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Darboux transformation and solitons for an integrable nonautonomous nonlinear integro-differential Schrödinger equation

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By modifying the scheme for an isospectral problem, the non-isospectral Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy is constructed via allowing the time varying spectrum. In this paper, we consider an integrable nonautonomous nonlinear integro-differential Schrödinger equation discussed before in "Multi-soliton management by the integrable nonautonomous nonlinear integro-differential Schrödinger equation" [Y. J. Zhang, D. Zhao and H. G. Luo, *Ann. Phys.* **350** (2014) 112]. We first analyze the integrability conditions and identify the model. Second, we modify the existing Darboux transformation (DT) for such a non-isospectral problem. Third, the nonautonomous soliton solutions are obtained via the resulting DT and basic properties of these solutions

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in the inhomogeneous media are discussed graphically to illustrate the influences of the variable coefficients. In the process, a technique by selecting appropriate spectral parameters instead of the variable inhomogeneities is employed to realize a different type of one-soliton management. Several novel optical solitons are constructed and their features are shown by some specific figures. In addition, four kinds of the special localized two-soliton solutions are obtained. The solitonic excitations localized both in space and time, which exhibit the feature of the so-called rogue waves but with a zero background, are discussed.

Keywords: Darboux transformation; non-autonomous soliton; non-isospectral AKNS system; nonlinear integro-differential Schrödinger equation.

1. Introduction

The standard nonlinear Schrödinger (NLS) equation

$$iq_t + \frac{1}{2}q_{xx} \pm |q|^2 q = 0 \tag{1}$$

is an important example of physically-significant nonlinear evolution equations that can be solved by the inverse scattering transform (IST) method.[1] It describes the evolution of generic small-amplitude, slowly varying wave packets in a nonlinear medium.[2] The NLS equation has been derived in many branches of sciences such as deep water waves, nonlinear optical fibers and magnetostatic spin waves.[3,4] In addition to its importance as a model of physical systems, the NLS equation has been attracting much attention from mathematicians and physicists due to its rich mathematical structures (see, e.g. Ref. 5 and references therein). The mathematical theory of the IST has been profitably extended to the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, which provides a pretty general kind of Lax pairs to find integrable systems.[6]

It is noted that the IST method of the AKNS hierarchy can also be applied to some evolution equations, which allow the time varying spectrum.[7–14] By modifying the scheme for the isospectral problem, Zhang *et al.* constructed the non-isospectral AKNS hierarchy to generate nonautonomous nonlinear equations.[15,16] They started from the 2×2 linear eigenvalue problem

$$\begin{cases} \Psi_x = U\Psi = \begin{pmatrix} -\eta(t) & q(x,t) \\ r(x,t) & \eta(t) \end{pmatrix} \Psi\,, \\[2ex] \Psi_t = V\Psi = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix} \Psi, \end{cases} \tag{2}$$

where the spectral parameter $\eta(t)$ is non-isospectral, obeying the equation

$$\frac{d\eta}{dt} = \sum_{j=0}^{n} \mu_j(t)\eta^{n-j}\,. \tag{3}$$

To obtain the nth AKNS system, V should be expanded in powers of the spectral parameter to degree n, and then the zero curvature condition $U_t - V_x + [U,V] = 0$

generates the corresponding integrable system.[17] To illustrate, they took $n = 2$, and the following transformation:

$$q(x,t) = \frac{\sqrt{2}}{2}\sqrt{\frac{g(t)}{f(t)}}Q(x,t)e^{\frac{i}{2}\theta(x,t)} \tag{4}$$

leads to a new integrable nonlocal NLS equation

$$\begin{aligned} iQ_t + (f(t)+h(t)x)\left(Q_{xx} + \frac{g(t)}{f(t)}|Q|^2Q\right) \\ + (v(x,t)+i\beta(x,t))Q + h(t)\left(2Q_x + \frac{g(t)}{f(t)}Q\int_{-\infty}^{x}|Q|^2dx'\right) = 0 \end{aligned} \tag{5}$$

with the time-dependent spectral parameter satisfying the Riccati equation

$$\frac{d\eta}{dt} = -2ih(t)\eta^2 + \left[\left(\ln\frac{g}{f}\right)_t - 2\gamma(t)\right]\eta - \frac{i}{2}v_1(t)\,. \tag{6}$$

