Partial Differential Equations in Applied Mathematics 4 (2021) 100161

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

Darboux transformation and dark vector soliton solutions for complex mKdV systems

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ARTICLE INFO

Keywords:
Dark vector soliton
Darboux transformation
Complex mKdV system

ABSTRACT

A Darboux transformation and general vector dark soliton solutions are constructed for multi-component complex modified Korteweg–de Vries (mKdV) system. Dark soliton solutions exist when nonlinearities are either defocusing or mixed focusing and defocusing. Single-dark–dark–dark and two-dark–dark–dark soliton solutions of three component complex mKdV system are explicitly presented, whose properties and dynamics are illustrated. It is also shown during the interaction between those solitons, energies in different components completely transmit through. Moreover, we perform an asymptotic analysis for the n-dark soliton solutions rigorously. Our results can be applicable to the study of dark solitons in different physical fields.

1. Introduction

During the past decades, it has been a difficult and significant problem to construct exact solutions to nonlinear systems of differential equations. Among integrable systems, the nonlinear Schrödinger (NLS) equation[1–5] has been recognized as an omnipresent mathematical model, which can be adopted to describe the dynamics of localized waves in many physical fields, such as Bose–Einstein condensate, nonlinear fibers, plasma physics, etc. Another interesting example is the mKdV equation, which has been studied extensively, due to its simplicity and physical essentiality. Some remarkable progresses are generalizations of the 2×2 AKNS matrix eigenvalue problem to 3×3, even more generally $n \times n$, in the recent literature.[6–9] In 1997, Iwao and Hirota[10] proposed a coupled version of the mKdV equation and constructed multi-soliton solutions of that system by using the Hirota bilinear method. Later, Tsuchida and Wadati[11] introduced a Lax representation for the matrix mKdV equation. Meanwhile, they presented the matrix NLS equation and performed an extension of the inverse scattering transformation to solve both the matrix mKdV equation and the matrix NLS equation.

Recently, one of the authors (Ma)[12,13] considered a 3×3 matrix spatial spectral problem and rederived the AKNS soliton hierarchy with four components. A typical nonlinear system in the corresponding soliton hierarchy is

$$
\begin{aligned}
&p_{j,t} + \frac{\beta}{\alpha^3}[p_{j,xxx} + 3(p_1q_1 + p_2q_2)p_{j,x} + 3(p_{1,x}q_1 + p_{2,x}q_2)p_j] = 0,\\
&q_{j,t} + \frac{\beta}{\alpha^3}[q_{j,xxx} + 3(p_1q_1 + p_2q_2)q_{j,x} + 3(p_1q_{1,x} + p_2q_{2,x})q_j] = 0,\\
&\quad 1 \le j \le 2,
\end{aligned}
\tag{1.1}
$$

which contains various mKdV equations, and constructed the multiple soliton solutions through a specific Riemann–Hilbert problem with an identity jump matrix.

In this work, we would like to discuss the following general multi-component complex mKdV integrable system

$$
\mathbf{p_t} + \mathbf{p_{xxx}} + 3\mathbf{p_x}\mathbf{p}^\dagger \mathbf{Y}\mathbf{p} + 3\mathbf{p}\mathbf{p}^\dagger \mathbf{Y}\mathbf{p_x} = \mathbf{0}, \tag{1.2}
$$

where

$$
\mathbf{p} = (p_1, p_2, \dots, p_N)^T, \quad Y = diag(y_1, y_2, \dots, y_N),
$$
$$
y_i = \begin{cases} 1, & 1 \le i \le k, \\ -1, & k+1 \le i \le N, \end{cases}
$$

where the symbol T denotes the transpose. It should be noted that when $k = 0$ and $Y = -I_N$, this system is the defocusing model that supports n-dark vector soliton solutions; when $k = N$ and $Y = I_N$, this system becomes the focusing model that supports n-bright soliton solutions; when $1 \le k \le N-1$, this system is the mixed focusing and defocusing system that supports n-dark vector soliton solutions, where I_N is the $N \times N$ identity matrix.

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https://doi.org/10.1016/j.padiff.2021.100161
Received 27 March 2021; Received in revised form 10 September 2021; Accepted 30 September 2021

The Darboux transformation (DT) is a powerful method to construct exact solutions for integrable systems.[14–18] In the present paper, by iterating a standard DT, dark vector soliton solutions for system (1.2) with the defocusing case and the mixed focusing and defocusing case are derived.

A brief outline of this paper is as follows. In Section 2, we present a Lax pair and construct a DT for the above multi-component complex mKdV system. In Section 3, we choose some plane-wave solutions as seed solutions to derive explicit formulas for one- and multi-dark soliton solutions in the defocusing case and the mixed focusing and defocusing case by a limiting process. Based on the obtained multi-dark soliton solutions, we analyze dynamics of single-dark–dark–dark and two-dark–dark–dark soliton solutions of the three-component system through detailed examination. The results are then summarized in Section 4.

2. Darboux transformation

In this section, we would like to construct a DT for the multi-component complex mKdV system (1.2). Firstly, we declare that the system (1.2) has the following Lax pair

$$\begin{aligned} \Phi_x &= U(\lambda;P)\Phi, \quad U(\lambda;P)=\frac{1}{2}i\lambda J+iJP,\\ \Phi_t &= V(\lambda;P)\Phi, \quad V(\lambda;P)=\frac{1}{2}i\lambda^3 J+i\lambda^2 JP+\lambda V_1+V_0, \end{aligned} \tag{2.1}$$

with

$$V_1=iJP^2+P_x, \quad V_0=P_xP-PP_x+2iJP^3-iJP_{xx},$$

$$J=\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0}^T & -I_N\end{pmatrix}, \quad P=\begin{pmatrix}0 & \mathbf{p}^T\\ -Y\mathbf{p}^* & \mathbf{O}\end{pmatrix},$$

where $\Phi=(\phi_1,\phi_2,\dots,\phi_{N+1})^T$ is a vector eigenfunction, λ is the spectral parameter, $\mathbf{0}$ is the $1\times N$ zero vector and $\mathbf{O}$ is the $N\times N$ zero matrix. It is straightforward to check that the compatibility condition $U_t-V_x+[U,V]=0$ exactly gives rise to the system (1.2).

