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Lump Solutions of the Modified Kadomtsev-Petviashvili-I Equation

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Abstract. The modified Kadomtsev-Petviashvili-I equation is studied by the Hirota bilinear method. Certain lump solutions of this equation are found via the ansatz technique. Rational solutions presented include plane bounded lumps, which do not decay in all directions in the space.

AMS subject classifications: 35Q51, 35Q53, 37K40

Key words: Lump, symbolic computation, Hirota bilinear method.

1. Introduction

As a $(2+1)$-dimensional integrable generalisation of the modified Korteweg-de Vries equation, the modified Kadomtsev-Petviashvili (mKP) equation

$$V_t + V_{xxx} - \frac{3}{2}V^2 V_x + 3\sigma^2 \partial_x^{-1} V_{yy} - 3\sigma V_x \partial_x^{-1} V_y = 0, \tag{1.1}$$

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where $\sigma^2 = \pm 1$, was introduced within the framework of the gauge-invariant description of the KP equation in [13]. In [11], it appeared as the first member of the first modified KP hierarchy. By introducing a new dependent variable defined as $V = \sigma U$, Eq. (1.1) becomes

$$U_t + U_{xxx} - 3\sigma^2 \left(\frac{1}{2} U^2 U_x - \partial_x^{-1} U_{yy} + U_x \partial_x^{-1} U_y \right) = 0, \tag{1.2}$$

which is classified as the modified Kadomtsev-Petviashvili-I (mKPI) equation when $\sigma = i$ and the modified Kadomtsev-Petviashvili-II (mKPII) equation when $\sigma = 1$ [44]. Both mKPI and mKPII equations are physically significant nonlinear evolution equations. They can be solved by the inverse scattering transform (IST) method, the Darboux transform method, the $\bar{\partial}$-dressing method, the Hirota bilinear method — cf. Refs. [1,2,6,14–17,25,29,45].

Lump solutions are a kind of analytic rational function solutions, localised in all directions in the space. General rational function solutions of the Korteweg-de Vries equation, the Boussinesq equation and the Toda lattice equation have been studied by using Wronskian and Casoratian determinant [5,22–24]. Special lumps also appear as solutions of KPI, BKP, Davey-Stewartson-II and Ishimori-I equations [3,7,10,12,18,28,31,36–39]. Although for mKP and KP equations the $2+1$-dimensional Miura transformation exists, it does not convert real solutions of mKPI and KPI equations into each other [15]. Therefore, it would interesting to find an efficient way for finding real rational solutions of the mKPI equation.

Based on the Hirota bilinear method [9], one of the authors (Ma) proposed a direct method for determining of positive quadratic function solutions to the $(2+1)$-dimensional bilinear KPI equation [20] and general Hirota bilinear equations [21]. The same approach applies to many other equations [8, 19, 26, 30, 32–35, 41, 42, 46–48]. The method has been also recently used to characterise the lump solutions of the KPI equation with a self-consistent source [43].

In this work, we employ Maple symbolic computation, to present two general classes of lump solutions of the mKPI equation (1.2). This equation (1.2) has a Hirota bilinear form and we use special ansätz to find real rational solutions. The solutions obtained contain free parameters, a special choice of which covers lump solutions generated from the IST. In addition, they also generate plane bounded lumps, which do not decay in all directions in the space. Finally, a few concluding remarks are given at the end of the paper.

2. Lump Solutions of mKPI Equation

Using the variable transformation $U = 2i(\ln(G/F))_x$, one can write the mKPI equation (1.2) as

$$\begin{aligned} &\left(D_t + D_x^3 + 3iD_x D_y\right) G \cdot F = 0, \\ &\left(D_x^2 - iD_y\right) G \cdot F = 0 \end{aligned} \tag{2.1}$$

with the D-operators

$$D_t^m D_x^n G \cdot F = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n G(x,t) F(x',t') \Big|_{x'=x, t'=t}.$$

