Darboux transformation and dark vector soliton solutions for complex mKdV systems

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Abstract

A Darboux transformation and general vector dark soliton solutions are constructed for multi-component complex modified Korteweg-de Vries (mKdV) system. Dark soliton solutions exist when nonlinearities are either defocusing or mixed focusing and defocusing. Single-dark–dark–dark and two-dark–dark–dark soliton solutions of three component complex mKdV system are explicitly presented, whose properties and dynamics are illustrated. It is also shown during the interaction between those solitons, energies in different components completely transmit through. Moreover, we perform an asymptotic analysis for the n-dark soliton solutions rigorously. Our results can be applicable to the study of dark solitons in different physical fields.

1. Introduction

During the past decades, it has been a difficult and significant problem to construct exact solutions to nonlinear systems of differential equations. Among integrable systems, the nonlinear Schrödinger (NLS) equation has been recognized as an omnipresent mathematical model, which can be adopted to describe the dynamics of localized waves in many physical fields, such as Bose–Einstein condensate, nonlinear fibers, plasma physics, etc. Another interesting example is the mKdV equation, which has been studied extensively, due to its nonlinear fibers, plasma physics, etc. Another interesting example is the mKdV equation, which has been studied extensively, due to its nonlinear fibers, plasma physics, etc.

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Recently, one of the authors (Ma) considered a 3 × 3 matrix spatial spectral problem and rederived the AKNS soliton hierarchy with four components. A typical nonlinear system in the corresponding soliton hierarchy is

\[ p_{j,x} + \beta [p_{j,x,x} + 3(p_1 q_1 + p_2 q_2) p_{j,x} + 3(p_1 q_1 + p_2 q_2) p_j] = 0, \]

\[ q_{j,x} + \beta [q_{j,x,x} + 3(p_1 q_1 + p_2 q_2) q_{j,x} + 3(p_1 q_1 + p_2 q_2) q_j] = 0, \]

\[ 1 \leq j \leq 2, \]

which contains various mKdV equations, and constructed the multiple soliton solutions through a specific Riemann–Hilbert problem with an identity jump matrix.

In this work, we would like to discuss the following general multi-component complex mKdV integrable system

\[ p_i + p_{xx} + 3p^T Y p + 3p^T Y p = 0, \]

where

\[ p = (p_1, p_2, \ldots, p_N)^T, \quad Y = \text{diag}(y_1, y_2, \ldots, y_N), \]

\[ y_i = \begin{cases} 1, & 1 \leq i \leq k, \\ -1, & k + 1 \leq i \leq N, \end{cases} \]

where the symbol \( T \) denotes the transpose. It should be noted that when \( k = 0 \) and \( Y = -I_N \), this system is the defocusing model that supports n-dark vector soliton solutions; when \( k = N \) and \( Y = I_N \), this system becomes the focusing model that supports n-bright soliton solutions; when \( 1 \leq k \leq N - 1 \), this system is the mixed focusing and defocusing system that supports n-dark vector soliton solutions, where \( I_N \) is the \( N \times N \) identity matrix.
The Darboux transformation (DT) is a powerful method to construct exact solutions for integrable systems. In the present paper, by iterating a standard DT, dark vector soliton solutions for system (2.1) with the defocusing case and the mixed focusing and defocusing case are derived.

A brief outline of this paper is as follows. In Section 2, we present a Lax pair and construct a DT for the above multi-component complex mKdV system. In Section 3, we choose some plane-wave solutions as seed solutions to derive explicit formulas for one- and multi-dark soliton solutions in the defocusing case and the mixed focusing and defocusing case by a limiting process. Based on the obtained multi-dark soliton solutions, we analyze dynamics of single-dark–dark–dark and two-dark–dark–dark soliton solutions of the three-component system through detailed examination. The results are then summarized in Section 4.

2. Darboux transformation

In this section, we would like to construct a DT for the multi-component complex mKdV system (1.2). Firstly, we declare that the system (1.2) has the following Lax pair

\[ \Phi_t = U(\lambda; P)\Phi, \quad \Phi = \begin{pmatrix} iJ & J \end{pmatrix}, \]

\[ \Phi_t = U(\lambda; P)\Phi, \quad \Phi = \begin{pmatrix} iJ + iJ^2P + iJ + J^2P \end{pmatrix}, \]

\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

(2.1)

with

\[ V_1 = J(\lambda P^2 + P), \quad V_0 = P - \lambda P, \quad 2iJ - iJ \sum_{k \neq 0} \lambda_k \]

\[ J = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \lambda \end{pmatrix}, \]

(2.2)

where \( \Phi = (\phi_1, \phi_2, \ldots, \phi_{N+1})^t \) is a vector eigenfunction, \( \lambda \) is the spectral parameter, \( \theta \) is the \( N \times N \) zero matrix. It is straightforward to check that the compatibility condition \( U_i - V_i + [U, V] = 0 \) exactly gives rise to the system (1.2).