It is evidently that the matrices $U(\lambda;P)$ and $V(\lambda;P)$ possess the reduction condition

$$U^\dagger(\lambda;P)=-SU(\lambda^*;P)S, \quad V^\dagger(\lambda;P)=-SV(\lambda^*;P)S, \tag{2.2}$$

where $\dagger$ denotes the Hermitian conjugation, and $S=diag(s_1,s_2,\dots,s_{N+1})$ with $s_i=1$ for $1\le i\le k+1$ and $s_i=-1$ for $k+2\le i\le N+1$.

The adjoint problem of the Lax pair (2.1) reads

$$\Psi_x=-\Psi U(\lambda;P), \quad \Psi_t=-\Psi V(\lambda;P), \tag{2.3}$$

where $\Psi=(\psi_1,\psi_2,\dots,\psi_{N+1})$. Due to the relation (2.2), a simple observation is that $\Phi^\dagger S$ satisfies the adjoint problem (2.3). Thus, through the loop group method,[19] we can find a Darboux matrix as follows:

$$T=I_{N+1}-\frac{\lambda_1-\lambda_1^*}{\lambda-\lambda_1^*}M, \quad M=\frac{\Phi_1\Phi_1^\dagger S}{\Phi_1^\dagger S\Phi_1}, \tag{2.4}$$

where Φ_1 is a special solution for the Lax pair (2.1) at $\lambda=\lambda_1$ and I_{N+1} is the $(N+1)\times(N+1)$ identity matrix.

Theorem 1. *The eigenfunction transformation*

$$\Phi(\lambda,P)\to\Phi[1](\lambda,P[1])=T\Phi(\lambda,P),$$

where T is defined by (2.4), converts the system (1.2) into a new system

$$\begin{aligned} \Phi[1]_x &= U[1]\Phi[1],\\ \Phi[1]_t &= V[1]\Phi[1], \end{aligned} \tag{2.5}$$

where $U[1]=U(\lambda;P[1])$ and $V[1]=V(\lambda;P[1])$ with

$$P[1]=\begin{pmatrix}0 & \mathbf{p[1]}^T\\ -Y\mathbf{p[1]}^* & \mathbf{O}\end{pmatrix}, \mathbf{p[1]}=(p_1[1],p_2[1],\dots,p_N[1])^T, \tag{2.6}$$

and the associated Bäcklund transformation for the potential matrix is

$$P[1]=P-\frac{1}{2}(\lambda_1-\lambda_1^*)[J,MJ]. \tag{2.7}$$

Proof. We merely need to prove

$$T_xT^{-1}+TU(\lambda)T^{-1}=U[1], \tag{2.8a}$$

$$T_tT^{-1}+TV(\lambda)T^{-1}=V[1], \tag{2.8b}$$

Firstly, we consider Eq. (2.8a) and use the residue analysis method to verify it. Expanding T and T^{-1} as the power series of λ at $\lambda=\infty$:

$$\begin{aligned} T &= I_{N+1}-M(\lambda_1-\lambda_1^*)(\frac{1}{\lambda}+\frac{\lambda_1^*}{\lambda^2}+\frac{\lambda_1^{*2}}{\lambda^3}+\cdots),\\ T^{-1} &= I_{N+1}+M(\lambda_1-\lambda_1^*)(\frac{1}{\lambda}+\frac{\lambda_1}{\lambda^2}+\frac{\lambda_1^2}{\lambda^3}+\cdots). \end{aligned} \tag{2.9}$$

Let

$$G_1(x,t;\lambda)=T_xT^{-1}+TU(\lambda)T^{-1}-U[1]. \tag{2.10}$$

Then, we want to show that the matrix function $G_1(x,t;\lambda)$ is holomorphic on the compact Riemann surface $S^2=\mathbb{C}\cup\{\infty\}$. Obviously, the residues for the matrix function $G_1(x,t;\lambda)$ at $\lambda=\lambda_1^*$ and $\lambda=\lambda_1$ can be computed as follows:

$$Res_{\lambda=\lambda_1^*}G_1(x,t;\lambda)=(\lambda_1^*-\lambda_1)[M_xT^{-1}+MU(\lambda)T^{-1}]|_{\lambda=\lambda_1^*}=0,$$

and

$$Res_{\lambda=\lambda_1}G_1(x,t;\lambda)=(\lambda_1-\lambda_1^*)[-M_xM+TU(\lambda)M]|_{\lambda=\lambda_1}=0.$$

Moreover, we can calculate the residue for the matrix function $G_1(x,t;\lambda)$ at $\lambda=\infty$ as follows:

$$G_1(x,t;\infty)=iJP-iJP[1]+\frac{1}{2}i(\lambda_1-\lambda_1^*)(JM-MJ)=0.$$

Therefore, $G_1(x,t;\lambda)$ is a holomorphic function on S^2. It follows from the above asymptotical behavior analysis that $G_1(x,t;\lambda)=0$. So Eq. (2.8a) holds. On the other hand, we continue to discuss the time evolution part (2.8b). A direct analysis can show that

$$G_2(x,t;\lambda)=T_tT^{-1}+TV(\lambda)T^{-1}-\hat{V}=0, \tag{2.11}$$

where

$$\begin{aligned} \hat{V} &= \frac{1}{2}i\lambda^3J+i\lambda^2JP[1]+\lambda\hat{V}_1+\hat{V}_0,\\ \hat{V}_1 &= iJP^2+P_x+\frac{1}{2}i(\lambda_1^*-\lambda_1)(\lambda_1^*MJ-\lambda_1JM)+i(\lambda_1^*-\lambda_1)[M,JP],\\ \hat{V}_0 &= P_xP-PP_x+2iJP^3-iJP_{xx}+i(\lambda_1^*-\lambda_1)[M,JP^2]+(\lambda_1^*-\lambda_1)[M,P_x]\\ &\quad+i(\lambda_1^*-\lambda_1)(\lambda_1^*MJP-\lambda_1JPM)+\frac{1}{2}i(\lambda_1^*-\lambda_1)(\lambda_1^{*2}MJ-\lambda_1^2JM). \end{aligned}$$

