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Rational solutions and lump solutions to the (3 + 1)-dimensional Mel'nikov equation

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In this paper, explicit representation of general rational solutions for the (3 + 1)-dimensional Mel'nikov equation is derived by employing the Hirota bilinear method. It is obtained in terms of determinants whose matrix elements satisfy some differential and difference relations. By selecting special value of the parameters involved, the first-order and second-order lump solutions are given and their dynamic characteristics are illustrated by two- and three-dimensional figures.

Keywords: Rational solutions; lump solutions; Mel'nikov equation; Hirota bilinear method.

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1. Introduction

Soliton equations are a kind of important partial differential equations and their solutions can explain many related physical phenomena.[1] Therefore, solving soliton equations have been paid close attention to by more and more researchers. Some exact solutions, such as the soliton solutions, periodic wave solutions, have been found by the inverse scattering method,[2–4] Darboux transformation method,[5–7] Bäcklund transformation method,[8,9] symmetry method,[10–12] Hirota bilinear method,[13] Painlevé expansion method,[14] etc.

As the integrable extension of soliton equations, the Kadomtsev–Petviashvili (KP) equation with self-consistent sources arose in the pioneering work of Mel'nikov for describing the interaction of waves on the x, y plane.[15] After that, the study of the KP equation with self-consistent sources has become a subject of intense investigation.[16–18] Furthermore, more and more soliton equations with self-consistent sources were studied by Sato's theory,[19] inverse scattering method,[20] generalized binary Darboux transformation method[21] and sources generation method.[22] It is found that solitary waves moving with nonconstant velocity are admitted by soliton equations with self-consistent sources.[23–30]

As a kind of rational function solutions localized in all directions in the space, the lump waves have attracted more and more attention in recent years.[31–38] Rational solutions to some higher-dimensional and parity-time-symmetric nonlocal nonlinear systems were also discussed.[39–44] However, in the $(3 + 1)$-dimensional case, the solutions obtained are often rationally localized in almost all but not all directions in space and are called only lump-type solutions.[45–47] Consequently, it is very important and interesting to investigate lump solutions to partial differential equations in $(3 + 1)$ dimensions.

In this work, we would like to discuss the general rational solutions of the $(3+1)$-dimensional Mel'nikov equation

$$\begin{cases} (u_t + 6uu_x + u_{xxx} + 8\kappa|\phi|^2_x)_x - u_{yy} + u_{zz} = 0\,, \\ i\phi_y = 2\phi_{xx} + 2u\phi\,, \\ i\phi_z = \phi_{xx} + u\phi\,, \end{cases} \tag{1}$$

where u is the long wave amplitude, ϕ is the complex short wave packet and κ satisfies the condition $\kappa^2 = 1$. The main content of this paper is as follows: in Sec. 2, the explicit expressions of general rational solutions are derived in terms of determinants by the Hirota bilinear method. In Sec. 3, the two- and three-dimensional figures of first-order and second-order lump wave solutions are presented to analyze their dynamics. In Sec. 4, the conclusions are given.

2. Rational Solutions of Eq. (1)

In this section, we focus on the general rational solutions of the $(3 + 1)$-dimensional Mel'nikov equation.

Through the dependent variable transformation $u = 2(\ln f)_{xx}$ and $\phi = \frac{g}{f}$, Eq. (1) is transformed into the bilinear equation

$$\begin{cases} (D_x^4 + D_xD_t - D_y^2 + D_z^2)f\cdot f - 8\kappa(f^2 - g\bar{g}) = 0\,, \\ (2D_x^2 - iD_y)g\cdot f = 0\,, \\ (D_x^2 - iD_z)g\cdot f = 0\,. \end{cases} \tag{2}$$

Here, f is a real function and g is a complex one with respect to the independent variables x, y, z, t and $\bar{g}$ denotes complex conjugation of g. Introducing a new independent variable s, Eq. (2) can be turned into

$$\begin{cases} (D_x^4 + D_xD_s - D_y^2 + D_z^2)f\cdot f = 0\,, \\ (D_xD_t - D_xD_s)f\cdot f = 8\kappa(f^2 - g\bar{g})\,, \\ (2D_x^2 - iD_y)g\cdot f = 0\,, \\ (D_x^2 - iD_z)g\cdot f = 0\,. \end{cases} \tag{3}$$

Moreover, by the variable transformation

$$x_1 = x, \quad x_2 = \sqrt{-1}y, \quad x_3 = \sqrt{-1}z, \quad x_4 = -(t+s), \quad x_{-1} = 4\kappa t\,, \tag{4}$$

then the bilinear form of Eq. (3) can be changed as the bilinear Bäcklund transformation of the (3+1)-dimensional KP equation prensented in Ref. 48 which reads as

$$(2D_{x_1}^2 + D_{x_2})\tau_{n+1}\cdot\tau_n = 0\,, \tag{5a}$$

$$(D_{x_1}^2 + D_{x_3})\tau_{n+1}\cdot\tau_n = 0\,, \tag{5b}$$

$$D_{x_1}D_{x_{-1}}\tau_n\cdot\tau_n + 2(\tau_{n+1}\cdot\tau_{n-1} - \tau_n^2) = 0\,, \tag{5c}$$

$$(D_{x_1}^4 + D_{x_2}^2 - D_{x_1}D_{x_4} - D_{x_3}^2)\tau_n\cdot\tau_n = 0\,. \tag{5d}$$

with $f = \tau_0$, $g = \tau_1$, $\bar{g} = \tau_{-1}$ and the complex conjugate condition becomes $\overline{\tau_n} = \tau_{-n}$.