It is evidently that the matrices \( U(\lambda; P) \) and \( V(\lambda; P) \) possess the reduction condition

\[ U^t(\lambda; P) = -SU(\lambda^*; P)S, \]

\[ V^t(\lambda; P) = -SV(\lambda^*; P)S, \]

(2.3)

where \( \Phi^t \) denotes the Hermitian conjugation, and \( S = diag(s_1, s_2, \ldots, s_{N+1}) \) with \( s_j = 1 \) for \( 1 \leq j \leq k + 1 \) and \( s_j = -1 \) for \( k + 2 \leq j \leq N + 1 \).

The adjoint problem of the Lax pair (2.1) reads

\[ \Psi_{t} = -PU(\lambda; P), \quad \Psi = -PV(\lambda; P). \]

(2.4)

where \( \Psi = (\psi_1, \psi_2, \ldots, \psi_{N+1}) \). Due to the relation (2.2), a simple observation is that \( \Phi^\dagger S \) satisfies the adjoint problem (2.3). Thus, through the loop group method, we can find a Darboux matrix as follows:

\[ T = I_{N+1} - \frac{\lambda - \lambda^*}{\lambda - \lambda^*} M, \quad M = \frac{\Phi_t}{\Phi^t S \Phi}, \]

(2.5)

where \( \Phi_t \) is a special solution for the Lax pair (2.1) at \( \lambda = \lambda_1 \) and \( I_{N+1} \) is the \( (N + 1) \times (N + 1) \) identity matrix.

Theorem 1. The eigenfunction transformation

\[ \Phi(\lambda, P) \rightarrow \Phi(\lambda, \lambda^* P)[1] \rightarrow T \Phi(\lambda, P), \]

where \( T \) is defined by (2.5), converts the system (1.2) into a new system

\[ \Phi(\lambda, P) \rightarrow V(\lambda; P)[1], \]

(2.6)

where \( U(\lambda; P)[1] \) and \( V(\lambda; P)[1] \) with

\[ P[1] = \begin{pmatrix} 0 & p[1] \\ -\gamma p[1]^t & 0 \end{pmatrix}, \quad p[1] = (p_1[1], p_2[1], \ldots, p_{N+1}[1])^t, \]

and the associated Bäcklund transformation for the potential matrix is

\[ P[1] = P - \frac{1}{2}(\lambda - \lambda^*)[J, M, J]. \]

(2.7)

Proof. We merely need to prove

\[ T_0^t T^{-1} + TU(\lambda) T^{-1} = U[1], \]

(2.8a)

\[ T_0^t T^{-1} + TV(\lambda) T^{-1} = V[1], \]

(2.8b)

Firstly, we consider Eq. (2.8a) and use the residue analysis method to verify it. Expanding \( T \) and \( T^{-1} \) as the power series of \( \lambda \) at \( \lambda = \infty \):

\[ T = I_{N+1} - M(\lambda - \lambda^*) \frac{1}{\lambda} \frac{\lambda^2}{\lambda_1} \frac{\lambda_1^2}{\lambda_3} + \cdots, \]

(2.9)

\[ T^{-1} = I_{N+1} + M(\lambda - \lambda^*) \frac{1}{\lambda} \frac{\lambda^2}{\lambda_1} \frac{\lambda_1^2}{\lambda_3} + \cdots. \]

Let

\[ G_1(t; \lambda) = T_0^t T^{-1} + TU(\lambda) T^{-1} - U[1]. \]

(2.10)

Then, we want to show that the matrix function \( G_1(t; \lambda) \) is holomorphic on the compact Riemann surface \( S^2 = C \cup \{\infty\} \). Obviously, the residues for the matrix function \( G_1(t; \lambda) \) at \( \lambda = \lambda^*_s \) and \( \lambda = \lambda_1 \) can be computed as follows:

\[ Res_{\lambda=\lambda^*_s} G_1(t; \lambda) = \langle \lambda^*_s - \lambda \rangle [M, T^{-1} + M U(\lambda) T^{-1}][1] = 0, \]

\[ Res_{\lambda=\lambda_1} G_1(t; \lambda) = \langle \lambda_1 - \lambda \rangle [M, T^{-1} - U(\lambda) T^{-1}][1] = 0. \]

