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# Lie symmetry analysis for a generalized Conde-Gordoa-Pickering equation via equivalence transformations

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ABSTRACT

In this paper, we consider a novel generalized Conde-Gordoa-Pickering equation with time-dependent coefficients by virtue of the Lie symmetry method. An equivalence group is constructed for the considered generalized nonlinear equation. A complete Lie group classification is performed with the aid of the equivalence group. A number of similarity reductions which reduce the generalized equation into ordinary differential equations are given, and several exact solutions are generated. Moreover, we discuss the integrable nature of the novel generalized equation, and particularly, determine cases of restricted coefficients under which the novel equation is Painlevé integrable.

## 1. Introduction

With the aim of defining a partial differential equation (PDE) having analogous properties to those of the second Painlevé equation, Conde *et al.* introduced a novel integrable system

$$\begin{cases} u_x - v^2 = 0, \\ v_{xt} - 2vu_t + 2\varepsilon v + 2g(t)xv - h(t) = 0, \end{cases} \tag{1}$$

and found an auto-Bäcklund transformation of ODE type [1]. Later, they considered a scalar version of the above system

$$vv_{xxt} - 4v^3v_t + 2g(t)v^2 - v_xv_{xt} + h(t)v_x = 0, \tag{2}$$

and obtained several exact solutions by three analytical methods [2].

In this paper, we consider an equation with time-dependent coefficients of the form

$$vv_{xxt} + A(t)v^3v_t + G(t)v^2 + B(t)v_xv_{xt} + H(t)v_x = 0, \tag{3}$$

where $A(t) \neq 0$, $B(t) \neq 0$, $G(t)$ and $H(t)$ are arbitrary smooth functions of $t$, and we will call it a generalized Conde-Gordoa-Pickering (gCGP) equation since it mimics the original gCGP one (2). It is widely known that Lie symmetries of differential equations provide a powerful tool for finding exact solutions [3,4]. For example, the nonlocal symmetries of a coupled KdV system was shown in Wang

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and Wang [5]. The invariance properties of the multiple-term fractional Kolmogorov-Petrovskii-Piskunov equation was considered in Qin et al. [6] by employing the Lie symmetry analysis method. The integrable form of Bratu equation was investigated through the Lie point transformation in Ali and Ma [7].

Over the last years, much attentions are paid to variable coefficient models due to the fact that variable coefficient equations can model certain real-world phenomena with more accuracy than their constant coefficient counterparts. The Lie symmetries of equations involving arbitrary functions seems rather difficult and equivalence transformations allow performing further study in a simpler way [8–13]. As far as we know, the equivalence group of Eq. (3) has not been reported before. Consequently, in this paper we firstly perform a Lie symmetry analysis for the novel equation via equivalence transformations which map one case in the class (3) into another case in the same class that may have simpler variable coefficients. This allows us to reduce the number of arbitrary coefficients and classify all nonequivalent reductions. Then based on four kinds of the resulting different equivalent subclasses, we exhaustively discover Lie point symmetries admitted by each subclass and compute the corresponding similarity reductions. In addition, integrable properties and several exact solutions are presented for restricted values of the involved parameters. A few concluding remarks will be given in the last section.

## 2. Equivalence transformations

In the following, for the class (3), we search for equivalence transformations of the form:

$$\tilde{x} = X(x, t, v), \quad \tilde{t} = T(x, t, v), \quad \tilde{v} = V(x, t, v), \tag{4}$$

which act on the independent and dependent variables in such a way that each maps (3) into another equation preserving the same differential structure

$$\tilde{v}\tilde{v}_{\tilde{x}\tilde{x}\tilde{t}} + \tilde{A}(\tilde{t})\tilde{v}^3\tilde{v}_{\tilde{t}} + \tilde{G}(\tilde{t})\tilde{v}^2 + \tilde{B}(\tilde{t})\tilde{v}_{\tilde{x}}\tilde{v}_{\tilde{x}\tilde{t}} + \tilde{H}(\tilde{t})\tilde{v}_{\tilde{x}} = 0, \tag{5}$$

except its arbitrary but simpler coefficients, $\tilde{A}(\tilde{t})$, $\tilde{B}(\tilde{t})$, $\tilde{G}(\tilde{t})$, and $\tilde{H}(\tilde{t})$.

