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Bound States of Dark Solitons in N-Coupled Complex Modified Korteweg-de Vries Equations

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Abstract
Multi-dark vector soliton solutions in the N-coupled complex modified Korteweg-de Vries (N-ccmKdV) equations are derived by the binary Darboux transformation. Dark solitons exist when nonlinearities are either defocusing or mixed focusing and defocusing. From obtained multi-dark vector soliton solutions, dark-dark-soliton bound states of 3-ccmKdV equations are provided graphically and it is shown that two dark solitons repel each other and all components of solitons have a double pole. Our results might be useful for applications about vector dark soliton solutions in other N-coupled integrable system.

Keywords Matrix spectral problem · Darboux transformation · Dark soliton solution · Bound states

Mathematics Subject Classification 37K15 · 37K35 · 37K40

1 Introduction

As we all known, there are many efficient solution generating approaches to integrable nonlinear evolution equations in soliton theory [1–9]. One of the powerful methods to construct soliton solutions is the binary Darboux transformation (bDT), which is based on the simultaneous analysis of spectral problem and the corresponding adjoint spectral problem [2, 10–13]. A key idea of bDT is to keep the both spectral problems and adjoint spectral problems associated with given equations invariant.

In 2018, one of the authors (Ma) considered a 3×3 matrix spatial spectral problem and rederived the AKNS soliton hierarchy with four components. A typical nonlinear system in

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the corresponding soliton hierarchy is

$$\begin{aligned} &p_{j,t}+\frac{\beta}{\alpha^3}[p_{j,xxx}+3(p_1q_1+p_2q_2)p_{j,x}+3(p_{1,x}q_1+p_{2,x}q_2)p_j]=0,\\ &q_{j,t}+\frac{\beta}{\alpha^3}[q_{j,xxx}+3(p_1q_1+p_2q_2)q_{j,x}+3(p_1q_{1,x}+p_2q_{2,x})q_j]=0,\quad 1\le j\le 2, \end{aligned} \tag{1}$$

which contains various modified Korteweg-de Vries (mKdV) equations, where α, β are constants. Multi-soliton solutions of equations (1) were generated by Riemann-Hilbert method [14]. Quite recently, soliton solutions for matrix mKdV equations and their reductions by using bDT with a new type of Darboux matrices have been derived in [12].

In this paper, we consider the following N-coupled complex mKdV (N-ccmKdV) equations

$$\mathbf{q}_t+\mathbf{q}_{xxx}+3\mathbf{q}\mathbf{Y}\mathbf{q}^{\dagger}\mathbf{q}_x+3\mathbf{q}_x\mathbf{Y}\mathbf{q}^{\dagger}\mathbf{q}=0, \tag{2}$$

where

$$\mathbf{q}=(q_1,q_2,\cdots,q_N),\quad \mathbf{Y}=diag(y_1,y_2,\cdots,y_N), \tag{3}$$

and

$$\begin{aligned} y_l&=1 \text{ for } l=1,2,\cdots,k,\\ &=-1 \text{ for } l=k+1,k+2,\cdots,N, \end{aligned} \tag{4}$$

where the symbol † denotes the Hermitian conjugate. When $k=0$ and $\mathbf{Y}=-I_N$, this system becomes the defocusing model that supports multi-dark vector soliton solutions. When $k=N$ and $\mathbf{Y}=I_N$, this system is the focusing model that supports multi-bright soliton solutions. When $1\le k\le N-1$, this system is the mixed focusing and defocusing system that supports multi-dark vector soliton solutions, where I_N is the $N\times N$ identity matrix.

A brief outline of this paper is as follows. In Sect. 2, we present a Lax pair and construct a bDT for the N-ccmKdV equations (2). In Sect. 3, we choose plane wave solutions as seed solutions, multi-dark vector soliton solutions in compact determinant forms are presented. In terms of introducing velocity resonance conditions, we analyze dynamics of dark-dark-soliton bound states of the 3-ccmKdV equations graphically. Section 4 contains some conclusions.