In order to find general rational solutions of the $(3+1)$-dimensional bilinear Mel'nikov equation (3), we first give the following elementary lemma.

Lemma 1. *Let* $m_{i,j}^{(n)}$, $\varphi_i^{(n)}$ *and* $\psi_j^{(n)}$ *be functions of variables* x_{-1}, x_1, x_2, x_3 *and* x_4 *satisfying the following relations*:

$$\begin{cases} \partial_{x_{-1}} m_{i,j}^{(n)} = -\varphi_i^{(n-1)}\psi_j^{(n+1)}\,, \\ \partial_{x_1} m_{i,j}^{(n)} = \varphi_i^{(n)}\psi_j^{(n)}\,, \\ \partial_{x_2} m_{i,j}^{(n)} = -2\Big(\varphi_i^{(n+1)}\psi_j^{(n)} + \varphi_i^{(n)}\psi_j^{(n-1)}\Big)\,, \\ \partial_{x_3} m_{i,j}^{(n)} = -\Big(\varphi_i^{(n+1)}\psi_j^{(n)} + \varphi_i^{(n)}\psi_j^{(n-1)}\Big)\,, \\ \partial_{x_4} m_{i,j}^{(n)} = 4\Big(\varphi_i^{(n+2)}\psi_j^{(n)} + \varphi_i^{(n+1)}\psi_j^{(n-1)} + \varphi_i^{(n)}\psi_j^{(n-2)}\Big)\,, \end{cases} \tag{6}$$

and

$$\begin{cases} m_{i,j}^{(n+1)} = m_{i,j}^{(n)} + \varphi_i^{(n)}\psi_j^{(n+1)}\,, \\ \partial_{x_1}\varphi_i^{(n)} = \varphi_i^{(n+1)},\ \partial_{x_1}\psi_j^{(n)} = -\psi_j^{(n-1)}\,, \\ \partial_{x_2}\varphi_i^{(n)} = -2\varphi_i^{(n+2)},\ \partial_{x_2}\psi_j^{(n)} = 2\psi_j^{(n-2)}\,, \\ \partial_{x_3}\varphi_i^{(n)} = -\varphi_i^{(n+2)},\ \partial_{x_3}\psi_j^{(n)} = \psi_j^{(n-2)}\,, \\ \partial_{x_4}\varphi_i^{(n)} = 4\varphi_i^{(n+3)},\ \partial_{x_4}\psi_j^{(n)} = -4\psi_j^{(n-3)}\,, \end{cases} \tag{7}$$

then the determinant

$$\tau_n = \det_{1\le i,j\le N}\left(m_{i,j}^{(n)}\right) \tag{8}$$

satisfies the bilinear equations Eqs. 5(a)–5(d).

Remark 1. We can find that the differential and difference relations with respect to variables x_1, x_{-1} are the same as the Lemma 3.1 in Ref. 27, so Eq. (5c) holds true without need for proof any more. Therefore, we should verify the remaining three equations.

Proof of Lemma 1. According to the differential of determinant and the expansion formula of bordered determinant, we can get the bordered determinants expression about derivatives of τ functions as follows:

$$\begin{aligned} &\partial_{x_1}\tau_n = \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n)} & 0 \end{vmatrix}, \quad \tau_{n+1} = \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n+1)} & 1 \end{vmatrix}, \\ &\partial_{x_1}\tau_{n+1} = \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix}, \end{aligned} \tag{9}$$

$$\partial_{x_1}^2\tau_{n+1} = \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix}, \tag{10}$$

$$\partial_{x_2}\tau_{n+1} = -2\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix} + 2\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix}, \tag{11}$$

$$\partial_{x_3}\tau_{n+1} = -\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n+1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n+1)} & 1 & 0 \end{vmatrix}, \tag{12}$$

$$\partial_{x_2}\tau_n = -2\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} - 2\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix}, \tag{13}$$

$$\partial_{x_3}\tau_n = -\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} - \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix}, \tag{14}$$

$$\partial_{x_4}\tau_n = 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} + 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix} + 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n-2)} & 0 \end{vmatrix}, \tag{15}$$

and

$$\partial_{x_1}\partial_{x_4}\tau_n = 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+3)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} + 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-3)} & 0 \end{vmatrix} - 4B, \tag{16}$$

$$\partial_{x_1}^2\tau_n = \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-1)} & 0 \end{vmatrix}, \tag{17}$$