To determine the continuous group of equivalence transformations, we apply the Lie infinitesimal criterion to the following equivalence class of operators:

$$Y = \xi_x\partial_x + \xi_t\partial_t + \eta_v\partial_v + \eta_A\partial_A + \eta_B\partial_B + \eta_G\partial_G + \eta_H\partial_H, \tag{6}$$

where $\xi_x$, $\xi_t$ and $\eta_v$ depend on $x$, $t$ and $v$ while $\eta_A$, $\eta_B$, $\eta_G$, $\eta_H$ depend on $x$, $t$, $v$, $A$, $B$, $G$ and $H$. In this case, we require not only the invariance of the class (3) but also the invariance of the auxiliary system

$$A_x = A_v = B_x = B_v = G_x = G_v = H_x = H_v = 0. \tag{7}$$

Henceforth the subscripts of capital letters denote partial derivatives. The invariance of the system of (3) and (7) under a one-parameter group of equivalence transformations with an infinitesimal generator in (6) requires a system of determining equations. Using the package GeM [14,15], we obtain the associated equivalence algebra which is spanned by

$$Y_1 = \partial_x, \tag{8}$$

$$Y_2 = v\partial_v - 2A\partial_A + H\partial_H, \tag{9}$$

$$Y_3 = x\partial_x - 2A\partial_A - 2G\partial_G - H\partial_H, \tag{10}$$

$$Y_4 = F\partial_t - GF_t\partial_G - HF_t\partial_H, \tag{11}$$

where $F = F(t)$ is an arbitrary smooth function with $F_t \neq 0$.

Then, the equivalence group of the class (3) consists of the transformations

$$\tilde{x} = \alpha_1 x + \alpha_2, \ \tilde{t} = T(t), \ \tilde{v} = \frac{v}{\alpha_1\alpha_3}, \tag{12}$$

$$\tilde{A} = \alpha_3^2 A, \ \tilde{B} = B, \ \tilde{G} = \frac{G}{\alpha_1^2 T'(t)}, \ \tilde{H} = \frac{H}{\alpha_1^2\alpha_3 T'(t)}, \tag{13}$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$ are arbitrary constants, $T(t)$ is an arbitrary smooth function of $t$, and we require that $\alpha_1\alpha_3 T'(t) \neq 0$ for getting nondegenerate transformations. The equivalence transformations enable us to simplify the class (3) by reducing the variable coefficients.

## 3. Lie symmetry classification

As there is an arbitrary function $T(t)$ in the group of transformations, we can set $\tilde{G} = 1$ when $G \neq 0$ or $\tilde{H} = 1$ when $H \neq 0$ to make a reqiued transformation. Therefore, the original Eq. (2) is equivalent to

$$vv_{xx\tilde{t}} - 4v^3v_{\tilde{t}} + v^2 - v_xv_{x\tilde{t}} + \tilde{h}(\tilde{t})v_x = 0, \tag{14}$$

via the transformation

$$\tilde{t} = \int 2g(t)dt, \quad \tilde{h}(\tilde{t}) = \frac{h(t)}{2g(t)}, \tag{15}$$

when $g(t) \neq 0$; and when $h(t) \neq 0$, the Eq. (2) is equivalent to

$$vv_{x x\tilde{t}} - 4v^3 v_{\tilde{t}} + \tilde{g}(\tilde{t})v^2 - v_x v_{x\tilde{t}} + v_x = 0, \tag{16}$$

via the transformation

$$\tilde{t} = \int h(t)dt, \quad \tilde{g}(\tilde{t}) = \frac{2g(t)}{h(t)}. \tag{17}$$

This implies the absence of one of the two arbitrary functions. Furthermore, it should be noticed that the two different non-trivial symmetry reductions considered in Conde et al. [2] showed $h(t) = cg(t)$ as denoted by Eqs. (17) and (23) there. Under this special condition, Eq. (2) is equivalent to an equation with constant coefficient, which means that the similarity reductions presented in two cases are incomplete.

Next, we try to perform a complete classification of Lie point symmetries for the gCGP Eq. (3) modulo the equivalence transformations. We continue our analysis using the following four subclasses, without presenting any detailed computation to keep brevity.