2 Binary Darboux Transformation for N-Coupled Complex Modified Korteweg-de Vries Equations

The Lax pair of N-ccmKdV equations (2) can be presented as follows

$$\begin{aligned} &\Phi_x=(\frac{1}{2}i\lambda J+iJQ)\Phi,\\ &\Phi_t=(\frac{1}{2}i\lambda^3J+i\lambda^2JQ+\lambda(iJQ^2+Q_x)+Q_xQ-QQ_x+2iJQ^3-iJQ_{xx})\Phi, \end{aligned} \tag{5}$$

with

$$J=\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0}^{\mathrm{T}} & -I_N\end{pmatrix},\quad Q=\begin{pmatrix}0 & \mathbf{q}\\ -\mathbf{Y}\mathbf{q}^{\dagger} & \mathbf{O}_N\end{pmatrix}, \tag{6}$$

where $\Phi = (\phi_1, \phi_2, \cdots, \phi_{N+1})^T$ is the vector eigenfunction, $\mathbf{0}$ is the $1 \times N$ zero vector, and $\mathbf{O}_N$ is the $N \times N$ zero matrix.

In what follows, our concern is to construct bDT for N-ccmKdV equations (2). Firstly, we consider the following lemma.

Lemma 1 *Suppose Φ_1 and Φ are the special solutions of Lax pair (5) with λ_1 and λ, respectively, then we have the total differential*

$$d\Omega(\Phi_1, \Phi) = \frac{1}{2}\Phi_1^\dagger JS\Phi dx + (\frac{1}{2}\Phi_1^\dagger JSQ\Phi(\lambda^2 + \lambda_1^{*2} + \lambda\lambda_1^*) + \Phi_1^\dagger JSQ\Phi(\lambda + \lambda_1^*) + \Phi_1^\dagger JSQ^2\Phi - i\Phi_1^\dagger SQ_x\Phi)dt. \tag{7}$$

In addition, we get

$$\Omega(\Phi_1, \Phi) = \frac{\Phi_1^\dagger S\Phi}{i(\lambda - \lambda_1^*)} + g. \tag{8}$$

If $\lambda_1 \in \mathbb{R}$, we obtain

$$\Omega(\Phi_1, \Phi_1) = \lim_{\lambda\to\lambda_1} \frac{\Phi_1^\dagger S\Phi}{i(\lambda - \lambda_1)} + g, \tag{9}$$

where g is a complex constant, and $S = diag(1, S_1, \cdots, S_k, S_{k+1}, \cdots, S_N)$, with $S_l = 1$ for $1 \le l \le k$ and $S_l = -1$ for $k+1 \le l \le N$.

In the following discussions, we choose the constant g as zero to keep the uniqueness. Based on the above lemma, we have the one-fold bDT for N-ccmKdV equations (2):

$$\begin{aligned}\Phi[1] &= \Phi - \frac{\Phi_1\Omega(\Phi_1, \Phi)}{\Omega(\Phi_1, \Phi_1)},\\ Q[1] &= Q + \frac{1}{2}i[J, \frac{\Phi_1\Phi_1^\dagger SJ}{\Omega(\Phi_1, \Phi_1)}].\end{aligned} \tag{10}$$

The following theorem shows the validity of the transformation (10).

Theorem 1 *Let Φ and Φ_1 be two vector solutions for Lax pair (5) with λ and λ_1, respectively, and $\Phi_1^\dagger S\Phi_1 = 0$ if $\lambda_1 \in \mathbb{R}$, then we have*

$$\begin{aligned}\Phi[1]_x =&(\frac{1}{2}i\lambda J + iJQ[1])\Phi[1],\\ \Phi[1]_t =&(\frac{1}{2}i\lambda^3 J + i\lambda^2 JQ[1] + \lambda(iJQ[1]^2 + Q[1]_x) + Q[1]_xQ[1] - Q[1]Q[1]_x\\ &+ 2iJQ[1]^3 - iJQ[1]_{xx})\Phi[1].\end{aligned} \tag{11}$$