The integrability conditions read as follows:

$$v(x,t) = -\frac{1}{4}(f+hx)\theta_x^2 + \frac{1}{2}\left[\left(\ln\frac{g}{f}\right)_t - 2\gamma(t)\right]x\theta_x - \frac{1}{2}\theta_t + xv_1(t) + v_0(t)\,, \tag{7}$$

$$\beta(x,t) = \frac{1}{2}(f+hx)\theta_{xx} + h\theta_x - \left(\ln\frac{g}{f}\right)_t + 2\gamma(t)\,, \tag{8}$$

where $\theta(x,t)$ is defined by

$$\theta(x,t) = \begin{cases} \dfrac{1}{h}\left[\left(\ln\dfrac{g}{f}\right)_t - 2\gamma(t)\right]\left[x - \dfrac{f}{h}\ln(2(f+hx))\right] & \text{if } h(t)\neq 0\,, \\ \dfrac{x^2}{2f}\left[\left(\ln\dfrac{g}{f}\right)_t - 2\gamma(t)\right] & \text{if } h(t) = 0\,. \end{cases} \tag{9}$$

When $h(t) = 0$, Eq. (5) becomes the generalized one-dimensional nonautonomous NLS equation

$$iQ_t + f(t)Q_{xx} + g(t)|Q|^2Q + v(x,t)Q + i\gamma(t)Q = 0\,, \tag{10}$$

which has been discussed before.[18] The case $h(t) \neq 0$ has been considered, and the multi-soliton management has been presented with the help of N-fold DT.[15] We would, however, like to remark on their results, especially in the nonvanishing case of $h(t)$. First, the term $\ln(2(f+hx))$ in the expression of $\theta(x,t)$ cannot hold for arbitrary f, h and x. Consequently, the extra condition $\left(\ln\frac{g}{f}\right)_t - 2\gamma(t) = 0$ is necessary in general. Second, let I be the identity matrix and S be a 2×2 matrix with parameters x and t. If η satisfies condition Eq. (3), then $M = \eta I - S$ could not be a Darboux matrix of system (2) since the trace of the transformed matrix $V' = MVM^{-1} + M_tM^{-1}$ is not equal to zero anymore. This fact has been demonstrated by many researchers.[19–24] Third, the captions of Figs. 1 and 2 in Ref. 15 show that they describe soliton solutions determined by Eqs. (3.7) and (3.8) for the case $h(t) \neq 0$, but the parameter of subfigure (a) was $h(t) = 0$. This is self-contradictory, besides their wrong analytic expression of solitons.

In view of those facts, we would like to reconsider the integrable nonautonomous Schrödinger equation to revise the existing DT. We will focus on the case of $h(t) \neq 0$. In this case, the condition $\left(\ln\frac{g}{f}\right)_t - 2\gamma(t) = 0$ must be satisfied, which suggests that $g(t) = cf(t)e^{2\int\gamma(t)dt}$. For simplicity, we take the integral constant $c = 2$ and get the equation

$$\begin{aligned} &iq_t + (f(t) + h(t)x)(q_{xx} + 2|q|^2 q) \\ &\quad + (v_1(t)x + v_0(t))q + 2h(t)\left(q_x + q\int_{-\infty}^{x} |q|^2 dx'\right) = 0\,, \end{aligned} \tag{11}$$

based on the inverse transformation of Eq. (4). The Lax pair of this focusing integrable equation reads

$$\begin{cases} \Psi_x = U\Psi = (-iJ\lambda + U_1)\Psi\,, \\ \Psi_t = V\Psi = (V_0\lambda^2 + V_1\lambda + V_2)\Psi\,, \end{cases} \tag{12}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix},$$

$$V_0 = \begin{pmatrix} -2i(f + hx) & 0 \\ 0 & 2i(f + hx) \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 2q(f + hx) \\ -2q^*(f + hx) & 0 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix},$$

with A and B being given by

$$A = \frac{i}{2}(v_1 x + v_0) + i(f + hx)|q|^2 + ih\int_{-\infty}^{x} |q|^2 dx'\,,$$