Hence, our aim is turned to verify that $\hat{V}=V[1]$. In view of the relation $G_1(x,t;\lambda)=G_2(x,t;\lambda)=0$, we achieve

$$U[1]_t-\hat{V}_x+[U[1],\hat{V}]=0. \tag{2.12}$$

By comparing the coefficients of λ, we get

$$\begin{aligned} \lambda^2:&\quad P[1]_x=\frac{1}{2}[J,J\hat{V}_1],\\ \lambda^1:&\quad \hat{V}_{1,x}=\frac{1}{2}i[J,\hat{V}_0]+i[JP[1],\hat{V}_1],\\ \lambda^0:&\quad \hat{V}_{0,x}=iJP[1]_t+i[JP[1],\hat{V}_0]. \end{aligned} \tag{2.13}$$

Based on the first equation of (2.13), one can derive

$$\hat{V}_1^{\,off}=P[1]_x, \tag{2.14}$$

where ^{off} represents the $(1,2)$, …, $(1,N+1)$, $(2,1)$, …, $(N+1,1)$ elements for the matrix. Moreover, with the help of the second equation of (2.13), one can find that

$$\hat{V}_{1,x}^{diag}=i[JP[1],P[1]_x], \tag{2.15}$$

where ^{diag} denotes the other elements except the ^{off} ones. Thus, one has

$$\hat{V}_1^{\,diag}=iJP[1]^2+g(t). \tag{2.16}$$

Without loss of generality, we can take $g(t) = 0$, and thus, we have $\hat{V}_1 = iJP[1]^2 + P[1]_x$. Similarly, we have $\hat{V}_0 = P[1]_x P[1] - P[1]P[1]_x + 2iJP[1]^3 - iJP[1]_{xx}$. This completes the proof. ■

For the sake of easy calculation, we set $|y_j\rangle = v_j\Phi_j$, where $v_j = v_j(x, t, \lambda_j)$ is the appropriate function. Then the elementary Darboux matrix can be rewritten as

$$T = I_{N+1} - \frac{\lambda_1 - \lambda_1^*}{\lambda - \lambda_1^*}\frac{|y_1\rangle\langle y_1|S}{\langle y_1|S|y_1\rangle}, \quad \langle y_1| = |y_1\rangle^{\dagger}. \tag{2.17}$$

It follows that $P[1]$ can be written as

$$P[1] = P - \frac{1}{2}(\lambda_1 - \lambda_1^*)[J, \frac{|y_1\rangle\langle y_1|S}{\langle y_1|S|y_1\rangle}J]. \tag{2.18}$$

Through the standard iterated step for the above-mentioned Darboux matrix, we can establish a general n-fold Darboux matrix for the multi-component complex mKdV system (1.2).

Theorem 2. *Suppose that Φ_i $(i = 1, 2, \ldots, n)$ are n linearly independent solutions of the spectral problem (2.1), corresponding to $\lambda = \lambda_i$, respectively, and denote $|y_i\rangle = v_i\Phi_i$ and $\langle y_i| = |y_i\rangle^{\dagger}$. Then, the n-fold Darboux matrix can be represented as*

$$T_n = I_{N+1} - YW^{-1}(\lambda I_n - G)^{-1}Y^{\dagger}S, \tag{2.19}$$

where $Y = (|y_1\rangle, |y_2\rangle, \ldots, |y_n\rangle)$, $W = [\frac{\langle y_i|S|y_j\rangle}{\lambda_j - \lambda_i^*}]_{n\times n}$, and $G = diag(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$. The associated Bäcklund transformation for the potential matrix reads

$$P[n] = P - \frac{1}{2}[J, YW^{-1}Y^{\dagger}SJ]. \tag{2.20}$$

3. One- and multi-dark soliton solutions

In this section, we derive general multi-dark soliton solutions for the complex mKdV system (1.2) through the presented DT and explore dynamics of single-dark–dark–dark and two-dark–dark–dark soliton solutions graphically.

We begin with a seed solution in the form of plane waves

$$p_j = c_j e^{i\theta_j} \quad (j = 1, 2, \ldots, N), \tag{3.1}$$

with

$$\theta_j = a_j x + [a_j^3 - 3a_j\sum_{l=1}^{N}\sigma_l c_l^2 - 3\sum_{l=1}^{N}\sigma_l a_l c_l^2]t,$$

where a_j, c_j $(j = 1, 2, \ldots, N)$ are all real parameters, and $\sigma_l = 1$, when $1 \le l \le k$, $\sigma_l = -1$, when $k + 1 \le l \le N$. In order to find the proper eigenfunction of the Lax pair (2.1), we make the gauge transformation

$$\Phi = D\Psi, \quad D = diag(1, e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_N}), \tag{3.2}$$

and then the linear system (2.1) becomes

$$\begin{aligned} \Psi_x &= i\tilde{U}\Psi, \\ \Psi_t &= i[\tilde{U}^3 + \frac{3}{2}\lambda\tilde{U}^2 + (\frac{3}{4}\lambda^2 - 3\sum_{l=1}^{N}\sigma_l c_l^2)\tilde{U} - (\frac{3}{8}\lambda^3 + \frac{3}{2}\lambda\sum_{l=1}^{N}\sigma_l c_l^2 \\ &\quad + 3\sum_{l=1}^{N}\sigma_l a_l c_l^2)I_{N+1}]\Psi, \end{aligned} \tag{3.3}$$

where

$$\tilde{U} = \begin{pmatrix} \frac{1}{2}\lambda & c^T \\ Yc & -\frac{1}{2}\lambda I_N + A \end{pmatrix}, \quad c = (c_1, c_2, \ldots, c_N)^T,$$

$$A = diag(a_1, a_2, \ldots, a_N).$$

Substituting the seed solution (3.1) into the spectral problem (2.1), we can determine its fundamental solution as follows:

$$\Phi = DHF, \tag{3.4}$$

where

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{c_1}{\chi_1 - a_1} & \frac{c_1}{\chi_2 - a_1} & \cdots & \frac{c_1}{\chi_{N+1} - a_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{c_k}{\chi_1 - a_k} & \frac{c_k}{\chi_2 - a_k} & \cdots & \frac{c_k}{\chi_{N+1} - a_k} \\ \frac{-c_{k+1}}{\chi_1 - a_{k+1}} & \frac{-c_{k+1}}{\chi_2 - a_{k+1}} & \cdots & \frac{-c_{k+1}}{\chi_{N+1} - a_{k+1}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{-c_N}{\chi_1 - a_N} & \frac{-c_N}{\chi_2 - a_N} & \cdots & \frac{-c_N}{\chi_{N+1} - a_N} \end{pmatrix},$$

$$F = diag(e^{i\xi(\chi_1)}, e^{i\xi(\chi_2)}, \ldots, e^{i\xi(\chi_{N+1})}),$$

$$\xi(z) = (z - \frac{1}{2}\lambda)x + (z^3 - 3z\sum_{l=1}^{N}\sigma_l c_l^2 - 3\sum_{l=1}^{N}\sigma_l a_l c_l^2 - \frac{1}{2}\lambda^3)t,$$

and $\chi_1, \chi_2, \ldots, \chi_{N+1}$ are $N+1$ distinct roots of the following $(N+1)$-th order algebraic equation

$$\prod_{l=1}^{N}(\chi - a_l)(\chi - \lambda - \sum_{l=1}^{N}\frac{\sigma_l c_l^2}{\chi - a_l}) = 0. \tag{3.5}$$

It is readily to see when $\sigma_l = 1$ $(1 \le l \le N)$, which corresponds to the focusing case, the constraint (3.5) cannot be satisfied, and thus n-dark vector soliton solutions cannot exist. When $\sigma_l = -1$ $(1 \le l \le N)$, which corresponds to the defocusing type, the constraint (3.5) can be satisfied, and thus n-dark soliton solutions can exist. When the system (1.2) has the mixed focusing and defocusing nonlinearities, the constraint (3.5) still can be satisfied, and hence, n-dark vector soliton solutions can exist. This phenomenon will be demonstrated in more detail in the next section.

3.1. Single-dark soliton solutions

Taking $v_1 = e^{i[\frac{1}{2}\lambda_1 x + (3\sum_{l=1}^{N}\sigma_l a_l c_l^2 + \frac{1}{2}\lambda_1^3)t]}$, we can choose

$$|y_1\rangle = D\begin{pmatrix} 1 & 1 \\ \frac{c_1}{\chi_1 - a_1} & \frac{c_1}{\chi_1^* - a_1} \\ \vdots & \vdots \\ \frac{c_k}{\chi_1 - a_k} & \frac{c_k}{\chi_1^* - a_k} \\ \frac{-c_{k+1}}{\chi_1 - a_{k+1}} & \frac{-c_{k+1}}{\chi_1^* - a_{k+1}} \\ \vdots & \vdots \\ \frac{-c_N}{\chi_1 - a_N} & \frac{-c_N}{\chi_1^* - a_N} \end{pmatrix}\begin{pmatrix} e^{i\kappa_1} \\ \alpha_1(\lambda_1 - \lambda_1^*)e^{i\kappa_1^*} \end{pmatrix}, \tag{3.6}$$

where α_1 is a real constant and $\kappa_1 = \chi_1 x + (\chi_1^3 - 3\chi_1\sum_{l=1}^{N}\sigma_l c_l^2)t$. In order to derive single-dark soliton solutions,[3,20] we need to take a limit process $\lambda_1 \to \lambda_1^*$. After tedious calculations, we can have $\frac{\langle y_1|S|y_1\rangle}{\lambda_1 - \lambda_1^*} = \frac{e^{-2Im(\kappa_1)} + \beta_1}{\chi_1 - \chi_1^*}$, where $\beta_1 = -4\alpha_1 Im(\chi_1)Im(\sum_{l=1}^{N}\frac{\sigma_l c_l^2}{(\chi_1^* - a_l)^2}) > 0$.

Based on Eq. (2.18), the corresponding single-dark soliton solution reads

$$p_j[1] = p_j[1 + \frac{\chi_1 - \chi_1^*}{2(\chi_1^* - a_j)} - \frac{\chi_1 - \chi_1^*}{2(\chi_1^* - a_j)}tanh(X_1)], \quad (j = 1, 2, \ldots, N), \tag{3.7}$$

with

$$X_1 = Im(\chi_1)(x + \eta_1 t) + \frac{1}{2}ln\beta_1,$$

$$\eta_1 = 3Re^2(\chi_1) - Im^2(\chi_1) - 3\sum_{l=1}^{N}\sigma_l c_l^2.$$

In the following, we investigate the asymptotic property of the solution. We assume $Im(\chi_1) < 0$ without loss of generality. When t is regarded as an evolution variable, we have $|p_j[1]| \to |p_j|$ as $x \to -\infty$, whereas one finds $p_j[1] \to \frac{\chi_1 - a_j}{\chi_1^* - a_j}p_j$ as $x \to +\infty$. Since $|\frac{\chi_1 - a_j}{\chi_1^* - a_j}| = 1$, one has $|p_j[1]| \to |p_j|$ as $x \to \pm\infty$. When x moves from $-\infty$ to $+\infty$, the phases of the components $p_j[1]$ acquire some shifts in the amount of φ_j, where φ_j is the constant $-iln(\frac{\chi_1 - a_j}{\chi_1^* - a_j})$. It is important to notice that the intensity functions move at velocity $-\eta_1$, to the right for $\eta_1 > 0$ and the left for $\eta_1 < 0$.

In addition, the trough of the single-dark soliton $p_j[1]$ is along the line

$$x + \eta_1 t + \frac{ln\beta_1}{2Im(\chi_1)} = 0, \tag{3.8}$$

and the depth of cavity $|p_j[1]|^2$ is

$$\frac{c_j^2 Im^2(\chi_1)}{(Re(\chi_1) - a_j)^2 + Im^2(\chi_1)}.$$

Note that through the above expression, one can see the intensity dips at the centers of $|p_j[1]|$ are characterized by the involved parameters of c_j, a_j and χ_1 and these parameters determine how dark the center is. In a similar way, we can also verify that $|p_j[1]|$ approaches the constant amplitude $|p_j|$ as $t \to \pm\infty$ when x is the evolution variable. We find the velocity is $-\frac{1}{\eta_1}$ and the center is still along the line given by Eq. (3.8) clearly.