$$\begin{aligned}\partial_{x_2}^2\tau_n = {} & 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+3)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} - 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n-2)} & 0 \end{vmatrix} \\ & + 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix} + 4\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-3)} & 0 \end{vmatrix} + 4B,\end{aligned} \tag{18}$$

$$\begin{aligned}\partial_{x_3}^2\tau_n = {} & \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+3)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} - \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n-2)} & 0 \end{vmatrix} \\ & - \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ \psi_j^{(n-1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-3)} & 0 \end{vmatrix} - 2B,\end{aligned} \tag{19}$$

$$\partial_{x_1}^3\tau_n = \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} + 2\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ \psi_j^{(n-1)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n-2)} & 0 \end{vmatrix}, \tag{20}$$

$$\begin{aligned}\partial_{x_1}^4\tau_n = {} & \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+3)} \\ -\psi_j^{(n)} & 0 \end{vmatrix} + 3\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+2)} \\ \psi_j^{(n-1)} & 0 \end{vmatrix} \\ & + 3\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n-2)} & 0 \end{vmatrix} + \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n-3)} & 0 \end{vmatrix} + 2B\,,\end{aligned} \tag{21}$$

where

$$B = \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} & \varphi_i^{(n+1)} \\ \psi_j^{(n)} & 0 & 0 \\ -\psi_j^{(n-1)} & 0 & 0 \end{vmatrix} . \tag{22}$$

So, we have

$$\begin{aligned} (2\partial_{x_1}^2 \tau_{n+1} + \partial_{x_2}\tau_{n+1})\tau_n &= 4(\partial_{x_1}\tau_{n+1})(\partial_{x_1}\tau_n) - \tau_{n+1}(2\partial_{x_1}^2\tau_n - \partial_{x_2}\tau_n)\,, \\ (\partial_{x_1}^2 \tau_{n+1} + \partial_{x_3}\tau_{n+1})\tau_n &= 2(\partial_{x_1}\tau_{n+1})(\partial_{x_1}\tau_n) - \tau_{n+1}(\partial_{x_1}^2\tau_n - \partial_{x_3}\tau_n)\,, \end{aligned} \tag{23}$$

which implies

$$\begin{aligned} (2D_{x_1}^2 + D_{x_2})\tau_{n+1} \cdot \tau_n &= 0\,, \\ (D_{x_1}^2 + D_{x_3})\tau_{n+1} \cdot \tau_n &= 0\,. \end{aligned} \tag{24}$$

Furthermore,

$$(\partial_{x_1}^4 \tau_n - \partial_{x_1}\partial_{x_4}\tau_n + \partial_{x_2}^2\tau_n - \partial_{x_3}^2\tau_n)\tau_n = 12B \tag{25}$$

and

$$\begin{aligned} &3(\partial_{x_1}^2\tau_n)^2 - 4(\partial_{x_1}^3\tau_n)(\partial_{x_1}\tau_n) + (\partial_{x_4}\tau_n)(\partial_{x_1}\tau_n) - (\partial_{x_2}\tau_n)^2 + (\partial_{x_3}\tau_n)^2 \\ &\quad = -12\left(\begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix} \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ \psi_j^{(n)} & 0 \end{vmatrix} - \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n)} \\ -\psi_j^{(n-1)} & 0 \end{vmatrix} \begin{vmatrix} m_{i,j}^{(n)} & \varphi_i^{(n+1)} \\ \psi_j^{(n)} & 0 \end{vmatrix} \right), \end{aligned} \tag{26}$$

then according to the Jacobi formula of determinants, we get the determinants satisfy the bilinear equations

$$(D_{x_1}^4 + D_{x_2}^2 - D_{x_1}D_{x_4} - D_{x_3}^2)\tau_n \cdot \tau_n = 0\,. \tag{27}$$

The proof is completed. □

Based on the above elementary lemma, the general rational solution of the (3+1)-dimensional Mel'nikov equation can be determined by the following theorem.

Theorem 1. *The (3+1)-dimensional Mel'nikov equation* (1) *has rational solutions* $u = 2(\ln f)_{xx}, \phi = \frac{g}{f}$ *by* $N \times N$ *determinants defined as*

$$f = \tau_0, \quad g = \tau_1\,, \tag{28}$$

where $\tau_n = \det_{1\le i,j\le N}(m_{i,j}^{(n)})$, *and the matrix elements are given by*

$$\begin{aligned} m_{i,j}^{(n)} &= \sum_{k=0}^{n_i} c_{i,k}(p_i\partial p_i + \xi_i + n)^{n_i-k} \sum_{l=0}^{n_j} \bar{c}_{j,l}(\bar{p}_j\partial\bar{p}_j + \bar{\xi}_j - n)^{n_j-l} \frac{1}{p_i + \bar{p}_j}\,, \\ \xi_i &= p_i x + Q_i y + R_i z + \Omega_i t\,, \end{aligned} \tag{29}$$

with

$$
\begin{aligned}
Q_i &= -4\sqrt{-1}p_i^2\,,\\
R_i &= -2\sqrt{-1}p_i^2\,,\\
\Omega_i &= -\frac{4\kappa}{p_i} - 12p_i^3\,.
\end{aligned} \tag{30}
$$

Here, p_i are complex constants, n_i, n_j are arbitrary positive integers.