**Case 1:** When $G(t) = 0$ and $H(t) = 0$, the Lie point symmetries admitted by the Eq. (3) for arbitrary $A(t)$ and $B(t)$ are generated by

$$X_1 = \partial_x, X_2 = v\partial_v - 2x\partial_x. \tag{18}$$

If $A_t \neq 0$ and $B(t) = B$ (an arbitrary nonzero constant, the same below), the Eq. (3) admits an additional Lie symmetry

$$X_3 = \frac{A}{A_t}\partial_t - \frac{x}{2}\partial_x. \tag{19}$$

If $A(t)$, $B(t)$ are both nonzero constants, then the Eq. (3) admits an additional Lie symmetry

$$X_3 = f(t)\partial_t, \tag{20}$$

where $f(t)$ is an arbitrary function.

**Case 2:** When $G(t) = 0$ and $H(t) = 1$, the Lie point symmetries admitted by the Eq. (3) for arbitrary $A(t)$ and $B(t)$ are generated by

$$X_1 = \partial_x. \tag{21}$$

If $B(t) = B$ and $A(t) = k(m - t)^n$ $(k \neq 0)$, then the Eq. (3) admits an additional Lie symmetry

$$X_2 = (n + 2)x\partial_x + 4(m - t)\partial_t + (n - 2)v\partial_v. \tag{22}$$

If $A(t)$, $B(t)$ are both nonzero constants, then the Eq. (3) admits an additional Lie symmetry

$$X_2 = \partial_t, \;\; X_3 = v\partial_v + 2t\partial_t - x\partial_x. \tag{23}$$

**Case 3:** When $G(t) = 1$ and $H(t) = 0$, the Lie point symmetries admitted by the Eq. (3) for arbitrary $A(t)$ and $B(t)$ are generated by

$$X_1 = \partial_x. \tag{24}$$

If $B(t) = B$ and $A(t) = k(m - t)^n$ $(k \neq 0)$, then the Eq. (3) admits an additional Lie symmetry

$$X_2 = x\partial_x + 2(m - t)\partial_t + (n - 1)v\partial_v. \tag{25}$$

If $A(t)$, $B(t)$ are both nonzero constants, then the Eq. (3) admits an additional Lie symmetry

$$X_2 = \partial_t, \;\; X_3 = v\partial_v + 2t\partial_t - x\partial_x. \tag{26}$$

**Case 4:** When $G(t) \neq 0$ is arbitrary and $H(t) = 1$, the Lie point symmetries admitted by the Eq. (3) for arbitrary $A(t)$ and $B(t)$ are generated by

$$X_1 = \partial_x. \tag{27}$$

If $A(t) = CG^2(t)(C \neq 0)$, $B(t) = B$ and $G(t) = k(m - t)^n$ $(k \neq 0)$, then the Eq. (3) admits an additional Lie symmetry

$$X_2 = (n + 1)x\partial_x + 2(m - t)\partial_t + (n - 1)v\partial_v. \tag{28}$$

If $A(t) = CG^2(t)(C \neq 0)$, $B(t) = B$ and $G(t) = e^{kt+m}$ $(k \neq 0)$, then the Eq. (3) admits an additional Lie symmetry

$$X_2 = kx\partial_x - 2\partial_t + kv\partial_v. \tag{29}$$

If $A(t)$, $B(t)$, $G(t)$ are all nonzero constants, then the Eq. (3) admits an additional Lie symmetry

$$X_2 = \partial_t, \;\; X_3 = v\partial_v + 2t\partial_t - x\partial_x. \tag{30}$$

## 4. Similarity reductions and exact solutions

From the Lie point symmetries obtained before, we note that only the translation and dilatation inviances are admitted by the gCGP Eq. (3). In order to find invariant solutions using the Lie group reduction method, we first reduce the Eq. (3) into ordinary differential equations (ODEs) by a one-dimensional sub-algebra of dilatations. We make the required reduction ansatz and present the corresponding reduced equations. Further, symmetries can be derived for some reduced equations which lead to a second reduction. In some cases, we give exact solutions of those reduced ODEs. Moreover, the Painlevé test is discussed to identify their integrability, and general rational solutions are also obtained by virtue of the Painlevé truncation method.