It is easy to prove the above theorem by direct calculations. As an example, we only consider the first identity in equations (11). It follows from Lemma 1 that we have

$$\begin{aligned}\Phi[1]_x &= \Phi_x - \frac{\Phi_{1x}\Omega(\Phi_1,\Phi)}{\Omega(\Phi_1,\Phi_1)} - \frac{\Phi_1[\Omega(\Phi_1,\Phi)]_x}{\Omega(\Phi_1,\Phi_1)} + \frac{\Phi_1\Omega(\Phi_1,\Phi)[\Omega(\Phi_1,\Phi_1)]_x}{[\Omega(\Phi_1,\Phi_1)]^2}\\ &= (\frac{1}{2}i\lambda J + iJQ)\Phi - \frac{(\frac{1}{2}i\lambda_1 J + iJQ)\Phi_1\Omega(\Phi_1,\Phi)}{\Omega(\Phi_1,\Phi_1)}\\ &\quad - \frac{\Phi_1\Phi_1^\dagger JS\Phi}{2\Omega(\Phi_1,\Phi_1)} + \frac{\Phi_1\Omega(\Phi_1,\Phi)\Phi_1^\dagger JS\Phi_1}{2[\Omega(\Phi_1,\Phi_1)]^2}\\ &= [\frac{1}{2}i\lambda J + iJ(Q + \frac{iJ\Phi_1\Phi_1^\dagger SJ}{2\Omega(\Phi_1,\Phi_1)} - \frac{i\Phi_1\Phi_1^\dagger S}{2\Omega(\Phi_1,\Phi_1)})](\Phi - \frac{\Phi_1\Omega(\Phi_1,\Phi)}{\Omega(\Phi_1,\Phi_1)}).\end{aligned} \tag{12}$$

Thus, we see that the corresponding Darboux matrix can be written in the form

$$D_1 = I_{N+1} - \frac{\Phi_1\Phi_1^\dagger S}{i(\lambda-\lambda_1^*)\Omega(\Phi_1,\Phi_1)}. \tag{13}$$

By iterating the above-mentioned bDT n times, we can construct the n-fold bDT for N-ccmKdV equations (2).

Theorem 2 *Suppose Φ_j $(j = 1, 2, \cdots, n)$ are n linearly independent solutions of the spectral problem (5) at $\lambda = \lambda_j$ $(j = 1, 2, \cdots, n)$, and $\Phi_j^\dagger S\Phi_j = 0$ if $\lambda_j \in \mathbb{R}$. Then, we can get the n-fold bDT:*

$$\Phi[n] = \Phi - RM^{-1}\Omega,\ R = (\Phi_1, \Phi_2, \cdots, \Phi_n), \tag{14}$$

where

$$M = \begin{pmatrix}\Omega(\Phi_1,\Phi_1) & \Omega(\Phi_1,\Phi_2) & \cdots & \Omega(\Phi_1,\Phi_n)\\ \Omega(\Phi_2,\Phi_1) & \Omega(\Phi_2,\Phi_2) & \cdots & \Omega(\Phi_2,\Phi_n)\\ \vdots & \vdots & \ddots & \vdots\\ \Omega(\Phi_n,\Phi_1) & \Omega(\Phi_n,\Phi_2) & \cdots & \Omega(\Phi_n,\Phi_n)\end{pmatrix}, \Omega = \begin{pmatrix}\Omega(\Phi_1,\Phi)\\ \Omega(\Phi_2,\Phi)\\ \vdots\\ \Omega(\Phi_n,\Phi)\end{pmatrix}. \tag{15}$$

The relation between the new potential and the old one is given by

$$Q[n] = Q + \frac{1}{2}i[J, RM^{-1}R^\dagger SJ]. \tag{16}$$

It follows that the n-fold Darboux matrix is

$$D_n = I_{N+1} + iRM^{-1}(\lambda I_n - G)^{-1}R^\dagger S, \quad G = diag(\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*). \tag{17}$$

In order to obtain the dark soliton solutions, we propose the limit form of the bDT [15, 16]. Let Φ_1 and Ψ_1 be two different solutions of linear problem (5) corresponding to $\lambda = \lambda_1$ such that $\Phi_1^\dagger S\Phi_1 = 0$ and $\Phi_1^\dagger S\Psi_1 = g_1 \neq 0$, where g_1 is a nonzero constant. If the spectral parameter $\lambda_1 \in \mathbb{R}$, then we have

$$\lim_{\rho\to\lambda_1}\left(\frac{\Phi_1^\dagger S\Theta(\rho)}{\rho-\lambda_1}\right)_x = \frac{1}{2}i\Phi_1^\dagger JS\Phi_1,$$
$$\lim_{\rho\to\lambda_1}\left(\frac{\Phi_1^\dagger S\Theta(\rho)}{\rho-\lambda_1}\right)_t = i\left(\frac{3}{2}\lambda_1^2\Phi_1^\dagger JSQ\Phi_1 + 2\lambda_1\Phi_1^\dagger JSQ\Phi_1 + \Phi_1^\dagger JSQ^2\Phi_1 - i\Phi_1^\dagger SQ_x\Phi_1\right), \tag{18}$$

with

$$\Theta(\rho) = \Phi_1(\rho) + \frac{\delta(\rho-\lambda_1)}{g_1}\Psi_1(\lambda_1). \tag{19}$$