Proof. According to Lemma 1, in order to verify this theorem, we only need to find the advisable matrix element $m_{i,j}^{(n)}$ which satisfies the differential and difference relations presented in Lemma 1. Hence, in order to get rational solutions we make an assumption for the functions $\varphi_i^{(n)}$, $\psi_j^{(n)}$ and $m_{i,j}^{(n)}$ as follows:

$$
\begin{aligned}
\varphi_i^{(n)} &= A_i[f(p_i)]^n e^{\xi_i'}\,,\\
\psi_j^{(n)} &= B_j[-f(q_j)]^{-n} e^{\eta_j'}\,,\\
m_{i,j}^{(n)} &= \int_{-\infty}^{x_1} \varphi_i^{(n)}\psi_j^n dx_1 = \frac{A_iB_j}{f(p_i)+f(q_j)}\left[-\frac{f(p_i)}{f(q_j)}\right]^n e^{\xi_i'+\eta_j'}\,,\\
\xi_i' &= \frac{1}{f(p_i)}x_{-1} + f(p_i)x_1 - 2[f(p_i)]^2x_2 - [f(p_i)]^2x_3 + 4[f(p_i)]^3x_4\,,\\
\eta_j' &= \frac{1}{f(q_j)}x_{-1} + f(q_j)x_1 + 2[f(q_j)]^2x_2 + [f(q_j)]^2x_3 + 4[f(q_j)]^3x_4\,,\\
A_i &= \sum_{k=0}^{n_i} c_{i,k}[f(p_i)\partial_{p_i}]^{n_i-k}\,,\\
B_j &= \sum_{l=0}^{n_j} d_{j,l}[f(q_j)\partial_{q_j}]^{n_j-l}\,,
\end{aligned} \tag{31}
$$

where $f(p_i)$ and $f(q_j)$ are arbitrary complex functions for p_i and q_j. And $c_{i,k}$, $d_{j,l}$ are arbitrary complex constants, n_i, n_j are arbitrary positive integers. First, we have $\partial_{x_m}A_i = A_i\partial_{x_m}$, $\partial_{x_m}B_j = B_j\partial_{x_m}$, and it is obvious that these ansatz satisfy Eqs. (6) and (7).

Next, by the operator relations, we can get

$$
\begin{aligned}
&[f(p_i)\partial_{p_i}]f(p_i)^n e^{\xi_i'} = C_if(p_i)^n e^{\xi_i'}[f(p_i)\partial p_i + n + \xi_i]\,,\\
&[f(q_j)\partial_{q_j}][-f(q_j)]^{-n} e^{\eta_j'} = D_j[-f(q_i)]^{-n} e^{\eta_j'}[f(q_j)\partial q_j - n + \eta_j]\,,\\
&C_i = \frac{df(p_i)}{dp_i}, \quad D_j = \frac{df(q_j)}{dq_j}\,,
\end{aligned} \tag{32}
$$

where

$$\begin{aligned}\xi_i &= -\frac{1}{f(p_i)}x_{-1} + f(p_i)x_1 - 4[f(p_i)]^2x_2 - 2[f(p_i)]^2x_3 + 12[f(p_i)]^3x_4\,,\\ \eta_j &= -\frac{1}{f(q_j)}x_{-1} + f(q_j)x_1 + 4[f(q_j)]^2x_2 + 2[f(q_j)]^2x_3 + 12[f(q_j)]^3x_4\,.\end{aligned} \tag{33}$$

Then, the matrix element $m_{i,j}^{(n)}$ becomes

$$m_{i,j}^{(n)} = C_iD_j\left[-\frac{f(p_i)}{f(q_j)}\right]^n e^{\xi_i'+\eta_j'}\frac{D}{f(p_i)+f(q_j)}\,, \tag{34}$$

where

$$D = \sum_{k=0}^{n_i} c_{i,k}(f(p_i)\partial_{p_i} + n + \xi_i)^{n_i-k}\sum_{l=0}^{n_j} d_{j,l}(f(q_j)\partial_{q_j} - n + \eta_j)^{n_j-l}\,. \tag{35}$$

In order to satisfy the complex conjugate condition $\overline{\tau_n} = \tau_{-n}$, we find that

$$f(q_j) = \overline{f(p_j)} = f(\overline{p_j}), \quad d_{j,l} = \overline{c_{j,l}}\,. \tag{36}$$

Without loss of generality, we can take the functions $f(p_i)$ and $f(q_j)$ as the following simple form:

$$f(p_i) = p_i = \overline{q_i}, \quad f(q_j) = q_j = \overline{p_j}\,, \tag{37}$$

then

$$\begin{aligned}\xi_i &= -\frac{1}{p_i}x_{-1} + p_ix_1 - 4p_i^2x_2 - 2p_i^2x_3 + 12p_i^3x_4\,,\\ \eta_j &= -\frac{1}{q_j}x_{-1} + q_jx_1 + 4q_j^2x_2 + 2q_j^2x_3 + 12q_j^3x_4\,.\end{aligned} \tag{38}$$