**Case 1:** We first consider the equation

$$vv_{xxt} + A(t)v^3v_t + B(t)v_xv_{xt} = 0. \tag{31}$$

For arbitrary nonzero functions $A(t)$ and $B(t)$, the generator $X_2$ leads to the corresponding invariant solution $v = F(t)/x$, where

$$A(t)F^2(t) + B(t) + 2 = 0. \tag{32}$$

If $A_t \neq 0$ and $B(t) = 1$, the corresponding similarity variables for $X_3$ are $v = V(z)$, $z = x^2A(t)$, and the reduced third-order ODE reads

$$4z^2(VV'')' + 10zVV'' + 4zV'^2 + zV^3V' + 2VV' = 0, \tag{33}$$

which admits an additional fourth Lie point symmetry $X_4 = V\partial_V - 2z\partial_z$ that reduces the Eq. (33) to a second-order ODE:

$$4\eta^2(\eta + 2U)^2U'' - 2\eta(3\eta + 4U)(\eta + 2U)U' + 8\eta^2(\eta + 2U)U'^2 + 8U^3 + 10\eta U^2 + \eta^3U + 6\eta^2U = 0, \tag{34}$$

with $U(\eta) = z^2VV'$, $\eta = zV^2$.

If $A(t) = A$ and $B(t) = B$ are both nonzero constants, the symmetry generators $X_2$, $X_3$ give rise to a group-invariant solution $v = F(t)V(z)$, $z = xF(t)$, with $F(t)$ being an arbitrary function, and reduce the Eq. (31) to the following third-order ODE:

$$zVV''' + AzV^3V' + BV'V'' + 3VV'' + 2BV'^2 + AV^4 = 0. \tag{35}$$

The above equation possesses a new symmetry $X_4 = V\partial_V - z\partial_z$, which results in a further reduction $U(\eta) = z^2V'$, $\eta = zV$, where

$$\eta UU'' + \eta^2U'' + BUU' - \eta U' + \eta U'^2 + A\eta^3 = 0. \tag{36}$$

Furthermore, in order to know about the integrable nature at this point, we can perform the Painlevé singularity structure analysis for the reduced ODEs, following the famous ARS algorithm [16]. Alternatively, here we directly handle the nonlinear PDE by the WTC method [17]. For the algorithm details, see [18] and references therein. We find that a general solution can be expanded as a generalized Laurent series in the neighborhood of the non-characteristic singularity manifold $f(x, t) = 0$:

$$v = \sum_{j=0}^{\infty} v_jf^{j-1}, \tag{37}$$

with the leading term coefficient $v_0^2 = -\frac{6 + 2B}{A}f_x^2$ and the resonances $j = -1, 4, B + 3$. The resonance $j = -1$ corresponds to the arbitrariness of the singularity manifold. Moreover, the number of resonances should equal to the order of the equation being considered, and $v_0$ must be nonzero. Thus $B \geq -2$ and $B \neq 1$ being an arbitrary integer is necessary for the principal expansion, which is a series with all the resonances nonnegative integers, except − 1 that should occur once only. Later, compatibility conditions at each of these resonances give rise to the restriction on the coefficient $A$. For illustration, we take $A = -4$, $B = -1$ for the Eq. (2). It is easy to find that it passes the test and is Painlevé integrable. By truncation, we obtain a general rational solution

$$v(x, t) = \frac{F_1'(x)F_2(t)}{F_1(x)F_2(t) + F_3(t)} - \frac{F_1''(x)}{2F_1'(x)}, \tag{38}$$

which becomes the solution presented by Eq. (31) in Conde et al. [2] of the Eq. (2) in the special case $g(t) = h(t) = 0$ when $F_1(x) = x^k$, $F_2(t) = c(t)$, $F_3(t) = 1$.