$$B = ihq + i(f + hx)q_x$$

and the spectral parameter λ satisfying

$$\lambda_t = 2h\lambda^2 - \frac{v_1}{2}\,. \tag{13}$$

We point out that the case when $h = \mu_1, f = \mu_2, v_0 = v_1 = 0$ was first considered in a more general form by Calogero and Degasperis[25] as a kind of solvable model by the spectral transformation. Moreover, Lakshmanan and Bullough showed the gauge equivalence between this system and the generalized Heisenberg ferromagnetic spin equation.[26] Furthermore, the inclusions of time-dependent terms have attracted widespread concern, since an extensive class of partial differential equations arises, to describe the propagation of waves in nonuniform media or in multi-dimensional cases. Many significant achievements have been made about this topic.[27–35] As the zero curvature condition of a non-isospectral AKNS hierarchy, one can usually solve the equation by the method of Darboux transformation (DT) and there have been many works devoted to construct DTs in a unified approach.[36–42] However, the DT for the non-isospectral AKNS hierarchy has an essential difference from the standard case and this essential difference is often neglected. Consequently, in this paper, we modify the existing DT for such a non-isospectral problem. Then, the nonautonomous soliton solutions are obtained via the modified DT, and basic properties of these solutions in the inhomogeneous media are discussed graphically to illustrate the influences of the variable coefficients.

2. Darboux Transformation

It is known that various results have been presented on the construction of DTs for non-isospectral problems, in which the spectral parameter depends on the time or space variables.[19–24] Here, we apply a method by Zhou[43,44] to construct a DT for the non-isospectral AKNS problem as follows.

Let q satisfy the rapidly vanishing asymptotic condition

$$\lim_{x\to\infty} |x|^k \partial_x^m q = 0 \tag{14}$$

for t and nonnegative integer k, m. If $\Psi_1 = (f_1, g_1)^T$ is a complex vector-valued eigenfunction of Eq. (2) corresponding to the parameter $\lambda_1 = \alpha_1(t) + i\beta_1(t)$, then $\Psi_2 = (-g_1^*, f_1^*)^T$ is a vector-valued eigenfunction of Eq. (2) corresponding to the parameter $\lambda_1^* = \alpha_1(t) - i\beta_1(t)$. Here, the symbol $*$ denotes the complex conjugation. Define

$$D(\lambda) = p(\lambda)G(t)T(\lambda)\,, \tag{15}$$

where

$$G(t) = \mathrm{diag}\left(e^{-2i\int_{-\infty}^{t} h\beta_1 dt'}, e^{2i\int_{-\infty}^{t} h\beta_1 dt'}\right), \tag{16}$$

$$p(\lambda) = [\det(\lambda I - H\Lambda H^{-1})]^{-1/2} = [(\lambda-\lambda_1)(\lambda-\lambda_1^*)]^{-1/2}\,, \tag{17}$$

$$T(\lambda) = \lambda I - S, \quad S = H\Lambda H^{-1}, \quad H = (\Psi_1, \Psi_2), \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_1^*)\,, \tag{18}$$

I being the identity matrix. Then, the gauge transformation

$$\hat{\Psi} = D(\lambda)\Psi \tag{19}$$

presents the DT for Eq. (2) and $\hat{\Psi}$ is a vector eigenfunction satisfying the eigenvalue equations

$$\begin{cases} \hat{\Psi}_x = \hat{U}\hat{\Psi}\,, \\ \hat{\Psi}_t = \hat{V}\hat{\Psi}\,, \end{cases} \tag{20}$$

with $\hat{U}$ and $\hat{V}$ have the same forms as those of U and V, except replacing $q^{[0]}$ with a new potential function $q^{[1]}$ in U and V.

As pointed in Refs. 45 and 46, the factors $p(\lambda)$ and $G(t)$ here are very important. For the standard isospectral AKNS hierarchy, they degenerate to vanish, while for equations with variable spectral parameters, the scalar factor $p(\lambda)$ is introduced to make $\hat{V} \in \mathrm{sl}(2)$. The gauge factor $G(t)$ keeps the integral constants invariant of V and $\hat{V}$. Moreover, the imaginary part of the spectral parameter should be negative to keep the asymptotic property of the elementary solution of the Lax pair.