Consequently, the general rational solutions of $(3+1)$-dimensional Mel'nikov equation can be obtained by the dependent variable transformation $u = 2(\ln f)_{xx}, \phi = \frac{g}{f}$ and the independent variable transformation Eq. (4) without considering the s dependence. This completes the proof of Theorem 1. □

3. Lump Waves of Eq. (1)

According to Theorem 1, we take $N = 1$, $n_1 = 1$ and $i = 1$, $p_1 = 1$, the fundamental first-order lump waves of $(3+1)$-dimensional Mel'nikov equation can be obtained as

$$u = 2(\ln f_{11})_{xx}, \quad \phi = \frac{g_{11}}{f_{11}}\,, \tag{39}$$

where

$$\begin{aligned}f_{11} &= \frac{1}{4} + 8t - \frac{x}{2} + 128t^2 - 16xt + 8y^2 + 8yz + 2z^2 + \frac{x^2}{2}\,,\\ g_{11} &= \frac{x^2}{2} - \frac{1}{4} - 16tx - \frac{x}{2} + 8y^2 + 8yz + 2z^2 + 128t^2 + 8t + \sqrt{-1}(2z+4y)\,.\end{aligned} \tag{40}$$

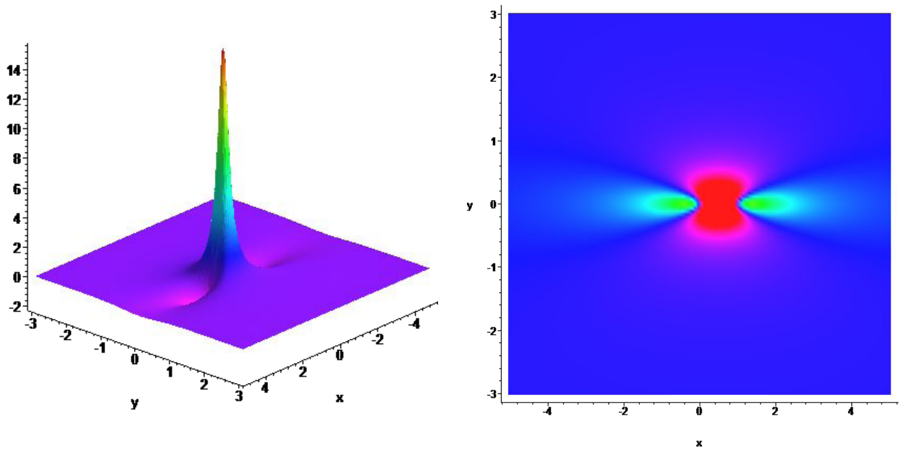

Fig. 1. Plots of first-order lump solution for u at $z = 0$, $t = 0$ for the parameter $p_1 = 1$.

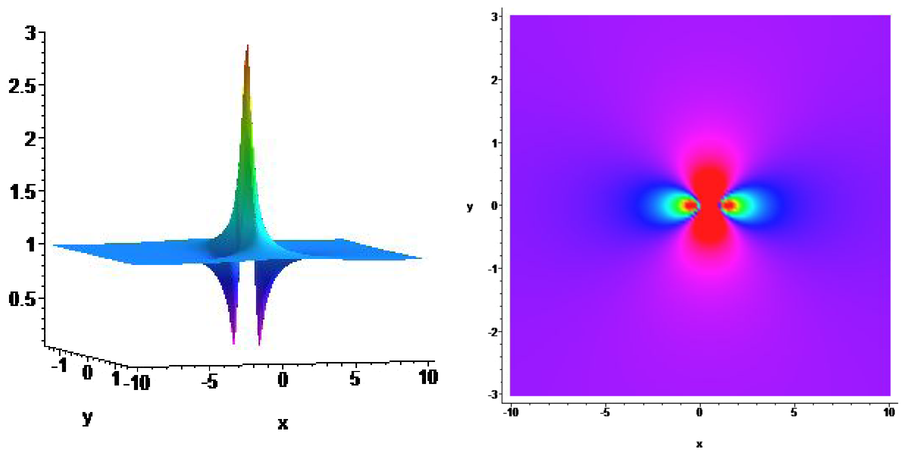

Fig. 2. Plots of first-order lump solution for $|\phi|$ at $z = 0$, $t = 0$ for the parameter $p_1 = 1$.

As shown in Figs. 1 and 2, the spatial structures of these solutions are drawn for a particular choice of the parameters when $z = t = 0$. It can be seen clearly that when $x^2 + y^2 \to \infty$, the lump solutions $u \to 0$, $|\phi| \to 1$.