Moreover, we can calculate the residue for the matrix function \( G_1(t; \lambda) \) at \( \lambda = \infty \) as follows:

\[ G_1(t; \infty) = \langle J P + i J P[1] + 1/2 (\lambda - \lambda^*) [M, J M - J M] \rangle = J[1]. \]

Therefore, \( G_1(t; \lambda) \) is a holomorphic function on \( S^2 \). It follows from the above asymptotic behavior analysis that \( G_1(t; \lambda) = 0 \). So Eq. (2.8a) holds. On the other hand, we continue to discuss the time evolution part (2.8b). A direct analysis can show that

\[ G_1(t; \lambda) = T_0^t T^{-1} + TV(\lambda) T^{-1} - \tilde{V} = 0, \]

(2.11)

where

\[ \tilde{V} = \frac{1}{2} i J M + i \lambda (J P[1] + J H), \]


(2.12)

Hence, our aim is turned to verify that \( \tilde{V} = V[1] \). In view of the relation \( G_1(t; \lambda) = G_2(t; \lambda) \), we achieve

\[ U[1] = \tilde{V} + [U[1], \tilde{V}]. \]

(2.13)

By comparing the coefficients of \( \lambda^k \), we get

\[ \lambda^2 : \quad P[1] = \frac{1}{2} [J, J P[1]], \]

\[ \lambda^1 : \quad \tilde{V}_1 = \frac{1}{2} (J, J P[1]) + (J P[1], \tilde{V}_1), \]

\[ \lambda^0 : \quad \tilde{V}_0 = i J P[1] + i J P[1], \]

(2.14)

Based on the first equation of (2.13), one can derive

\[ P_o^{off} = i J P[1], \]

(2.15)

where \( P_o^{off} \) represents the \( (1, 2), \ldots, (1, N + 1), (2, 1), \ldots, (N + 1, 1) \) elements for the matrix. Moreover, with the help of the second equation of (2.13), one can find that

\[ P_o^{diag} = i J P[1], \]

(2.16)
Without loss of generality, we can take \( g(t) = 0 \), and thus, we have \( \dot{V}_1 = iJ P[1] P + P[1] + P P[1] + 2i J P[1]^2 - J P[1]^{\ast} \). This completes the proof.

For the sake of easy calculation, we set \( |y_j| = \nu_j \phi_j \), where \( \nu_j = \nu_j(x, t, \lambda_j) \) is the appropriate function. Then the elementary Darboux matrix can be rewritten as

\[
T = I \gamma_{N+1} - \frac{\lambda_j - \lambda_i}{\lambda_j} |y_j| |y_i| S |y_j|^{\ast} |y_i|^{\ast}. \tag{2.17}
\]

It follows that \( P[1] \) can be written as

\[
P[1] = P - \frac{1}{2}(A - A^{\ast}) J |y_j| |y_i| S |y_j|^{\ast} |y_i|^{\ast} J. \tag{2.18}
\]

Through the standard iterated step for the above-mentioned Darboux matrix, we can establish a general \( n \)-fold Darboux matrix for the multi-component complex mKdV system (1.2).

**Theorem 2.** Suppose that \( \phi_j \) \( (i = 1, 2, \ldots, n) \) are \( n \) linearly independent solutions of the spectral problem (2.1), corresponding to \( \lambda = \lambda_j \), respectively, and denote \( |y_j| = \nu_j \phi_j \) and \( |y_i| = |y_i|^{\ast} \). Then, the \( n \)-fold Darboux matrix can be represented as

\[
T_n = I \gamma_{N+1} - Y W^{-1} |y_j|^2 |y_i|^2 W^{-1} Y, \tag{2.19}
\]

where \( Y = (|y_j|, |y_j|^2, \ldots, |y_j|^n) \), \( W = (\frac{|y_j|^n |y_i|^n}{\nu_j |y_i|^n}) \) loc, and \( G = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_j) \). The associated Bäcklund transformation for the potential matrix reads

\[
P[n] = P - \frac{1}{2} J Y W^{-1} Y W^{-1} S J, \tag{2.20}
\]

3. One- and multi-dark soliton solutions

In this section, we deduce general multi-dark soliton solutions for the complex mKdV system (1.2) through the presented DT and explore dynamics of single-dark–dark–dark and two-dark–dark–dark soliton solutions graphically.