Under the DT, it is easy to find out that

$$\hat{U} = G(U_1 + i[S, J])G^{-1}. \tag{21}$$

Hence, we obtain the representation of one-fold DT

$$q^{[1]} = e^{-4i\int_{-\infty}^{t} h\beta_1 dt'}\left[q^{[0]} - 2i\frac{(\lambda_1 - \lambda_1^*)f_1 g_1^*}{|f_1|^2 + |g_1|^2}\right]. \tag{22}$$

Iterating the transformation N times, N being an arbitrary positive integer, we can have the Nth iterated potential transformation

$$q^{[N]} = e^{-4i\sum_{j=1}^{N}\int_{-\infty}^{t} h\beta_j dt'}\left[q^{[0]} - 2i\sum_{k=1}^{N} e^{4i\sum_{m=1}^{k-1}\int_{-\infty}^{t} h\beta_m dt'}\frac{(\lambda_k - \lambda_k^*)f_k g_k^*}{|f_k|^2 + |g_k|^2}\right], \tag{23}$$

where $\lambda_k = \alpha_k(t) + i\beta_k(t)$, $1 \le k \le N$, are different non-isospectral parameters with negative imaginary parts and $\Psi_k = (f_k, g_k)^T$ is the vector eigenfunction of Eq. (2) corresponding to the parameter λ_k and $q = q^{[k-1]}$ for each $1 \le k \le N$. This formula will be used to construct multi-soliton solutions in the following section.

3. Nonautonomous Solitons

It is now generally accepted that solitary waves in nonautonomous nonlinear and dispersive systems can propagate in the form of so-called nonautonomous solitons or solitonlike similaritons. Nonautonomous solitons interact elastically and generally move with varying amplitudes and speeds.[47–59] In this section, we take $q^{[0]} = 0$ to construct soliton solutions in explicit forms.

To obtain the one-soliton solution for Eq. (1), we take the spectral parameter $\lambda_1 = \alpha_1 + i\beta_1$, which satisfies

$$\alpha_{1t} = 2h(\alpha_1^2 - \beta_1^2) - \frac{v_1}{2}, \quad \beta_{1t} = 4h\alpha_1\beta_1 \tag{24}$$

and obtain the following eigenfunctions by directly solving the corresponding Lax pair

$$f_1 = e^{-i\rho_1}, \quad g_1 = e^{i\rho_1} \tag{25}$$

with

$$\rho_1 = \lambda_1 x + \int\left(2f\lambda_1^2 - \frac{v_0}{2}\right)dt.$$

After applying the DT, we can get the following one-soliton solution for Eq. (1):

$$q^{[1]} = 2\beta_1 \operatorname{sech}(2\theta_1) e^{-2i(\zeta_1 + 2\int_{-\infty}^{t} h\beta_1 dt')} \tag{26}$$

with

$$\theta_1 = \beta_1 x + 4\int f\alpha_1\beta_1 dt\,,$$

$$\zeta_1 = \alpha_1 x + \int \left[2h\beta_1 + 2f(\alpha_1^2 - \beta_1^2) - \frac{v_0}{2}\right] dt\,.$$

It is evident that the amplitude of the bright soliton grows and decays with time. The v_0 term in the external potential contributes only to the frequency shift of the one-soliton and it can be removed by a time-dependent phase in the wave function. The linear external potential v_1 term contributes to both the frequency shift and the central position of the one-soliton. These two terms do not influence the integrability of the system.

A soliton can be well defined by its amplitude, frequency, phase and time position. So, it is possible to control soliton dynamics by defining the parameter functions. This is the meaning of soliton management. We can determine that the width of the obtained nonautonomous soliton is

$$W(t) \sim \frac{1}{2\beta_1}\,, \tag{27}$$

the evolution of its peak is

$$|q^{[1]}|^2_{\max} = 4\beta_1^2 \tag{28}$$

and the trajectory of its wave center is

$$x_c(t) = -\frac{4\int f\alpha_1\beta_1 dt}{\beta_1}\,. \tag{29}$$

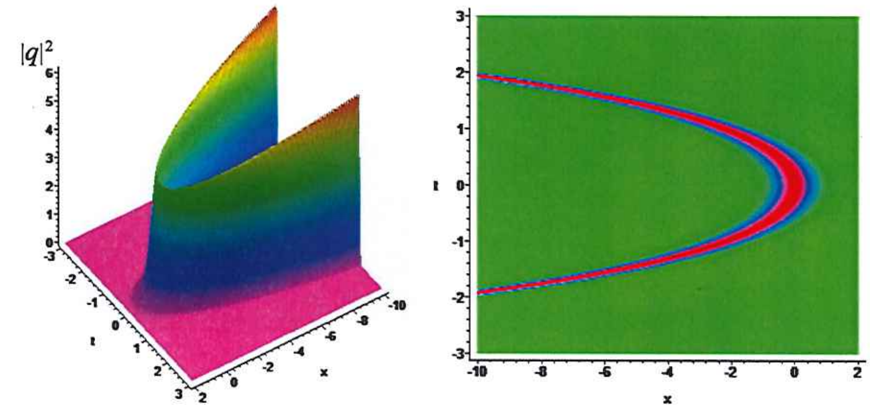
Based on these expressions, soliton management can be realized. To demonstrate this, we take the following idea. Rewrite Eq. (24) as