In addition, by selecting $N = 1$, $n_1 = 1$, $c_{10} = 1$, $c_{11} = 0$ and taking $p_1 = p_{1r} + \sqrt{-1}p_{1i}$, it is easy to see that the structures of u and $|\phi|^2$ depend on the values of p_{1r}, p_{1i}. For convenience, we only discuss the dynamic characteristics of function $|\phi(x, y, 0, 0)|^2$ here. It is found that $|\phi(x, y, 0, 0)|^2$ is symmetric with respect to $y = 0$. And we can obtain one extreme point $\frac{1}{2p_{1r}}$ and two symmetric extreme points $\frac{1+\sqrt{3}}{2p_{1r}}$, $\frac{1-\sqrt{3}}{2p_{1r}}$ which correspond to the same amplitudes. Moreover, as shown in Fig. 3, by fixing $p_{1r} = 1$, in the case of $-p_{1r} < p_{1i} < p_{1r}$, the extreme point $\frac{1}{2p_{1r}}$ is a maximal point and it is a bright rouge wave. The amplitude of the wave

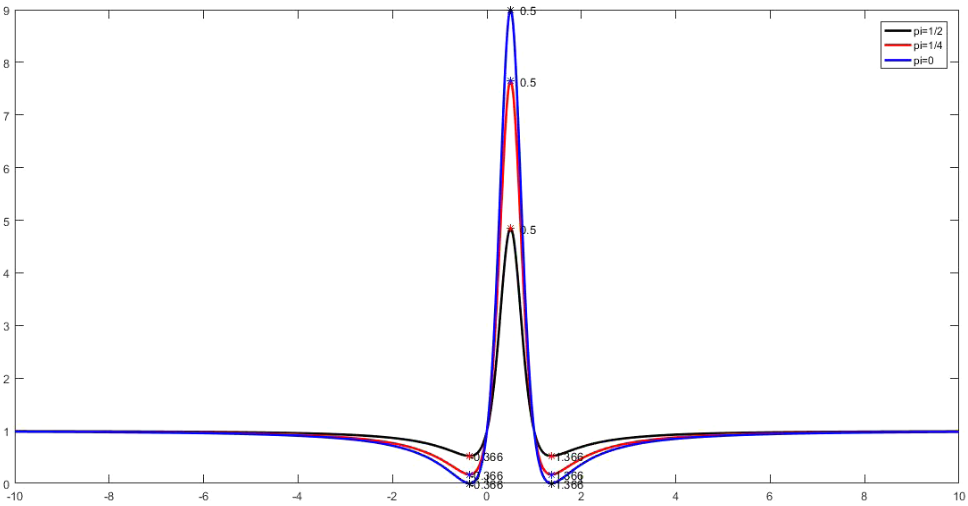

Fig. 3. Plots of first-order lump solution for $|\phi|^2$ at $y = 0$, $z = 0$, $t = 0$ for the parameter $p_{1r} = 1$.

decreases with the increase of p_{1i}. On the other hand, in the case of $p_{1i} > p_{1r}$ (or $p_{1i} < -p_{1r}$), the extreme point $\frac{1}{2p_{1r}}$ is a minimum point value and it represents a type of dark rouge wave of the $(3+1)$-dimensional Mel'nikov equation.

Furthermore, taking $N = 2$, $p_1 = 2$, $p_2 = 1$, we can get the second-order lump waves described by the following solution:

$$u = 2(\ln f_{22})_{xx}, \quad \phi = \frac{g_{22}}{f_{22}}, \tag{41}$$

where

$$\begin{aligned} f_{22} = &-\frac{1808}{9}xyzt - \frac{217}{3888}x + \frac{5531}{1944}t + \frac{217}{23328} + \frac{195608}{27}t^3 - \frac{17}{108}x^3 \\ &+ \frac{307328}{9}t^4 + \frac{512}{9}y^4 + \frac{32}{9}z^4 + \frac{1}{18}x^4 - \frac{13033}{18}xt^2 + \frac{4652}{27}y^2t \\ &+ \frac{1163}{27}z^2t + \frac{701}{36}x^2t - \frac{332}{27}xy^2 - \frac{83}{27}xz^2 - \frac{50960}{9}xt^3 + \frac{27400}{9}y^2t^2 \\ &+ \frac{6850}{9}z^2t^2 + \frac{1931}{6}x^2t^2 - \frac{65}{9}x^3t + \frac{1024}{9}y^3z + \frac{256}{3}y^2z^2 + \frac{40}{9}x^2y^2 \\ &+ \frac{256}{9}yz^3 + \frac{10}{9}x^2z^2 - \frac{452}{9}xz^2t - \frac{1808}{9}xy^2t + \frac{4652}{27}yzt + \frac{40}{9}x^2yz \\ &+ \frac{27400}{9}yzt^2 - \frac{332}{27}xyz + \frac{89}{9}y^2 + \frac{89}{36}z^2 \\ &+ \frac{145393}{324}t^2 - \frac{5531}{324}xt + \frac{217}{1296}x^2 + \frac{89}{9}yz, \end{aligned} \tag{42}$$