Bases on above analysis, we obtain

$$\Phi[1] = \lim_{\rho\to\lambda_1}\left(I_{N+1} + \frac{\lambda_1-\rho}{\lambda-\lambda_1}\frac{\Phi_1\Phi_1^\dagger S}{\Phi_1^\dagger S\Theta(\rho)}\right)\Phi,$$
$$Q[1] = Q + \frac{1}{2}\lim_{\rho\to\lambda_1}\left[J, \frac{(\lambda_1-\rho)\Phi_1\Phi_1^\dagger SJ}{\Phi_1^\dagger S\Theta(\rho)}\right]. \tag{20}$$

It should be mentioned that we require $\lim_{\rho\to\lambda_1}\frac{\Phi_1^\dagger S\Phi_1(\rho)}{\rho-\lambda_1} + \delta \neq 0$ for any $(x,t)\in\mathbb{R}^2$ to keep the non-singularity of bDT.

3 Multi-Dark Vector Soliton Solutions for N-Coupled Complex Modified Korteweg-de Vries Equations

To obtain the multi-dark vector soliton solutions of N-ccmKdV equations (2), we choose the general plane wave as the seed solutions

$$q_j = c_j e^{i\theta_j}, \tag{21}$$

with

$$\theta_j = a_j x + \left(a_j^3 - 3a_j\sum_{l=1}^{N}\sigma_l c_l^2 - 3\sum_{l=1}^{N}\sigma_l a_l c_l^2\right)t, \tag{22}$$

where a_j, $c_j (j = 1, 2, \cdots, N)$ are all real parameters, and $\sigma_l = 1$, when $1 \le l \le k$, $\sigma_l = -1$, when $k+1 \le l \le N$. In order to solve the spectral problem (5), we make the gauge transformation

$$\Phi = D\Psi, \quad D = diag(1, e^{-i\theta_1}, e^{-i\theta_2}, \cdots, e^{-i\theta_N}), \tag{23}$$

then the spectral problem (5) leads to

$$\Psi_x = i\tilde{U}\Psi,$$
$$\Psi_t = i\left[\tilde{U}^3 + \frac{3}{2}\lambda\tilde{U}^2 + \left(\frac{3}{4}\lambda^2 - 3\sum_{l=1}^{N}\sigma_l c_l^2\right)\tilde{U} - \left(\frac{3}{8}\lambda^3 + \frac{3}{2}\lambda\sum_{l=1}^{N}\sigma_l c_l^2 + 3\sum_{l=1}^{N}\sigma_l a_l c_l^2\right)I_{N+1}\right]\Psi, \tag{24}$$

where $\tilde{U} = iD^{-1}D_x - iD^{-1}UD$, with $U = \frac{1}{2}i\lambda J + iJQ$.

Substituting seed solution (21) into Lax pair (5), we have the fundamental solution as follows:

$$\Phi_l = \begin{pmatrix} e^{i\omega_l} \\ \frac{c_1}{\xi_l - a_1 + \frac{1}{2}\lambda_l} e^{i(\omega_l - \theta_1)} \\ \vdots \\ \frac{c_k}{\xi_l - a_k + \frac{1}{2}\lambda_l} e^{i(\omega_l - \theta_k)} \\ \frac{-c_{k+1}}{\xi_l - a_{k+1} + \frac{1}{2}\lambda_l} e^{i(\omega_l - \theta_{k+1})} \\ \vdots \\ \frac{-c_N}{\xi_l - a_N + \frac{1}{2}\lambda_l} e^{i(\omega_l - \theta_N)} \end{pmatrix} \quad (l = 1, 2, \cdots, n), \tag{25}$$

where

$$\omega_l = \xi_l x + [\xi_l^3 + \frac{3}{2}\lambda_l \xi_l^2 + (\frac{3}{4}\lambda_l^2 - 3\sum_{l=1}^{N} \sigma_l c_l^2)\xi_l - \frac{3}{8}\lambda_l^3 - \frac{3}{2}\lambda_l \sum_{l=1}^{N} \sigma_l c_l^2 - 3\sum_{l=1}^{N} \sigma_l a_l c_l^2]t, \tag{26}$$

and ξ_i is the root for the following algebraic equation

$$\xi_i - \frac{1}{2}\lambda_i - \sum_{l=1}^{N} \frac{\sigma_l c_l^2}{\xi_i - a_l + \frac{1}{2}\lambda_i} = 0. \tag{27}$$