$$h = \frac{\beta_{1t}}{4\alpha_1\beta_1}\,, \quad v_1 = -2\alpha_{1t} + \beta_{1t}\left(\frac{\alpha_1}{\beta_1} - \frac{\beta_1}{\alpha_1}\right), \tag{30}$$

other than solving it for α_1 and β_1. This idea can help us visualize various dynamics of the one-soliton much more easily. Moreover, in order to avoid the singularity of α_1, β_1 and keep the negative property of β_1, we assume

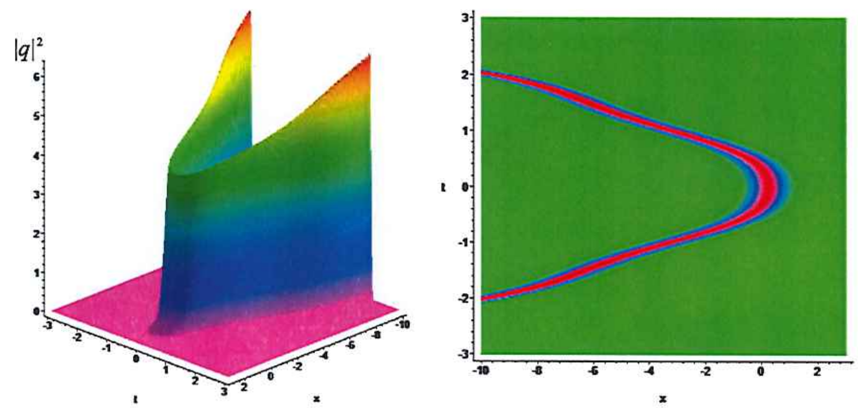
$$\alpha_1 = 1 + n(t)^2\,, \quad \beta_1 = -1 - m(t)^2\,, \tag{31}$$

where $m(t), n(t)$ are arbitrary real functions. Then, the management of the one-soliton can be easily completed by choosing the functions $m(t), n(t)$ and $f(t)$. Many different soliton shapes can be achieved through manipulation of inhomogeneous terms. As special cases, some types of functions will be taken as examples to illustrate how the variable coefficients affect the evolution of the soliton. For example, to achieve a soliton with periodic oscillation, the related operation can be presented



(a) (b)

Fig. 1. (Color online) Parabolic type one-soliton via solution (26) when $m(t) = \frac{t}{4}$, $n(t) = \frac{t}{2}$, $f(t) = t$. (a) Intensity $|q|^2$ and (b) contour plot of Fig. 1(a).

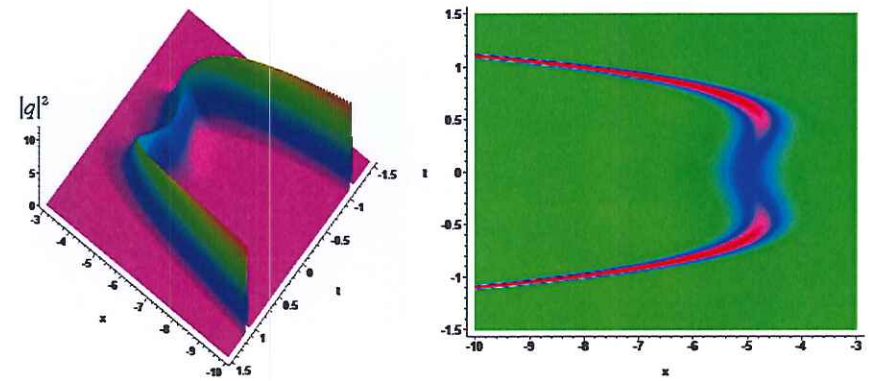


(a) (b)

Fig. 2. (Color online) Parabolic type one-soliton with oscillation via solution (26) when $m(t) = \frac{t}{4}$, $n(t) = -\sin(2t)$, $f(t) = t$. (a) Intensity $|q|^2$ and (b) contour plot of Fig. 2(a).