$$
\begin{aligned}
g_{22} = &-\frac{1808}{9}txyz - \frac{199}{3888}x + \frac{8357}{1944}t + \frac{217}{23328} + \frac{195608}{27}t^3 - \frac{17}{108}x^3 + \frac{307328}{9}t^4 \\
&+ \frac{512}{9}y^4 + \frac{32}{9}z^4 + \frac{1}{18}x^4 - \frac{13033}{18}t^2x + \frac{4652}{27}ty^2 + \frac{1163}{27}tz^2 + \frac{701}{36}tx^2 \\
&- \frac{332}{27}xy^2 - \frac{83}{27}xz^2 - \frac{50960}{9}t^3x + \frac{27400}{9}t^2y^2 + \frac{6850}{9}t^2z^2 + \frac{1931}{6}t^2x^2 - \frac{65}{9}tx^3 \\
&+ \frac{1024}{9}y^3z + \frac{256}{3}y^2z^2 + \frac{40}{9}y^2x^2 + \frac{256}{9}yz^3 + \frac{10}{9}z^2x^2 - \frac{452}{9}txz^2 - \frac{1808}{9}txy^2 \\
&+ \frac{4652}{27}tyz + \frac{40}{9}yzx^2 + \frac{27400}{9}t^2yz - \frac{332}{27}xyz + \frac{23}{9}y^2 + \frac{23}{36}z^2 + \frac{101023}{324}t^2 \\
&- \frac{3623}{324}tx + \frac{127}{1296}x^2 + \frac{23}{9}yz + \sqrt{-1}\left(\frac{1}{6}y + \frac{40}{9}z^3 + \frac{1}{12}z - \frac{34}{27}yx + \frac{5314}{9}t^2z\right. \\
&+ \frac{160}{3}y^2z + \frac{214}{27}ty + \frac{80}{3}yz^2 - \frac{520}{9}tyx + \frac{8}{9}yx^2 + \frac{4}{9}zx^2 - \frac{17}{27}zx + \frac{320}{9}y^3 \\
&\left.+ \frac{107}{27}tz + \frac{10628}{9}t^2y - \frac{260}{9}tzx\right). \qquad (43)
\end{aligned}
$$

The second-order lump solutions are drawn for a particular choice of the parameters in Figs. 4 and 5. And in order to get more dynamic properties of the lump solutions, we plot the two-dimensional figure of $|\phi(x,0,0,0)|$ in Fig. 6. We can find that the maximum peak coordinate is about (1.1590, 6.5398) which is more than six times the constant background. In Fig. 7, there are two lump waves which have a collision. The lump with smaller peak is faster and the two lump waves separate after their collision. Compared with the first-order lump solutions, the properties of higher-order solutions may have more applications in physics.

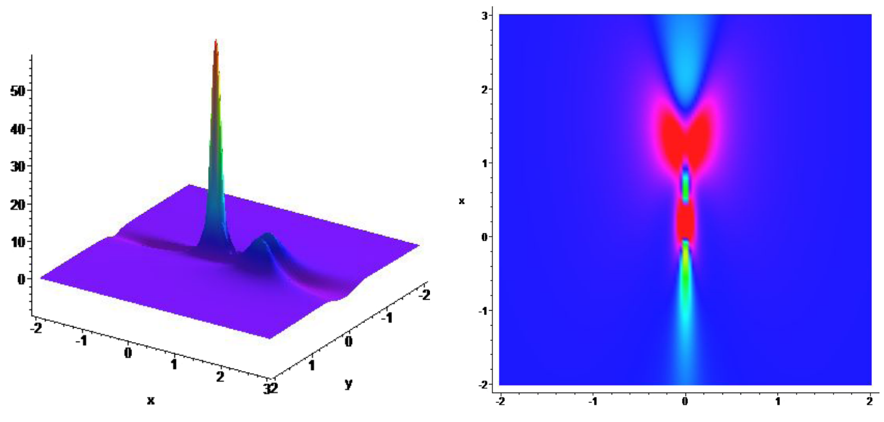

Fig. 4. Plots of second-order lump solution for u at $z = 0$, $t = 0$ for the parameter $p_1 = 2$, $p_2 = 1$.

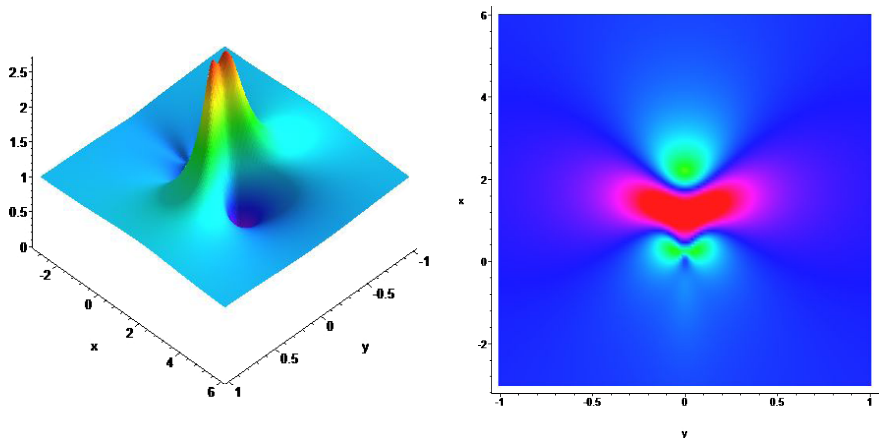

Fig. 5. Plots of second-order lump solution for $|\phi|$ at $z = 0$, $t = 0$ for the parameter $p_1 = 2$, $p_2 = 1$.