If $\lambda_j \in \mathbb{R}$, then we have

$$\xi_j^* - \frac{1}{2}\lambda_j - \sum_{l=1}^{N} \frac{\sigma_l c_l^2}{\xi_j^* - a_l + \frac{1}{2}\lambda_j} = 0. \tag{28}$$

From equations (27)-(28), we can obtain

$$(\xi_i - \frac{1}{2}\lambda_i) - (\xi_j^* - \frac{1}{2}\lambda_j) + \sum_{l=1}^{N} \frac{\sigma_l c_l^2[(\xi_i + \frac{1}{2}\lambda_i) - (\xi_j^* + \frac{1}{2}\lambda_j)]}{(\xi_i - a_l + \frac{1}{2}\lambda_i)(\xi_j^* - a_l + \frac{1}{2}\lambda_j)} = 0, \tag{29}$$

which leads to

$$\frac{(\xi_i - \frac{1}{2}\lambda_i) - (\xi_j^* - \frac{1}{2}\lambda_j)}{(\xi_i + \frac{1}{2}\lambda_i) - (\xi_j^* + \frac{1}{2}\lambda_j)} + \sum_{l=1}^{N} \frac{\sigma_l c_l^2}{(\xi_i - a_l + \frac{1}{2}\lambda_i)(\xi_j^* - a_l + \frac{1}{2}\lambda_j)} = 0. \tag{30}$$

It follows that

$$\frac{\Phi_j^\dagger S \Phi_i}{\lambda_i - \lambda_j} = \frac{e^{i(\omega_i - \omega_j^*)}}{(\xi_i + \frac{1}{2}\lambda_i) - (\xi_j^* + \frac{1}{2}\lambda_j)}. \tag{31}$$

Then, we come back to the one-fold bDT (10) and choose $\delta = \frac{e^{2\alpha_1 Im(\xi_1)}}{\xi_1 - \xi_1^*}$, where $\alpha_1 \in \mathbb{R}$. After that, one can get

$$\lim_{\rho \to \lambda_1} \frac{\Phi_1^\dagger S \Theta(\rho)}{\rho - \lambda_1} = \frac{e^{i(\omega_1 - \omega_1^*)} + e^{2\alpha_1 Im(\xi_1)}}{\xi_1 - \xi_1^*}. \tag{32}$$

Thus, we deduce the Darboux matrix formula

$$D_1 = I_{N+1} - \frac{(\xi_1 - \xi_1^*)\Phi_1\Phi_1^\dagger S}{(\lambda - \lambda_1)(e^{i(\omega_1-\omega_1^*)} + e^{2\alpha_1 Im(\xi_1)})}. \tag{33}$$

Consequently, we obtain the bDT associated with the single-dark vector soliton solution of N-ccmKdV equations (2) as follows

$$\begin{aligned} \Phi[1] &= (I_{N+1} - \frac{(\xi_1 - \xi_1^*)\Phi_1\Phi_1^\dagger S}{(\lambda - \lambda_1)(e^{i(\omega_1-\omega_1^*)} + e^{2\alpha_1 Im(\xi_1)})})\Phi, \\ Q[1] &= Q + \frac{\xi_1^* - \xi_1}{2}[J, \frac{\Phi_1\Phi_1^\dagger SJ}{e^{i(\omega_1-\omega_1^*)} + e^{2\alpha_1 Im(\xi_1)}}]. \end{aligned} \tag{34}$$