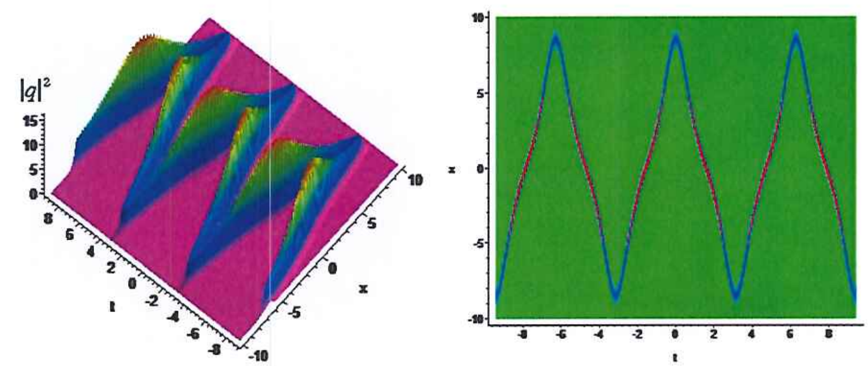
as $m(t) = \sin(\varepsilon t)$, while $m(t) = e^{\varepsilon t}$ means that the amplitude of the corresponding nonautonomous soliton increases (or decreases) exponentially. Furthermore, the trajectory described by $\theta_1 = 0$ exhibits rich diversity. To illustrate, the propagation and evolution of some $|q^{[1]}|^2$ are shown in Figs. 1–4.

To get solitons as in Fig. 1, the inhomogeneous parameters could be designed as $m(t) = \frac{t}{4}, n(t) = \frac{t}{2}, f(t) = t$. We can see that the soliton reveals the parabolic-type propagation trajectory with the increasing amplitude and continuously changeable



(a) (b)

Fig. 3. (Color online) M-type one-soliton via solution (26) when $m(t) = \sin(t)$, $n(t) = -2t$, $f(t) = t$. (a) Intensity $|q|^2$ and (b) contour plot of Fig. 3(a).



(a) (b)

Fig. 4. (Color online) Periodicity oscillated one-soliton via solution (26) when $m(t) = \sin(t)$, $n(t) = \cos(t)$, $f(t) = \sin(t)$. (a) Intensity $|q|^2$ and (b) contour plot of Fig. 4(a).

velocity. However, the propagation trajectory of the soliton presents oscillation when $n(t)$ is chosen as the trigonometric function $\sin(-2t)$ as shown in Fig. 2. Furthermore, Fig. 3 describes an M-type evolution through choosing $m(t) = \sin(t)$, $n(t) = -2t$ and $f(t) = t$. For comparison, the propagation trajectory of the soliton presents the periodicity oscillation when all the inhomogeneous parameters are chosen as the trigonometric functions as shown in Fig. 4. Likewise, if the variable coefficients are taken as other forms, the corresponding evolutions will present different characters. Therefore, this provides an appropriate way to improve the optical soliton transmission quality.

After the one-soliton dynamics has been clearly explored, below we mainly discuss the propagation and interaction between two solitons. To calculate a two-soliton solution, we need a solution of Eq. (3) with $q = q^{[1]}$ and $\lambda = \lambda_2 = \alpha_2 + i\beta_2$ to obtain $\Psi_2 = (f_2, g_2)^T$. The DT in Eq. (12) gives the required solution and one can easily obtain the explicit two-soliton solution, whose interactions under different cases are portrayed in Figs. 5–8. It should be noted that the technique above for one-soliton management fails here. Consequently, one has to solve the Riccati-type equation for the spectral parameters and complicated integrals are necessary.