We begin with a seed solution in the form of plane waves

\[
p_j = c_j e^{i \psi_j} \quad (j = 1, 2, \ldots, N), \tag{3.1}
\]

with

\[
\psi_j = a_j x + |a_j|^2 \sum_{j=1}^N \sigma_j c_j^2 + \sum_{j=1}^N \sigma_j |a_j c_j|^2 J, \tag{3.2}
\]

where \( a_j, c_j \) \((j = 1, 2, \ldots, N)\) are all real parameters, and \( \sigma_j = 1 \), when \( 1 \leq j \leq k \), \( \sigma_j = -1 \), when \( k + 1 \leq j \leq N \). In order to find the proper eigenfunction of the Lax pair (2.1), we make the gauge transformation

\[
\Phi = DHF, \tag{3.3}
\]

and then the linear system (2.1) becomes

\[
\Psi_i = \iota (1, e^{-i \theta_1}, e^{-i \theta_2}, \ldots, e^{-i \theta_N}), \tag{3.4}
\]

where \( \theta_j = a_j x + |a_j|^2 \sum_{j=1}^N \sigma_j c_j^2 + \sum_{j=1}^N \sigma_j |a_j c_j|^2 J \), \( \iota = \left( \begin{array}{c} 1 \iota \iota \iota \end{array} \right) \), \( \iota = \left( \begin{array}{c} \iota \iota \iota \iota \end{array} \right) \), \( \iota = \left( \begin{array}{c} \iota \iota \iota \iota \end{array} \right) \), \( \iota = \left( \begin{array}{c} \iota \iota \iota \iota \end{array} \right) \).

Substituting the seed solution (3.1) into the spectral problem (2.1), we can determine its fundamental solution as follows:

\[
\Phi = DHF. \tag{3.5}
\]
the intensity functions move at velocity \(-\eta_j\), to the right for \(\eta_j > 0\) and the left for \(\eta_j < 0\).

In addition, the trough of the single-dark soliton \(p_j[n]\) is along the line

\[ x + \eta_j t + \frac{\text{Im}\eta_j}{2\text{Im}(\chi)} = 0, \tag{3.8} \]

and the depth of cavity \(|p_j[n]|^2\) is

\[ c_j^2 \text{Im}^2(\chi) \left( \frac{\text{Re}(\chi_j) - a_j^2}{\eta_j} \right)^2 + \text{Im}(\chi_j). \]

Note that through the above expression, one can see the intensity dips at the centers of \(|p_j[n]|^2\) are characterized by the involved parameters of \(c_j, \eta_j\) and \(\chi_j\) and these parameters determine how dark the center is. In a similar way, we can also verify that \(|p_j[n]|^2\) approaches the constant amplitude \(|p_j|\) as \(t \to \pm \infty\) when \(x\) is the evolution variable. We find the velocity is \(-\eta_j\) and the center is still along the line given by Eq. (3.8) clearly.

In what follows, we take a system of three-component complex mKdV equations as an example to exhibit the dynamics property explicitly. If \(a_1 = a_2 = a_3\), then \(\varphi_1 = \varphi_2 = \varphi_3\), and hence the \(p_1[n], p_2[n]\) and \(p_3[n]\) components are proportional to each other, i.e. \(p_1[n]: p_2[n]: p_3[n] = c_1 : c_2 : c_3\). Indeed, dark–dark–dark soliton solutions of the three-component complex mKdV system are equivalent to a scalar dark soliton of the single-component complex mKdV system, and thus are degenerate. In order to obtain non-degenerate single-dark–dark–dark soliton solutions, we pick \(a_1 \neq a_2 \neq a_3\) to keep \(\varphi_1 \neq \varphi_2 \neq \varphi_3\), and thus the \(p_1[n], p_2[n]\) and \(p_3[n]\) components are not proportional to each other. Under this condition, they are not reducible to scalar single-dark soliton solutions in the system (1.2).

Due to the fact that degenerate and non-degenerate single-dark–dark–dark soliton solutions in the mixed focusing and defocusing case qualitatively resemble the defocusing type, here we merely consider the non-degenerate case of the defocusing system while making plots.