In order to compute real solutions to the mKPI equation (1.2), we take G as the complex conjugation of F, so that

$$U = 2i\left(\ln\frac{F^*}{F}\right)_x = 4\frac{(\mathfrak{Re}F)(\mathfrak{Im}F)_x-(\mathfrak{Re}F)_x(\mathfrak{Im}F)}{(\mathfrak{Re}F)^2+(\mathfrak{Im}F)^2}, \tag{2.2}$$

where $\mathfrak{Re}F$ and $\mathfrak{Im}F$ are the real and imaginary parts of F, respectively. In order to obtain rational solutions, we assume that

$$F = X^TAX + C_1X + a + i\left(X^TBX + C_2X + b\right), \quad X = (x,y,t)^T, \tag{2.3}$$

where a and b are real constants, $A = (a_{jk})_{3\times 3}$ and $B = (b_{jk})_{3\times 3}$ real symmetric matrixes, and $C_m = (c_{mn})_{1\times 3}$, $m = 1,2$, real row vectors. These are also parameters we would like to determine.

Substituting the Eq. (2.3) into (2.1) and extracting the coefficients at x,y,t, we obtain an algebraic system for the parameters. Firstly, we notice that the conditions $a_{jk}b_{jk} - a_{mn}b_{mn} = 0$ hold for any subscripts. Without loss of generality, we can take $A = 0$ and reduce F as

$$F = C_1X + a + i\left(X^TBX + C_2X + b\right). \tag{2.4}$$

This implies that $\mathfrak{Re}F$ is a linear function and $\mathfrak{Im}F$ is at most a quadratic function of independent variables. Moreover, the non-singularity condition $F \neq 0$ must be considered. After detailed analysis, we found that two solutions of the determining equations — viz.

Case 1.

$$B = C_1 = 0, \quad c_{22} = -\frac{c_{21}^2}{k_1}, \quad c_{23} = \frac{3c_{21}^3}{k_1^2}.$$

Case 2.

$$\begin{aligned}
b_{11} &= \frac{24b_{12}^3}{b_{22}(18b_{12}-c_{13})}, \quad b_{13} = \frac{b_{22}(6b_{12}-c_{13})}{4b_{12}},\\
b_{23} &= \frac{b_{22}^2(18b_{12}-c_{13})}{8b_{12}^2}, \quad b_{33} = \frac{3b_{22}^3(18b_{12}-c_{13})}{8b_{12}^3},\\
c_{11} &= \frac{-48b_{12}^4}{b_{22}^2(18b_{12}-c_{13})}, \quad c_{12} = \frac{-4b_{12}^2(6b_{12}-c_{13})}{b_{22}(18b_{12}-c_{13})},\\
c_{23} &= \frac{b_{22}(2\,b_{12}c_{22}-b_{22}c_{21})(18b_{12}-c_{13})}{8b_{12}^3},\\
k_1 &= \frac{24b_{12}^3c_{22}}{b_{22}^2(18b_{12}-c_{13})} - \frac{2c_{21}b_{12}}{b_{22}},\\
k_2 &= \frac{\Delta}{b_{12}^2b_{22}^3(6b_{12}+c_{13})(18b_{12}-c_{13})},
\end{aligned}$$

where

$$\begin{aligned}\Delta =&2304b_{12}^8+432c_{22}^2b_{22}^2b_{12}^4-24c_{13}b_{12}^3c_{22}^2b_{22}^2-648c_{21}c_{22}b_{12}^3b_{22}^3+c_{21}^2c_{13}^2b_{22}^4\\&+72c_{13}c_{21}c_{22}b_{12}^2b_{22}^3+324b_{12}^2c_{21}^2b_{22}^4-36b_{12}c_{13}c_{21}^2b_{22}^4-2b_{12}c_{21}c_{22}b_{22}^3c_{13}^2,\end{aligned}$$

and the other parameters are arbitrary provided that the solutions are well defined.

Until now, we can conclude that these two cases of solutions for the parameters lead to two classes of lump solutions defined by (2.4), to the mKPI equation (1.2) through the transformation (2.2).