In what follows, we take a system of three-component complex mKdV equations as an example to exhibit the dynamics property explicitly. If $a_1 = a_2 = a_3$, then $\varphi_1 = \varphi_2 = \varphi_3$, and hence the $p_1[1]$, $p_2[1]$ and $p_3[1]$ components are proportional to each other, i.e. $p_1[1] : p_2[1] : p_3[1] = c_1 : c_2 : c_3$. Indeed, dark–dark–dark soliton solutions of the three-component complex mKdV system are equivalent to a scalar dark soliton of the single-component complex mKdV system, and thus are degenerate. In order to obtain non-degenerate single-dark–dark–dark soliton solutions, we pick $a_1 \neq a_2 \neq a_3$ to keep $\varphi_1 \neq \varphi_2 \neq \varphi_3$, and thus the $p_1[1]$, $p_2[1]$ and $p_3[1]$ components are not proportional to each other. Under this condition, they are not reducible to scalar single-dark soliton solutions in the system (1.2).

Due to the fact that degenerate and non-degenerate single-dark–dark–dark soliton solutions in the mixed focusing and defocusing case qualitatively resemble the defocusing type, here we merely consider the non-degenerate case of the defocusing system while making plots.

Solving the algebraic equation (3.5), we can obtain all of the parameters for the single dark soliton. For instance, we consider the three-component complex mKdV equation with the defocusing case (i.e. $N = 3$, $\sigma_1 = \sigma_2 = \sigma_3 = -1$). We choose the parameters:

$$\begin{aligned} &a_1 = 0, \quad -\beta_1 = -a_2 = a_3 = -c_1 = -c_2 = -c_3 = -1, \\ &\lambda_1 = \frac{1}{2}, \quad \chi_1 \approx 0.301755798158048 - 1.54163830779653i. \end{aligned} \tag{3.9}$$

The resulting single-dark–dark–dark soliton solutions are plotted in Fig. 1, which shows that solutions $|p_1[1]|, |p_2[1]|, |p_3[1]|$ have different degrees of darkness at the centers and their trajectories are along the same line. Meanwhile, it indicates that the $|p_1[1]|$ component is black, but the $|p_3[1]|$ component is only gray at its center.

3.2. Multi-dark soliton solutions

In this subsection, we present multi-dark soliton solutions and their asymptotic analysis. Similar to the process of obtaining single-dark soliton solutions, we take

$$|y_\iota\rangle = D\begin{pmatrix} 1 & 1 \\ \frac{c_1}{\chi_\iota - a_1} & \frac{c_1}{\chi_\iota^* - a_1} \\ \vdots & \vdots \\ \frac{c_k}{\chi_\iota - a_k} & \frac{c_k}{\chi_\iota^* - a_k} \\ \frac{-c_{k+1}}{\chi_\iota - a_{k+1}} & \frac{-c_{k+1}}{\chi_\iota^* - a_{k+1}} \\ \vdots & \vdots \\ \frac{-c_N}{\chi_\iota - a_N} & \frac{-c_N}{\chi_\iota^* - a_N} \end{pmatrix} \begin{pmatrix} e^{i\kappa_\iota} \\ \alpha_\iota(\lambda_\iota - \lambda_\iota^*)e^{i\kappa_\iota^*} \end{pmatrix},$$

where

$$\kappa_\iota = \chi_\iota x + (\chi_\iota^3 - 3\chi_\iota \sum_{l=1}^{N} \sigma_l c_l^2)t,$$

and α_ι is a real constant. To obtain multi-dark soliton solutions, taking a limit process $\lambda_\iota \to \lambda_\iota^*$ ($\iota = 1, 2\cdots, n$) is needed. Via the iterative algorithm, together with Theorem 2, general multi-dark vector soliton solutions of Eq. (1.2) can be represented as follows:

Theorem 3. *The n-fold Darboux matrix for dark soliton solutions of the system (1.2) can be rewritten as*

$$\begin{aligned} &T_n = I_{N+1} - YZ^{-1}(\lambda I_n - G)^{-1}Y^\dagger S, \\ &p_j[n] = c_j e^{i\theta_j}\frac{|Z_j|}{|Z|}, \quad (j = 1, 2, \dots, N), \end{aligned} \tag{3.10}$$

where

$$Z_j = \begin{pmatrix} Z & N_j^\dagger \\ R & 1 \end{pmatrix}, \quad N_j = (\frac{-e^{i\kappa_1}}{\chi_1 - a_j}, \frac{-e^{i\kappa_2}}{\chi_2 - a_j}, \dots, \frac{-e^{i\kappa_n}}{\chi_n - a_j}),$$

$$R = (e^{i\kappa_1}, e^{i\kappa_2}, \dots, e^{i\kappa_n}),$$

and

$$Z = (\frac{e^{i(\kappa_j - \kappa_i^*)} + \delta_{ij}\beta_j}{\chi_j - \chi_i^*})_{1\le i,j\le n}, \quad \beta_i = -4\alpha_i Im(\chi_i) Im(\sum_{l=1}^{N}\frac{\sigma_l c_l^2}{(\chi_i^* - a_l)^2}) > 0,$$

δ_{ij} being the Kronecker's delta.

Based on the n-dark soliton solution (3.10), we investigate the collision dynamics between two vector dark soliton solutions for the three-component complex mKdV system. For an illustrative purpose, we investigate the three-component complex mKdV equation with the mixed of focusing and defocusing case (i.e. $N = 3$, $\sigma_1 = \sigma_2 = -\sigma_3 = 1$). By solving the algebraic equation (3.5), the parameters are chosen as follows:

$$\begin{aligned} &a_1 = \frac{1}{2}, a_3 = \frac{3}{2}, \quad \beta_1 = \beta_2 = a_2 = c_1 = c_2 = c_3 = 1, \quad \lambda_1 = -2, \lambda_2 = 0, \\ &\chi_1 \approx 1.33924933170325 - 0.369190128736683i, \\ &\chi_2 \approx 1.54243168261534 - 0.537284734536601i. \end{aligned} \tag{3.11}$$

Graphs of two-dark–dark–dark soliton solutions are displayed in Fig. 2. We can see that after collision, the two solitons pass through each other without any change of shape, darkness or velocity in its three components, and hence there is no energy transfer between the two solitons after collision. In addition, since the parameters β_1 and β_2 determine the initial position of solitons, we can adjust different values to distinguish two solitons.