Solving the algebraic equation (3.5), we can obtain all of the parameters for the single dark soliton. For instance, we consider the three-component complex mKdV equation with the defocusing case (i.e. \(N = 3\), \(\sigma_1 = \sigma_2 = \sigma_3 = -1\)). We choose the parameters:

\[ a_1 = 0, \quad -\beta_1 = a_2 = a_3 = c_1 = c_2 = c_3 = 1, \quad \lambda_1 = 1/2, \quad \chi_1 \approx 0.301755798158048 - 1.54163830779653i. \tag{3.9} \]

The resulting single-dark–dark–dark soliton solutions are plotted in Fig. 1, which shows that solutions \(|p_1[n]|^2, |p_2[n]|^2, |p_3[n]|^2\) have different degrees of darkness at the centers and their trajectories are along the same line. Meanwhile, it indicates that the \(|p_3[n]|^2\) component is black, but the \(|p_j[n]|^2\) component is only gray at its center.

### 3.2. Multi-dark soliton solutions

In this subsection, we present multi-dark soliton solutions and their asymptotic analysis. Similar to the process of obtaining single-dark soliton solutions, we take

\[ |y_j| = D \begin{pmatrix} \frac{1}{\delta_0 - \delta_1} & \frac{1}{\delta_0 - \delta_2} \\ \vdots & \vdots \\ \frac{1}{\delta_0 - \delta_N} & \frac{1}{\delta_0 - \delta_{N+1}} \end{pmatrix} \begin{pmatrix} \delta_1 - \delta_0 & \cdots & \delta_{N+1} - \delta_0 \\ \cdots & \cdots & \cdots \\ \delta_1 - \delta_N & \delta_2 - \delta_N & \cdots & \delta_{N+1} - \delta_N \end{pmatrix} \begin{pmatrix} e^{i\kappa_3} \\ a_3(\lambda_3 - \lambda_j^*)e^{i\kappa_3} \\ \vdots \\ a_N(\lambda_N - \lambda_j^*)e^{i\kappa_3} \end{pmatrix}, \]

where

\[ \kappa_j = \chi_j x + (\chi_j^2 - 3 \chi_j) \sum_{n=1}^{N} \sigma_n c_j^2 t, \]

and \(a_j\) is a real constant. To obtain multi-dark soliton solutions, taking a limit process \(\lambda_i \rightarrow \lambda_j^* \quad (i = 1, 2, \ldots, n)\) is needed. Via the iterative algorithm, together with Theorem 2, general multi-dark vector soliton solutions of Eq. (1.2) can be represented as follows:

**Theorem 3.** The \(N\)-fold Darboux matrix for dark soliton solutions of the system (1.2) can be rewritten as

\[ T_n = I_{N+1} - Y \begin{pmatrix} \lambda \text{ln} \beta, 0, 0, \ldots, 0 \end{pmatrix} Y^{-1} \begin{pmatrix} \lambda \text{ln} \beta, 0, 0, \ldots, 0 \end{pmatrix}, \tag{3.10} \]

where

\[ Z = \begin{pmatrix} Z \end{pmatrix} = \begin{pmatrix} Z_1 \end{pmatrix} = \begin{pmatrix} \varphi_1 - \varphi_2, \varphi_1 - \varphi_3, \ldots, \varphi_1 - \varphi_N \end{pmatrix}^T, \]

and

\[ R = (e^{\varphi_1}, e^{\varphi_2}, \ldots, e^{\varphi_N}). \]

(3.11)

Graphs of two-dark–dark–dark soliton solutions are displayed in Fig. 2. We can see that after collision, the two solitons pass through each other without any change of shape, darkness or velocity in its three components, and hence there is no energy transfer between the two solitons after collision. In addition, since the parameters \(\beta_1\) and \(\beta_2\) determine the initial position of solitons, we can adjust different values to distinguish two solitons.

To elucidate the interaction of multi-component soliton solutions, by introducing the following Cauchy’s determinant identity

\[ C(x_1, x_2, \ldots, x_n) = \det \begin{pmatrix} (x_1 - x_2)^{-1} & \cdots & (x_1 - x_n)^{-1} \\ \cdots & \cdots & \cdots \\ (x_1 - x_n)^{-1} & \cdots & (x_1 - x_n)^{-1} \end{pmatrix} = \prod_{j<n} (x_j - x_n)^+ \prod_{j<n} (x_j - x_n)^- \]

we give an asymptotic property and its proof below.