**Case 2:** We consider the equation

$$vv_{xxt} + t^nv^3v_t + v_xv_{xt} + v_x = 0. \tag{39}$$

If $n = -2$, then we have $v(x, t) = tV(z)$, $z = x$ and

$$VV'' + V' + V'^2 + V^4 = 0, \tag{40}$$

which can be implicitly solved to obtain

$$z = \int \frac{1}{g(\xi)}d\xi, \quad V = \xi, \tag{41}$$

where $g(\xi)$ satisfies

$$\xi gg' + g + g^2 + \xi^4 = 0. \tag{42}$$

If $n = 2$, then $v(x, t) = V(z)$, $z = xt$ and

$$zV'V'' + V'^2 + zVV''' + 2VV'' + zV^3V' + V' = 0. \tag{43}$$

If $n \neq \pm 2$, then $v = x^{\frac{n-2}{n+2}}V(z)$, $z = tx^{\frac{4}{n+2}}$ and

$$(n^2-4)(V+VV') + 16z^2(VV''' + V'V'') + z^n(n+2)^2V^3V' + 4z(n+2)(V' + V'^2 + 2VV'') = 0. \tag{44}$$

**Case 3:** For the equation

$$vv_{xxt} + t^n v^3 v_t + v^2 + v_x v_{xt} = 0, \tag{45}$$

the similarity variables are $v(x, t) = x^{n-1}V(z)$, $z = tx^2$ and the reduced ODE is

$$2z(n+1)V'^2 + (n+1)(2n-1)VV' + z^nV^3V' + 4z^2VV''' + 2(3n+2)zVV'' + V^2 + 4z^2V'V'' = 0. \tag{46}$$

**Case 4:** Finally, we consider the equation

$$vv_{xxt} + A(t)v^3v_t + G(t)v^2 - v_xv_{xt} + v_x = 0. \tag{47}$$

When $A(t) = G^2(t)$ and $G(t) = 1/t$, we have $v(x, t) = tV(z)$, $z = x$ and

$$VV'' + V^4 + V^2 - V'^2 + V' = 0, \tag{48}$$

which can be solved implicitly to obtain

$$z = \int \frac{1}{g(\xi)}d\xi, \quad V = \xi, \tag{49}$$

where $g(\xi)$ satisfies

$$\xi gg' + g - g^2 + \xi^4 + \xi^2 = 0. \tag{50}$$

When $A(t) = G^2(t)$ and $G(t) = t^n (n \neq -1)$, then $v = x^{\frac{n-1}{n+1}}V(z)$, $z = tx^{\frac{2}{n+1}}$ and

$$\begin{aligned}&(n^2-1)(V - VV') + (n+1)^2z^nV^2(1 + z^nVV')\\ &+ (2n+2)z(V' - V'^2) + 4z^2(VV''' - V'V'') + 8zVV'' = 0.\end{aligned} \tag{51}$$

When $A(t) = G^2(t)$ and $G(t) = e^t$, then $v = xV(z)$, $z = x^2e^t$ and

$$12z^2VV'' + 3zVV' + 4z^3VV''' + z^3V^3V' + zV^2 - 6z^2V'^2 - 4z^3V'V'' + V + 2zV' = 0. \tag{52}$$

Similar to the above analysis for the Eq. (31), we have carried out the Painlevé analysis for the Eq. (47) to identify its integrability with $A(t)$ being an arbitrary nonzero constant. This confirms the integrable nature of the original system (2). In addition, the equivalence group of (47) is formed by the transformations

$$\tilde{x} = \alpha_1 x + \alpha_2,\ \tilde{t} = \beta_1 t + \beta_2,\ \tilde{v} = \alpha_1\beta_1 v,\ \tilde{A} = \frac{A}{\alpha_1^4\beta_1^2},\ \tilde{G} = \frac{G}{\alpha_1^2\beta_1}, \tag{53}$$

where $\alpha_1$, $\alpha_2$ and $\beta_1$, $\beta_2$ are arbitrary constants. Consequently, any integrable equation in the class (47) with a nonzero constant coefficient $A(t)$ is equivalent to the Eq. (16), which is equivalent to the Eq. (2).

## 5. Concluding remarks

In this paper, we have presented a complete Lie group classification for the gCGP Eq. (3) with the aid of its equivalence group. The equivalence transformations can be used to show that the number of variable coefficients can be reduced. Moreover, integrable properties for the reduced equations have been discussed by means of the Painlevé analysis method. Furthermore, a few examples of symmetry reductions and exact solutions have been obtained. There is a connection between conservation laws and pairs of symmetries and adjoint symmetries (see, for example, Ma [19], Tian et al. [20], Ma [21]). It is hoped that our classification of equivalence groups could also help us determine conservation laws for the considered equations. In addition, the asymptotic analysis [22–24] of solutions will be discussed in the future.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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