm{T}}$ associated with λ_1 is given by

$$\begin{aligned}
&f_{11}=(-\mathrm{i}\lambda_1x-\mathrm{i}\lambda_1^2t)\,\mathrm{e}^{-\mathrm{i}\lambda_1x-\mathrm{i}\lambda_1^2t}\,,\\
&f_{12}=(\mathrm{i}\lambda_1x+\mathrm{i}\lambda_1^2t)\,\mathrm{e}^{\mathrm{i}\lambda_1x+\mathrm{i}\lambda_1^2t}\,,\\
&f_{13}=\mathrm{e}^{-\mathrm{i}\lambda_1x-\mathrm{i}\lambda_1^2t}\,,\\
&f_{14}=\mathrm{e}^{\mathrm{i}\lambda_1x+\mathrm{i}\lambda_1^2t}\,.
\end{aligned}\tag{37}$$

Meanwhile, we take $\lambda_2=\lambda_1^*, f_{21}=-f_{12}^*, f_{22}=f_{11}^*, f_{23}=-f_{14}^*, f_{24}=f_{13}^*$. Furthermore, choosing $\lambda_1=\xi_1+\mathrm{i}\eta_1$, where ξ_1 and η_1 are both real numbers, we have

$$\begin{aligned}
&f_{11}=(\alpha_1-\mathrm{i}\beta_1)\,\mathrm{e}^{\alpha_1-\mathrm{i}\beta_1}\,,\\
&f_{12}=(-\alpha_1+\mathrm{i}\beta_1)\,\mathrm{e}^{-\alpha_1+\mathrm{i}\beta_1}\,,\\
&f_{13}=\mathrm{e}^{\alpha_1-\mathrm{i}\beta_1}\,,\\
&f_{14}=\mathrm{e}^{-\alpha_1+\mathrm{i}\beta_1}\,,
\end{aligned}\tag{38}$$

where $\alpha_1=\eta_1x+2\xi_1\eta_1t, \beta_1=\xi_1x+(\xi_1^2-\eta_1^2)t$. So we have

$$\begin{aligned}
&p^{[1]}=2\eta_1\,\mathrm{e}^{-2\mathrm{i}\beta_1}\,\mathrm{sech}\,(2\alpha_1)\,,\\
&r^{[1]}=2\eta_1(1-2\mathrm{i}\beta_1)\,\mathrm{e}^{-2\mathrm{i}\beta_1}\,\mathrm{sech}\,(2\alpha_1)-4\alpha_1\eta_1\\
&\qquad\times\,\mathrm{e}^{-2\mathrm{i}\beta_1}\tanh(2\alpha_1)\,\mathrm{sech}\,(2\alpha_1)\,,
\end{aligned}\tag{39}$$

which are one-soliton solutions of the ICNLS equations (34).

5 Conclusions and Discussions

DT has been widely applied to many notable integrable equations, and several literatures can be found to study DT for integrable couplings of soliton equations, for example Refs. [29–34]. After some careful comparisons, we found some differences and advantages between these references and our paper. On one hand, comparing Ref. [29] with our paper, the main difference include that the spectral problem of ICAKNS system is different. Therefore, the representation of DT, the numbers of basic solutions and the representation of new solutions are also different. On the other hand, there are some advantages of our paper when compared with Ref. [34]. In our paper, the determinant representations of N-fold DT and N-transformed solutions for the ICAKNS equations (13) are constructed in Eqs. (28) and (30), respectively. While in Ref. [34], they only obtained the representations of one-fold DT and one-transformed solutions. And the other advantage of our paper is to derive the determinant representation of N-fold DT for the ICAKNS equations (13) without iterating.

Above all, in this paper, we have constructed determinant representations of N-fold DT (28) and N-times transformed solutions (30) for the ICAKNS equations (13). Imposing the constraint conditions $q=-p^*$, $s=-r^*$, we have obtained ones of the ICNLS equations (34). After choosing the initial values $p=r=0$, we derived soliton solutions (39) of the ICNLS equations (34). We believe that this method will be successfully applied to the other integrable couplings of soliton equations. And there maybe exist the other solutions for integrable couplings of equations. Both of these questions will be considered in our future paper.

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