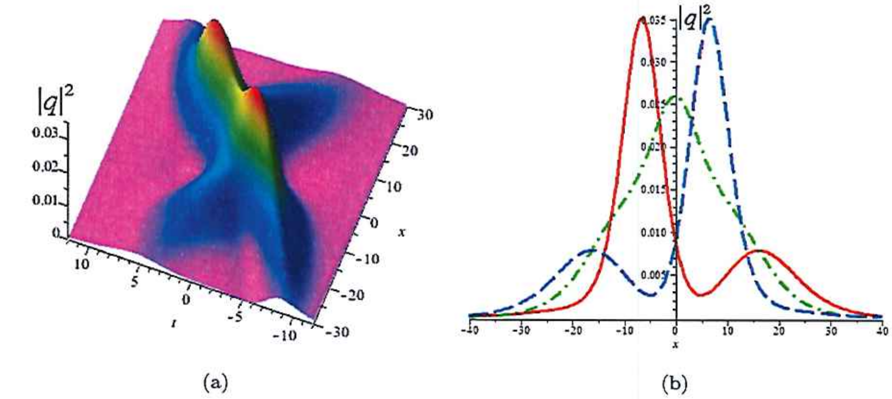


Fig. 5. (Color online) Head-on collision between one right-going soliton and one left-going soliton when $\alpha_1 = -\frac{t}{128+2t^2}$, $\beta_1 = -\frac{4}{t^2+64}$, $\alpha_2 = -\frac{9}{2}\frac{t}{9t^2+256}$, $\beta_2 = -\frac{24}{9t^2+256}$, $h = 1, f = t$. (a) Intensity $|q|^2$ and (b) one-dimensional profiles at different times, $t = -3$ (solid), 0 (dashdot), 3 (dash).

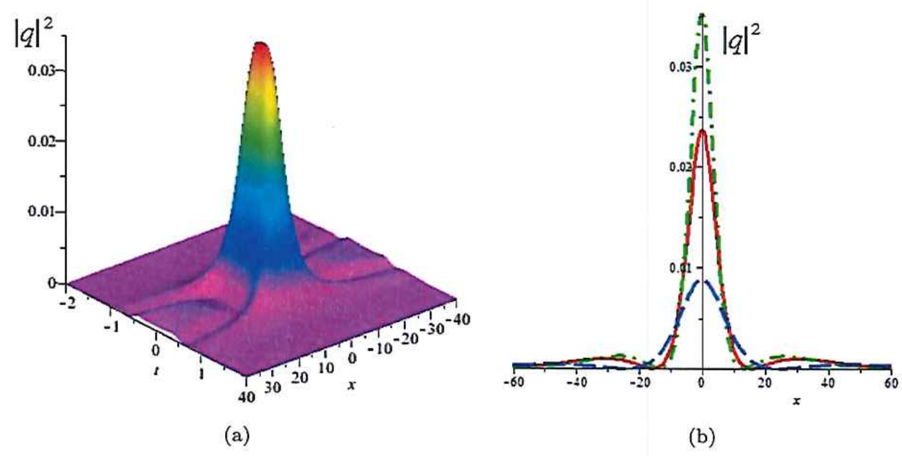
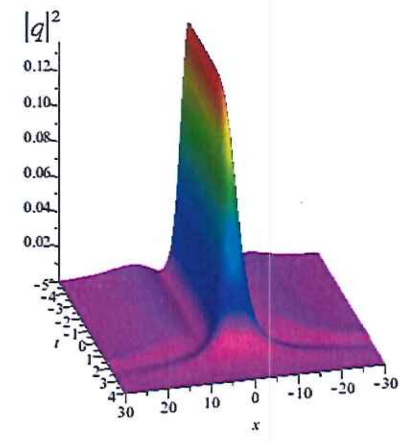
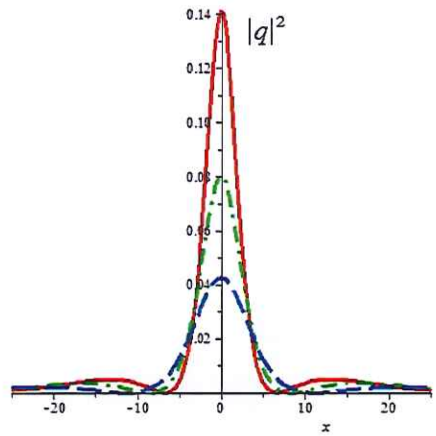


Fig. 6. (Color online) Double localized two-soliton when $\alpha_1 = -\frac{5}{16}\frac{t^2}{25t^4+1}$, $\beta_1 = -\frac{1}{16(25t^4+1)}$, $\alpha_2 = -\frac{5}{16}\frac{t^2}{25t^4+4}$, $\beta_2 = -\frac{1}{8(25t^4+4)}$, $h = 80t, f = 0$. (a) Intensity $|q|^2$ and (b) one-dimensional profiles at different times, $t = -\frac{1}{3}$ (solid), 0 (dashdot), $\frac{1}{2}$ (dash).