Due to the above analysis, the transformations between field variables can be neatly reformed

$$q_j[1] = c_j e^{i\theta_j}\{1 + \frac{T_j}{2} - \frac{T_j}{2}\tanh[Im(\xi_1)(x + \eta_1 t + \alpha_1)]\}, \quad (j = 1, 2, \cdots, N), \tag{35}$$

with

$$\begin{aligned} T_j &= \frac{\xi_1 - \xi_1^*}{\xi_1^* - a_j + \frac{\lambda_1}{2}}, \\ \eta_1 &= \frac{3}{4}\lambda_1^2 + 3Re(\xi_1)\lambda_1 + 3Re^2(\xi_1) - Im^2(\xi_1) - 3\sum_{l=1}^{N}\sigma_l c_l^2. \end{aligned} \tag{36}$$

In what follows, we study the asymptotic property of the solution. We suppose $Im(\xi_1) < 0$ without loss of generality. It is easy to see that the dark soliton $q_j[1]$ moves at velocity $-\eta_1$. In addition, as x varies from $-\infty$ to ∞, we find

$$\begin{aligned} q_j[1] &\to c_j e^{i\theta_j}, \quad x \to -\infty, \\ q_j[1] &\to c_j e^{i(\theta_j+\chi_j)}, \quad x \to +\infty, \end{aligned} \tag{37}$$

where

$$i\chi_j = \ln\frac{\xi_1 - a_j + \frac{\lambda_1}{2}}{\xi_1^* - a_j + \frac{\lambda_1}{2}}. \tag{38}$$

The centre of the single-dark soliton $q_j[1]$ is along the line $x + \eta_1 t + \alpha_1 = 0$, and the depth of cavity of $|q_j[1]|^2$ is

$$\frac{c_j^2 Im^2(\xi_1)}{(\frac{\lambda_1}{2} - a_j + Re(\xi_1))^2 + Im^2(\xi_1)}. \tag{39}$$

It follows from expression (39) that we can see that the intensity dips at the centers of $|q_j[1]|^2$ $(j = 1, 2, \cdots, N)$ are characterized by involved parameters of c_j, a_j, λ_1 and ξ_1 and these parameters determine how dark the center is.

Via the iterative algorithm based on the bDT, we give the general multi-dark vector soliton of N-ccmKdV equations (2).

Theorem 3 *The n-fold bDT for dark vector soliton solutions of N-ccmKdV equations can be represented as*

$$\begin{aligned}\Phi[n] &= [I_{N+1} - RZ^{-1}(\lambda I_n - G)^{-1}R^{\dagger}S]\Phi,\\ q_j[n] &= c_j e^{i\theta_j}\frac{|Z_j|}{|Z|}\ (j=1,2,\cdots,N),\end{aligned} \tag{40}$$

with

$$Z = \begin{pmatrix} \frac{e^{i(\omega_1-\omega_1^*)}+e^{2\alpha_1 Im(\xi_1)}}{\xi_1-\xi_1^*} & \frac{e^{i(\omega_2-\omega_1^*)}}{(\xi_2+\frac{\lambda_2}{2})-(\xi_1^*+\frac{\lambda_1}{2})} & \cdots & \frac{e^{i(\omega_n-\omega_1^*)}}{(\xi_n+\frac{\lambda_n}{2})-(\xi_1^*+\frac{\lambda_1}{2})} \\ \frac{e^{i(\omega_1-\omega_2^*)}}{(\xi_1+\frac{\lambda_1}{2})-(\xi_2^*+\frac{\lambda_2}{2})} & \frac{e^{i(\omega_2-\omega_2^*)}+e^{2\alpha_2 Im(\xi_2)}}{\xi_2-\xi_2^*} & \cdots & \frac{e^{i(\omega_n-\omega_2^*)}}{(\xi_n+\frac{\lambda_n}{2})-(\xi_2^*+\frac{\lambda_2}{2})} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e^{i(\omega_1-\omega_n^*)}}{(\xi_1+\frac{\lambda_1}{2})-(\xi_n^*+\frac{\lambda_n}{2})} & \frac{e^{i(\omega_2-\omega_n^*)}}{(\xi_2+\frac{\lambda_2}{2})-(\xi_n^*+\frac{\lambda_n}{2})} & \cdots & \frac{e^{i(\omega_n-\omega_n^*)}+e^{2\alpha_n Im(\xi_n)}}{\xi_n-\xi_n^*} \end{pmatrix}, \tag{41}$$