To elucidate the interaction of multi-component soliton solutions, by introducing the following Cauchy's determinant identity

$$C(\chi_1, \chi_2, \dots, \chi_n) = det(\frac{1}{\chi_i - \chi_j^*})_{1\le i,j\le n} = \frac{\prod_{i=2}^{n}\prod_{j=1}^{i-1}(\chi_i - \chi_j)(\chi_j^* - \chi_i^*)}{\prod_{i=1}^{n}\prod_{j=1}^{n}(\chi_i - \chi_j^*)}, \tag{3.12}$$

we give an asymptotic property and its proof below.

Proposition 1. *As $t \to \pm\infty$, $p_j[n]$ can be expressed as a sum of single-dark soliton solutions*

$$p_j[n] = c_j e^{i\theta_j}[1 + S_1^{[j]\pm} + (S_2^{[j]\pm} - G_1^{[j]\pm}) + \cdots + (S_n^{[j]\pm} - G_{n-1}^{[j]\pm})] + O(e^{-\delta|t|}), \tag{3.13}$$

where $\delta = min(|\chi_i|)min_{i\neq j}(|\kappa_i - \kappa_j|)$, and

$$S_\tau^{[j]-} = \frac{\sum_{l=1}^{\tau}(-1)^{\tau+1}\frac{1}{\chi_l^* - a_j}e^{i(\kappa_\tau - \kappa_\tau^*)}A(\tau, l) + \sum_{l=1}^{\tau-1}(-1)^{\tau}\frac{1}{\chi_l^* - a_j}\frac{\beta_\tau}{\chi_\tau - \chi_\tau^*}A(\tau - 1, l)}{e^{i(\kappa_\tau - \kappa_\tau^*)}C(\chi_1, \chi_2, \dots, \chi_\tau) + \frac{\beta_\tau}{\chi_\tau - \chi_\tau^*}C(\chi_1, \chi_2, \dots, \chi_{\tau-1})},$$

$$S_\tau^{[j]+} = \frac{\sum_{l=\tau}^{n}(-1)^{n-\tau}\frac{1}{\chi_l^* - a_j}e^{i(\kappa_\tau - \kappa_\tau^*)}D(\tau, l) + \sum_{l=\tau+1}^{n}(-1)^{n-\tau+1}\frac{1}{\chi_l^* - a_j}\frac{\beta_\tau}{\chi_\tau - \chi_\tau^*}D(\tau + 1, l)}{e^{i(\kappa_\tau - \kappa_\tau^*)}C(\chi_\tau, \chi_{\tau+1}, \dots, \chi_n) + \frac{\beta_\tau}{\chi_\tau - \chi_\tau^*}C(\chi_{\tau+1}, \chi_{\tau+2}, \dots, \chi_n)},$$

$$G_\tau^{[j]-} = \frac{\sum_{l=1}^{\tau}(-1)^{\tau+1}\frac{1}{\chi_l^* - a_j}A(\tau, l)}{C(\chi_1, \chi_2, \dots, \chi_\tau)}, \quad G_\tau^{[j]+} = \frac{\sum_{l=\tau+1}^{n}(-1)^{n-\tau+1}\frac{1}{\chi_l^* - a_j}D(\tau + 1, l)}{C(\chi_{\tau+1}, \chi_{\tau+2}, \dots, \chi_n)},$$