**Proposition 1.** As \(t \to \pm \infty\), \(p_j[n]\) can be expressed as a sum of single-dark soliton solutions

\[ p_j[n] = c_j e^{\theta_j} [1 + S_j(\text{Im} \gamma_j)] \left( \frac{1}{x_j - x_n} \right)^{\text{Im} \gamma_j - \text{Im} \gamma_n}, \tag{3.12} \]

where

\[ \theta_j = \min(x_j) \min(\sigma_j) (|\kappa_j - \kappa_n|), \]

and

\[ S_j(\text{Im} \gamma_j) = \sum_{l=0}^{\text{Im} \gamma_j} \left( x_j^l - x_n^l \right)^{\text{Im} \gamma_j} \frac{\text{Im} \gamma_j}{x_j - x_n}, \]

and

\[ R = \sum_{j=1}^{N} \left( x_j^l - x_n^l \right)^{\text{Im} \gamma_j} \frac{\text{Im} \gamma_j}{x_j - x_n}, \]

\[ \Gamma_j = \sum_{j=1}^{N} \left( x_j^l - x_n^l \right)^{\text{Im} \gamma_j} \frac{\text{Im} \gamma_j}{x_j - x_n}, \]

and

\[ \Gamma_j = \sum_{j=1}^{N} \left( x_j^l - x_n^l \right)^{\text{Im} \gamma_j} \frac{\text{Im} \gamma_j}{x_j - x_n}, \]

(3.13)
Proof. We only consider the case $t \to -\infty$ here, and the asymptotical behavior as $t \to +\infty$ is similarly. Fix the value of $\text{Im}(\chi_i)$,

$$\text{Im}(\chi_i) = \text{Im}(\chi_i)(s + m_i t) = \text{const.}, \quad m_i = 3 \text{Re}^2(\chi_i) - \text{Im}^2(\chi_i) - 3 \sum_{i=1}^{N} c_i^2.$$  

(3.14)

Without loss of generality, we assume $\text{Im}(\chi_i) < 0$ and $m_1 < m_2 < \cdots < m_n$. From $\text{Im}(\chi_i) = \text{Im}(\chi_i)(s + m_i t + (m_i - m_j)r)$, we obtain $\text{Im}(\chi_i) \to -\infty$ for $1 \leq i \leq \tau - 1$ and $\text{Im}(\chi_i) \to +\infty$ for $\tau + 1 \leq i \leq n$. The determinants $Z$ and $Z_j$ of Theorem 3 are given explicitly by

$$\det(Z) = e^{2\text{Im}(\chi_i) s_n} e^{2\text{Re}(\chi_i) r_n} \det(Z_{\tau-1}) + O(e^{-\theta|m_i|}),$$

$$\det(Z_j) = e^{2\text{Im}(\chi_i) s_n} e^{2\text{Re}(\chi_i) r_n} \det(Z_{j\tau-1}) + O(e^{-\theta|m_i|}),$$

(3.15)

where

$$\det(Z) = \begin{vmatrix}
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\end{vmatrix}$$

$$\det(Z_j) = \begin{vmatrix}
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\end{vmatrix}$$

Finally, the asymptotic behavior when $t \to -\infty$ along $\theta$ can be represented as follows

$$\rho_n = \rho_n e^{\theta r_n} \left[1 + \frac{1}{2} \text{Re}(r_n) \text{Re}(x_n) + \frac{1}{2} \text{Re}(r_n) \text{Re}(x_n) + \frac{1}{2} \text{Re}(r_n) \text{Re}(x_n) + O(e^{-\theta|m_i|})\right].$$

(3.16)

The proof is completed. ■

4. Conclusions

In this paper, we have investigated a multi-component complex mKdV system and constructed a Darboux transformation for the system. With a nonzero seed solution, we have presented a single-dark soliton solution and multi-dark soliton solutions in the defocusing case and the mixed focusing and defocusing case. Moreover, we have shown that the energies in two components of the resulting solitons completely transmit through when those two solitons pass through each other. A detailed dynamical analysis has also been provided for the single-dark–dark–dark and two-dark–dark–dark soliton solutions. Based on
Fig. 2. Two-dark–dark–dark soliton solutions: Parameters $\sigma_1 = \sigma_2 = -\sigma_3 = \beta_1 = \beta_2 = a_2 = c_1 = c_2 = c_3 = 1, a_1 = \frac{1}{2}, a_3 = 3^2, \lambda_1 = -2, \lambda_2 = 0, \chi_1 = 1.3392493170325 - 0.369190128736683i, \chi_2 = 1.54234168261534 - 0.537284734536601i$. Upper row: $t$ is the evolution variable; lower row: $x$ is the evolution variable.

the compact determinant form of the solutions, we have performed an asymptotic analysis for the $n$-dark soliton solutions. The results would further enrich our understanding of dark solitons in integrable models. It is an interesting topic to study soliton solutions of other types for the coupled complex mKdV equation, including breathers and rogue waves.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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