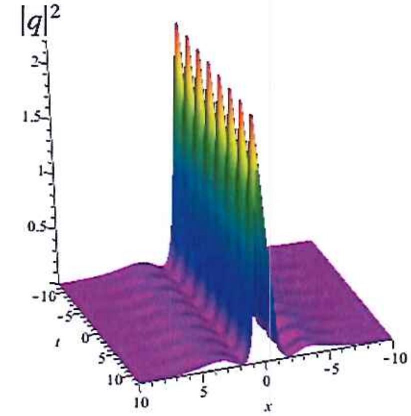


(a)

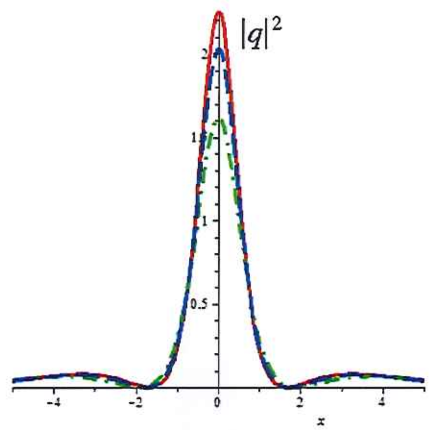


(b)

Fig. 7. (Color online) Spatial localized two-soliton with exponential decay when $\alpha_1 = -\frac{1}{2}\frac{e^t}{e^{2t}+64}$, $\beta_1 = -\frac{4}{e^{2t}+64}$, $\alpha_2 = -\frac{1}{2}\frac{e^t}{e^{2t}+16}$, $\beta_2 = -\frac{2}{e^{2t}+16}$, $h = e^t, f = 0$. (a) Intensity $|q|^2$ and (b) one-dimensional profiles at different times, $t = -3$ (solid), 1 (dashdot), $\frac{3}{2}$ (dash).



(a)



(b)

Fig. 8. (Color online) Spatial localized two-soliton with periodic behavior when $\alpha_1 = \frac{\cos(t)}{16+\cos(t)^2}$, $\beta_1 = -\frac{8}{\cos(2t)+33}$, $\alpha_2 = \frac{\cos(t)}{4+\cos(t)^2}$, $\beta_2 = -\frac{4}{\cos(2t)+9}$, $h = \frac{1}{2}\sin(t), f = 0$. (a) Intensity $|q|^2$ and (b) one-dimensional profiles at different times, $t = -\frac{3}{2}$ (solid), 0 (dashdot), 1 (dash).

For simplicity, we take $v_0 = v_1 = 0$. One notes that in Fig. 5, a head-on collision between one right-going soliton and one left-going soliton occurs. Furthermore, both the propagation of pulses own the S-type trajectories which imply temporary change of propagation directions. Also, Fig. 6 shows a special two-soliton solution localized both in space and time. It exhibits the feature of the so-called rogue waves, but with a zero background rather than a plane wave background. This figure indicates a sharp compression and strong amplification of the nonautonomous soliton under the action of inhomogeneity. Moreover, two different spatial localized solitons are also presented. Figure 7 exhibits a solution with exponential decay profile and Fig. 8 shows periodic propagation of the analytic bound localized solitonic excitations. Such localized modes might be useful in explaining nonlinear phenomena in both BECs and optical systems.

4. Conclusion

In this paper, an integrable nonautonomous nonlinear integro-differential Schrödinger equation was discussed. The integrability conditions were analyzed and identified. After that, the representation of a modified DT was given by a detailed deduction. Based on the DT, the analytic one- and two-soliton solutions were obtained. All of these solutions have parameters denoting the contribution of inhomogeneity, which can be used to control dynamics of the solitons. The characteristic contributions of different control parameters to the soliton dynamics have been clearly presented, which is of significance to guide experiment to control the soliton dynamics. We expect that our results can have applications in some real physical cases. However, unlike the homogeneous NLS equation, we cannot get breather and rogue wave solutions by virtue of the modified DT since the rapidly vanishing condition of plane seed solutions is not satisfied. Consequently, it is necessary to discuss the existing conditions and solving methods of more solutions for inhomogeneous NLS-type equations like Eq. (1).

Moreover, by using the Sylvester identity and its generalization, a compact representation of the DT for an isospectral AKNS system by determinants has been obtained.[60] The advantage of the determinant representation is more direct and universal to get the solutions and then soliton surfaces for the AKNS system. We also hope that the determinant representation of the N-fold DT is applicable for non-isospectral integrable equations. Furthermore, since the defocusing NLS equation has dark soliton solutions with nonvanishing boundary, it is a highly nontrivial problem to get smooth dark N-solitons by using the DT method in the non-isospectral cases. These problems are left for future research.

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