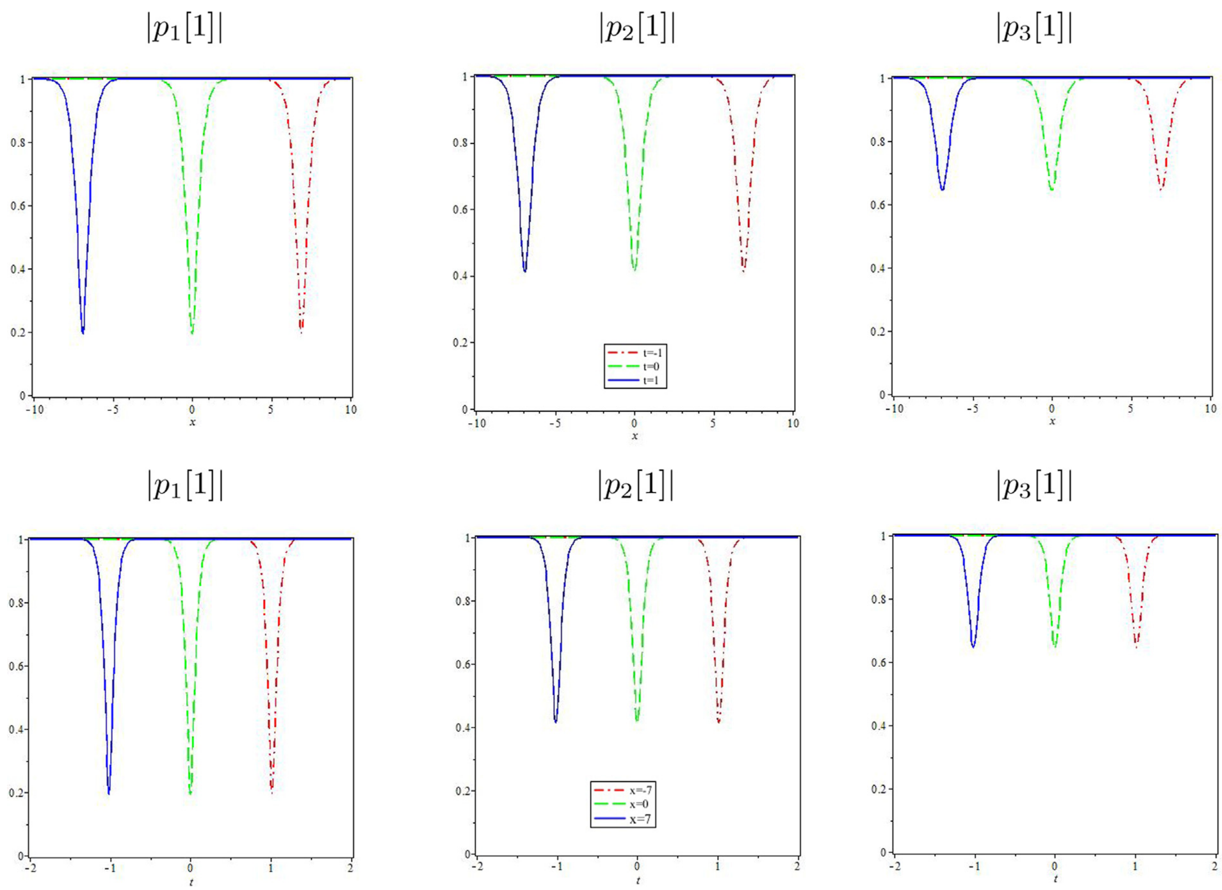

Fig. 1. Single-dark–dark–dark soliton solutions: Parameters $\sigma_1 = \sigma_2 = \sigma_3 = -\beta_1 = -a_2 = a_3 = -c_1 = -c_2 = -c_3 = -1, a_1 = 0, \lambda_1 = \frac{1}{2}, \chi_1 = 0.301755798158048 - 1.54163830779653i$. Upper row: t is the evolution variable; lower row: x is the evolution variable.

$$A(\tau,l) = C(\chi_1,\chi_2,\dots,\chi_n)\frac{\prod_{j=1}^{\tau}(\chi_j-\chi_l^*)}{\prod_{j=1,j\neq l}^{\tau}(\chi_l^*-\chi_j^*)},$$
$$D(\tau,l) = C(\chi_\tau,\chi_{\tau+1},\dots,\chi_n)\frac{\prod_{j=\tau}^{n}(\chi_j-\chi_l^*)}{\prod_{j=\tau,j\neq l}^{n}(\chi_l^*-\chi_j^*)}.$$

Proof. We only consider the case $t \to -\infty$ here, and the asymptotical behavior as $t \to +\infty$ is similarly. Fix the value of $Im(\kappa_\tau)$,

$$Im(\kappa_\tau) = Im(\chi_\tau)(x + m_\tau t) = const, \quad m_\tau = 3Re^2(\chi_\tau) - Im^2(\chi_\tau) - 3\sum_{l=1}^{N}\sigma_l c_l^2. \tag{3.14}$$

Without loss of generality, we assume $Im(\chi_\iota) < 0$ and $m_1 < m_2 < \cdots < m_n$. From $Im(\kappa_\iota) = Im(\chi_\iota)(x + m_\tau t + (m_\iota - m_\tau)t)$, we obtain $Im(\kappa_\iota) \to -\infty$ for $1 \le \iota \le \tau - 1$ and $Im(\kappa_\iota) \to +\infty$ for $\tau + 1 \le \iota \le n$. The determinants Z and Z_j of Theorem 3 are given explicitly by

$$\begin{aligned} det(Z) &= e^{-2Im(\kappa_1+\kappa_2+\cdots+\kappa_{\tau-1})}[det(Z_\tau) + O(e^{-\delta|t|})],\\ det(Z_j) &= e^{-2Im(\kappa_1+\kappa_2+\cdots+\kappa_{\tau-1})}[det(Z_{j,\tau}) + O(e^{-\delta|t|})], \end{aligned} \tag{3.15}$$

where

$$det(Z_\tau) = \begin{vmatrix} \frac{1}{\chi_1-\chi_1^*} & \cdots & \frac{1}{\chi_{\tau-1}-\chi_1^*} & \frac{e^{i\kappa_\tau}}{\chi_\tau-\chi_1^*} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\chi_1-\chi_{\tau-1}^*} & \cdots & \frac{1}{\chi_{\tau-1}-\chi_{\tau-1}^*} & \frac{e^{i\kappa_\tau}}{\chi_\tau-\chi_{\tau-1}^*} & 0 & \cdots & 0 \\ \frac{e^{-i\kappa_\tau^*}}{\chi_1-\chi_\tau^*} & \cdots & \frac{e^{-i\kappa_\tau^*}}{\chi_{\tau-1}-\chi_\tau^*} & \frac{e^{i(\kappa_\tau-\kappa_\tau^*)}+\beta_\tau}{\chi_\tau-\chi_\tau^*} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \frac{\beta_{\tau+1}}{\chi_{\tau+1}-\chi_{\tau+1}^*} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{\beta_n}{\chi_n-\chi_n^*} \end{vmatrix},$$

$$det(Z_{j,\tau}) = \begin{vmatrix} \frac{1}{\chi_1-\chi_1^*} & \cdots & \frac{1}{\chi_{\tau-1}-\chi_1^*} & \frac{e^{i\kappa_\tau}}{\chi_\tau-\chi_1^*} & 0 & \cdots & 0 & \frac{-1}{\chi_1^*-a_j} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\chi_1-\chi_{\tau-1}^*} & \cdots & \frac{1}{\chi_{\tau-1}-\chi_{\tau-1}^*} & \frac{e^{i\kappa_\tau}}{\chi_\tau-\chi_{\tau-1}^*} & 0 & \cdots & 0 & \frac{-1}{\chi_{\tau-1}^*-a_j} \\ \frac{e^{-i\kappa_\tau^*}}{\chi_1-\chi_\tau^*} & \cdots & \frac{e^{-i\kappa_\tau^*}}{\chi_{\tau-1}-\chi_\tau^*} & \frac{e^{i(\kappa_\tau-\kappa_\tau^*)}+\beta_\tau}{\chi_\tau-\chi_\tau^*} & 0 & \cdots & 0 & \frac{-e^{-i\kappa_\tau^*}}{\chi_\tau^*-a_j} \\ 0 & \cdots & 0 & 0 & \frac{\beta_{\tau+1}}{\chi_{\tau+1}-\chi_{\tau+1}^*} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \frac{\beta_n}{\chi_n-\chi_n^*} & 0 \\ 1 & \cdots & 1 & e^{i\kappa_\tau} & 0 & \cdots & 0 & 1 \end{vmatrix}.$$