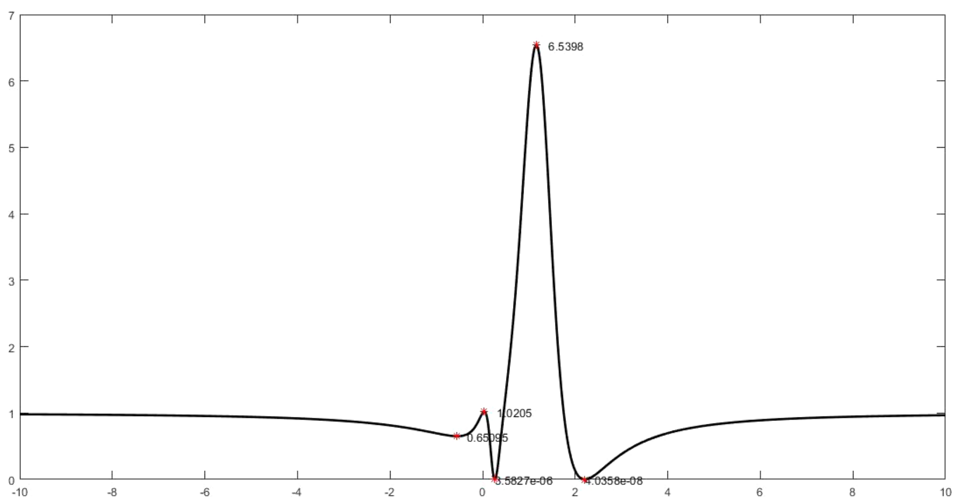

Fig. 6. Plots of second-order lump solution for $|\phi|$ at $y = 0$, $z = 0$, $t = 0$ for the parameter $p_1 = 2$, $p_2 = 1$.

4. Conclusion

In this paper, we have found the rational solutions expressed in terms of determinants for the (3+1)-dimensional Mel'nikov equation by the Hirota bilinear method. The method used here is general and can be applied to other nonlinear soliton equations. Also, we studied graphically some dynamic behaviors of first-order and second-order lump waves. Moreover, we can get more richer superposition patterns if we consider some larger N and special value of parameters involved by virtue of Theorem 1. In the future, we will further investigate other types of interactive wave solutions.

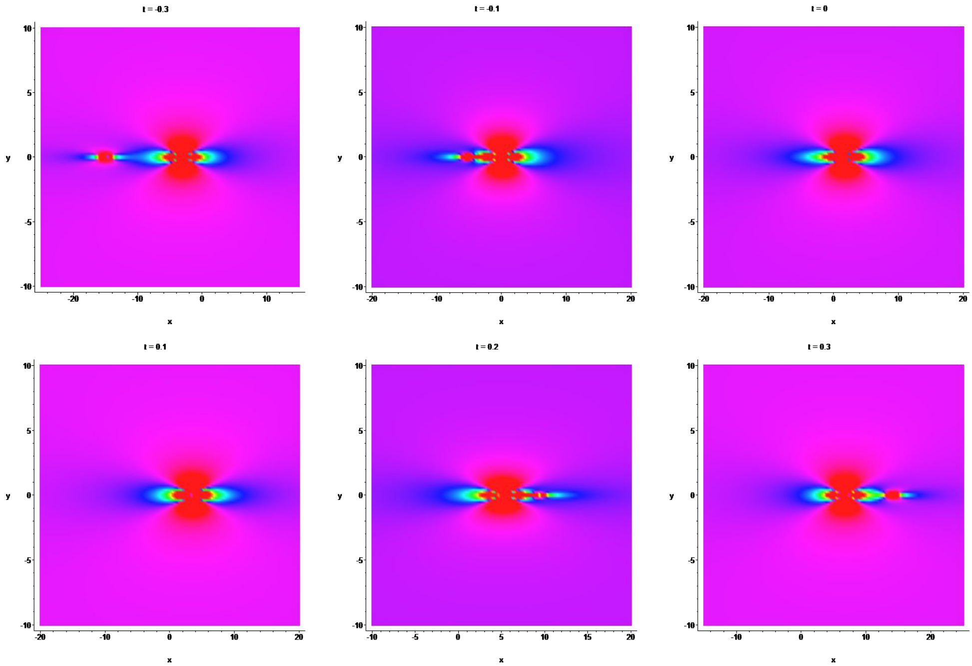

Fig. 7. Plots of second-order lump solution for $|\phi|$ at $t = -0.3,\ -0.1,\ 0, 0.1,\ 0.2,\ 0.3$.

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