In Case 1, we obtain the plane bounded lump solution

$$U=\frac{4k_1c_{21}}{(c_{21}x-c_{21}^2y/k_1+3c_{21}^3t/k_1^2+k_2)^2+k_1^2}, \tag{2.5}$$

where k_1,k_2 and c_{21} are arbitrary real constants. If we further set $c_{21}=1$ and $k_1=\lambda/2$, this produces the plane lump solution from [15]. However, it does not tend to zero in the direction $c_{21}x-c_{21}^2y/k_1+3c_{21}^3t/k_1^2+k_2=$const.

In Case 2, we set

$$b_{12}=b_{22}=1, \quad c_{13}=c_{21}=c_{22}=2,$$

and obtain the lump solution

$$\begin{aligned}U=&4\big(4y^2-18x^2+88t^2-12xy+24xt+56yt-12x+8y+56t+58\big)\big/\\&\big(4(3x+y-2t+1)^2+(3x^2+4xy+4xt+2y^2+8yt+12t^2+4x+4y+8t+11)^2\big),\end{aligned} \tag{2.6}$$

which decays in all directions in the (x,y)-plane. This solution is analytic, since the denominator becomes $[9(x-2t)^2+9]^2$ if $3x+y-2t+1=0$. Fig. 1 shows three-dimensional profiles of the two classes of plane lump and lump solutions. Their plots when $y=0$ for different times are depicted in Fig. 2, respectively.

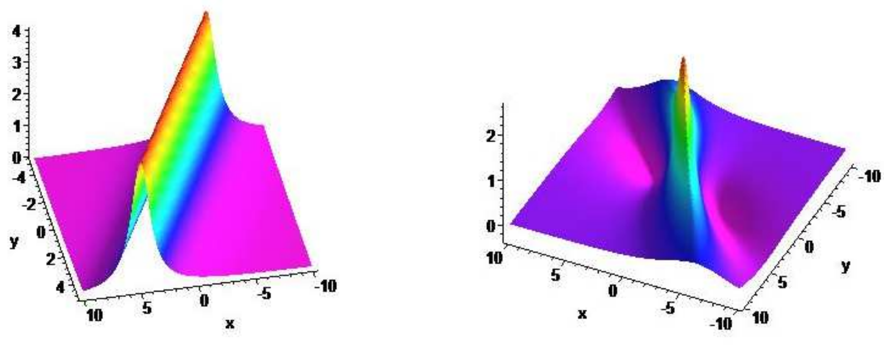
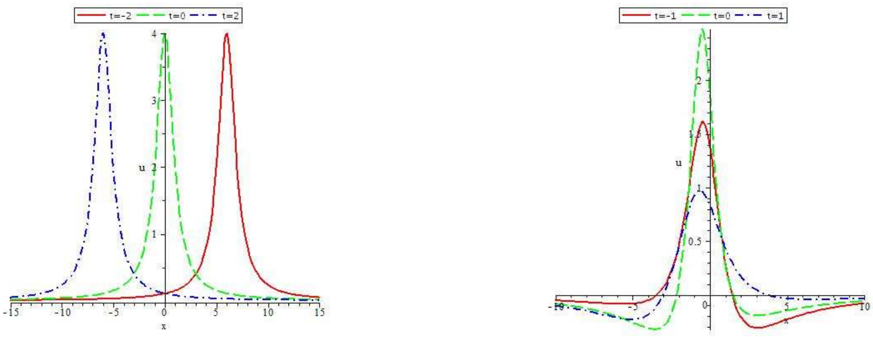


Figure 1: Lump solutions (2.5) and (2.6), $t=0$. a) Plane lump. b) Lump.



a) b)

Figure 2: Profiles of lump solutions (2.5) and (2.6) for $y=0$. a) Plane lump: solid, $t=-2$; dash, $t=0$; dashdot, $t=2$. b) Lump: solid, $t=-1$; dash, $t=0$; dashdot, $t=1$.

3. Conclusion

We study lump solutions for the mKPI equation. Constraint conditions for the existence of such polynomial solutions are given. The solutions presented include plane bounded lumps, which do not decay in all directions in the space. Note that in general high-order equations such as the (3+1)-dimensional Jimbo-Miwa equation and the (3+1)-dimensional potential YTSF equation [4,40] have bounded plane lumps. It is known that linear partial differential equations have lump solutions [27] and it will be interesting to find lump solutions of nonlinear partial differential equations.

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