Finally, the asymptotic behavior when $t \to -\infty$ along θ_j can be represented as follows

$$p_j[n] = c_j e^{i\theta_j}[1 + \frac{\sum_{l=1}^{\tau}(-1)^{\tau+1}\frac{1}{\chi_l^*-a_j}e^{i(\kappa_\tau-\kappa_\tau^*)}A(\tau,l) + \sum_{l=1}^{\tau-1}(-1)^{\tau}\frac{1}{\chi_l^*-a_j}\frac{\beta_\tau}{\chi_\tau-\chi_\tau^*}A(\tau-1,l)}{e^{i(\kappa_\tau-\kappa_\tau^*)}C(\chi_1,\chi_2,\dots,\chi_\tau) + \frac{\beta_\tau}{\chi_\tau-\chi_\tau^*}C(\chi_1,\chi_2,\dots,\chi_{\tau-1})}] + O(e^{-\delta|t|}). \tag{3.16}$$

The proof is completed. ■

4. Conclusions

In this paper, we have investigated a multi-component complex mKdV system and constructed a Darboux transformation for the system. With a nonzero seed solution, we have presented a single-dark soliton solution and multi-dark soliton solutions in the defocusing case and the mixed focusing and defocusing case. Moreover, we have shown that the energies in two components of the resulting solitons completely transmit through when those two solitons pass through each other. A detailed dynamical analysis has also been provided for the single-dark–dark–dark and two-dark–dark–dark soliton solutions. Based on

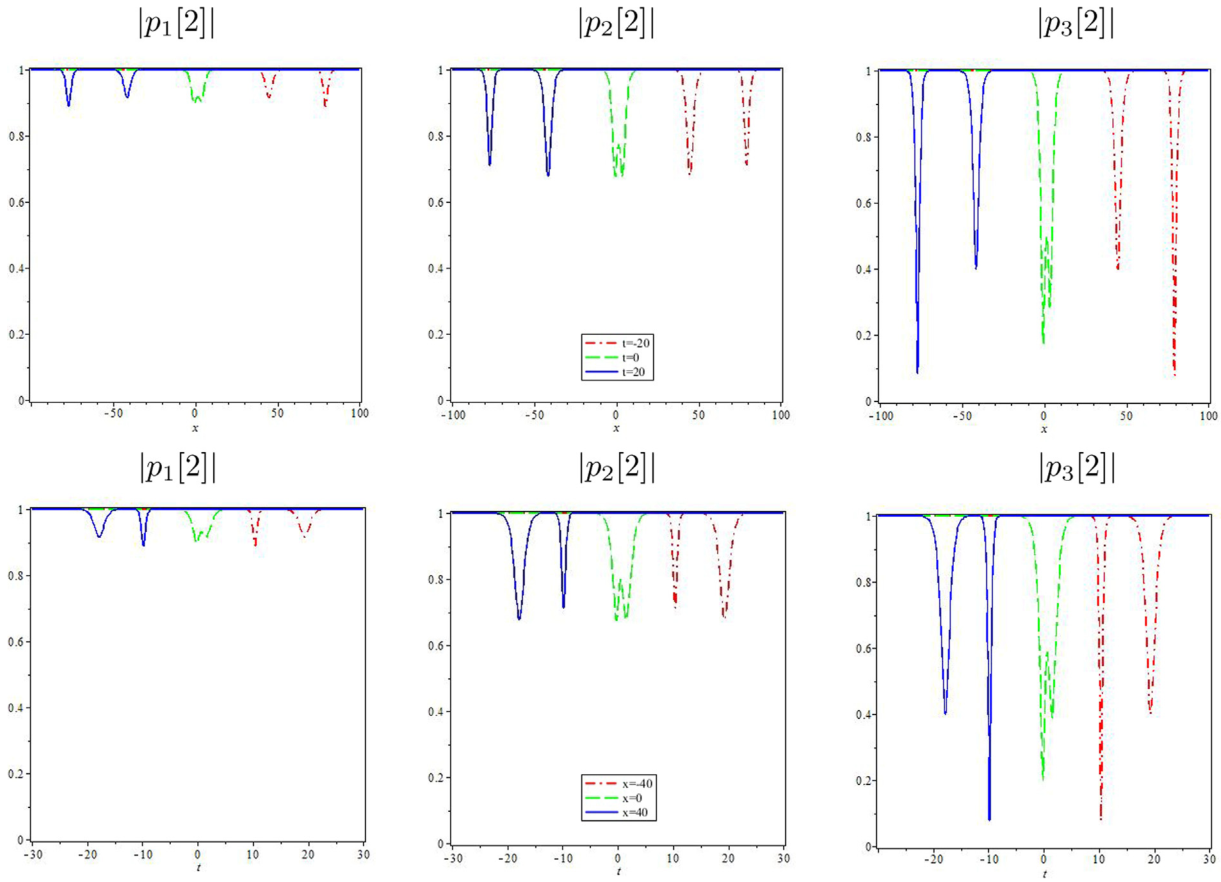

Fig. 2. Two-dark–dark–dark soliton solutions: Parameters $\sigma_1 = \sigma_2 = -\sigma_3 = \beta_1 = \beta_2 = a_2 = c_1 = c_2 = c_3 = 1, a_1 = \frac{1}{2}, a_3 = \frac{3}{2}, \lambda_1 = -2, \lambda_2 = 0, \chi_1 = 1.33924933170325 - 0.369190128736683i, \chi_2 = 1.54243168261534 - 0.537284734536601i$. Upper row: t is the evolution variable; lower row: x is the evolution variable.

the compact determinant form of the solutions, we have performed an asymptotic analysis for the n-dark soliton solutions. The results would further enrich our understanding of dark solitons in integrable models. It is an interesting topic to study soliton solutions of other types for the coupled complex mKdV equation, including breathers and rogue waves.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 11371326 and No. 11975145).

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