$$Z_j = \begin{pmatrix} \frac{e^{i(\omega_1-\omega_1^*)}+e^{2\alpha_1 Im(\xi_1)}}{\xi_1-\xi_1^*} & \frac{e^{i(\omega_2-\omega_1^*)}}{(\xi_2+\frac{\lambda_2}{2})-(\xi_1^*+\frac{\lambda_1}{2})} & \cdots & \frac{e^{i(\omega_N-\omega_1^*)}}{(\xi_n+\frac{\lambda_n}{2})-(\xi_1^*+\frac{\lambda_1}{2})} & \frac{-e^{-i\omega_1^*}}{\xi_1^*-a_j+\frac{\lambda_1}{2}} \\ \frac{e^{i(\omega_1-\omega_2^*)}}{(\xi_1+\frac{\lambda_1}{2})-(\xi_2^*+\frac{\lambda_2}{2})} & \frac{e^{i(\omega_2-\omega_2^*)}+e^{2\alpha_2 Im(\xi_2)}}{\xi_2-\xi_2^*} & \cdots & \frac{e^{i(\omega_n-\omega_2^*)}}{(\xi_n+\frac{\lambda_n}{2})-(\xi_2^*+\frac{\lambda_2}{2})} & \frac{-e^{-i\omega_2^*}}{\xi_2^*-a_j+\frac{\lambda_2}{2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{e^{i(\omega_1-\omega_n^*)}}{(\xi_1+\frac{\lambda_1}{2})-(\xi_n^*+\frac{\lambda_n}{2})} & \frac{e^{i(\omega_2-\omega_n^*)}}{(\xi_2+\frac{\lambda_2}{2})-(\xi_n^*+\frac{\lambda_n}{2})} & \cdots & \frac{e^{i(\omega_n-\omega_n^*)}+e^{2\alpha_n Im(\xi_n)}}{\xi_n-\xi_n^*} & \frac{-e^{-i\omega_n^*}}{\xi_n^*-a_j+\frac{\lambda_n}{2}} \\ e^{i\omega_1} & e^{i\omega_2} & \cdots & e^{i\omega_n} & 1 \end{pmatrix}. \tag{42}$$

In studies of dark solitons, multi-dark-soliton bound states have attracted considerable attention [17–19]. To obtain dark-dark-soliton bound states, the three dark-dark solitons in the solution should have the same velocity, i.e. $\eta_1 = \eta_2$, where $\eta_j = \frac{3}{4}\lambda_j^2 + 3Re(\xi_j)\lambda_j + 3Re^2(\xi_j) - Im^2(\xi_j) - 3\sum_{l=1}^{N}\sigma_l c_l^2$ $(j = 1, 2, \cdots, n)$. It follows that the two constituent dark solitons can stay together for all times. By choosing appropriate parameters $\lambda_1, \lambda_2, \xi_1, \xi_2$ and adjusting the parameters α_1, α_2, we can obtain bound states where $|q_j[2]|(j = 1, 2, 3)$ are double-dipped (i.e., have a double pole). For an illustrative purpose, we investigate the 3-ccmKdV equations, i.e. $N = 3$ and we give two examples which correspond to defocusing case and mixed focusing and defocusing case, respectively.

Example 1 We investigate dark-dark-soliton bound states with the defocusing case (i.e. $\sigma_1 = \sigma_2 = \sigma_3 = -1$). Solving the algebraic equation (27), the parameter values are chosen as follows:

$$\begin{aligned} &a_1 = -1, a_2 = 1, a_3 = 0, \quad c_1 = c_2 = c_3 = 1, \quad \alpha_1 = 3, \alpha_2 = -3,\\ &\lambda_1 = -1, \xi_1 = -0.102951066822494 + 1.50504559366611i,\\ &\lambda_2 = 1, \xi_2 = 0.102951066822494 - 1.50504559366611i. \end{aligned} \tag{43}$$

From (43), we can see that $\lambda_1 = -\lambda_2, \xi_1 = -\xi_2$. The corresponding dark-dark-soliton bound state is plotted in Fig. 1. As shown in Fig. 1, we can see that two dark solitons repel each other and all components $|q_j[2]|(j = 1, 2, 3)$ have a double pole.

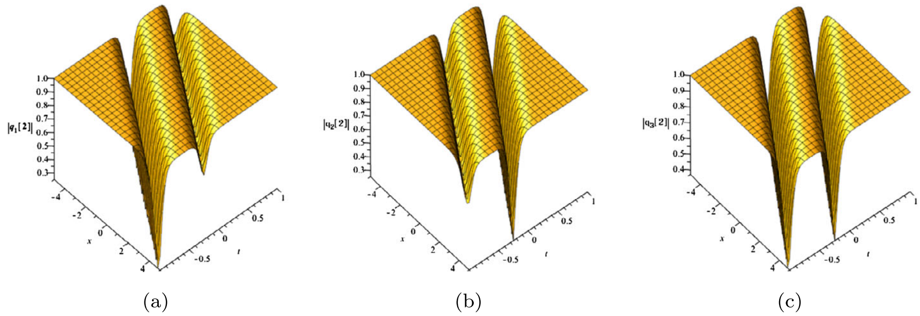

Fig. 1 (**a**)-(**c**): Dark-dark-soliton bound states with the defocusing case (i.e. $\sigma_1 = \sigma_2 = \sigma_3 = -1$): Parameters $a_1 = -1$, $a_2 = 1$, $a_3 = 0$, $c_1 = c_2 = c_3 = 1$, $\alpha_1 = 3$, $\alpha_2 = -3$, $\lambda_1 = -1$, $\xi_1 = -0.102951066822494 + 1.50504559366611i$, $\lambda_2 = 1$, $\xi_2 = 0.102951066822494 - 1.50504559366611i$.

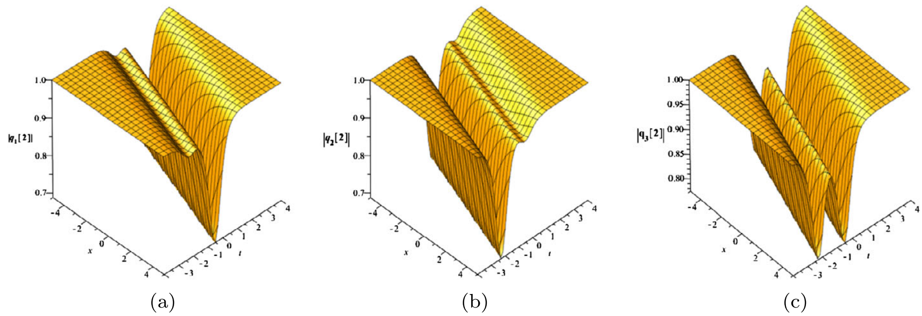

Fig. 2 (**a**)-(**c**): Dark-dark-soliton bound states with the mixed of focusing and defocusing case (i.e. $\sigma_1 = \sigma_2 = -\sigma_3 = -1$): Parameters $a_1 = -1$, $a_2 = 1$, $a_3 = 0$, $c_1 = c_2 = c_3 = 1$, $\alpha_1 = 3$, $\alpha_2 = -3$, $\lambda_1 = -1$, $\xi_1 = 1.06612094115595 + 0.458821464672557i$, $\lambda_2 = 1$, $\xi_2 = -1.06612094115595 + 0.458821464672557i$.

Example 2 We consider the dark-dark-soliton bound states with the mixed of focusing and defocusing case (i.e. $\sigma_1 = \sigma_2 = -\sigma_3 = -1$). By solving the algebraic equation (27), we take parameter values

$$\begin{aligned} &a_1 = -1, a_2 = 1, a_3 = 0, c_1 = c_2 = c_3 = 1, \alpha_1 = 3, \alpha_2 = -3, \\ &\lambda_1 = -1, \xi_1 = 1.06612094115595 + 0.458821464672557i, \\ &\lambda_2 = 1, \xi_2 = -1.06612094115595 + 0.458821464672557i. \end{aligned} \tag{44}$$

From (44), one can find that $\lambda_1 = -\lambda_2$, $\xi_1 = -\xi_2^*$. The corresponding graph of the dark-dark soliton bound state is displayed in Fig. 2. Obviously, due to the change of parameters, the depth of the poles in Fig. 2 has changed compared with that in Fig. 1.

4 Conclusions

In this paper, we have discussed the bDT of N-ccmKdV equations (2). Multi-dark vector soliton solutions in the compact determinant form are derived. Interesting, we have shown that bound states of dark solitons in N-ccmKdV equations (2) can exist when nonlinearities are either defocusing or mixed focusing and defocusing. By choosing appropriate parameters, two examples of dark-dark-soliton bound states in 3-ccmKdV equations are presented graphically. Our results might be helpful for understanding bound states of dark solitons in different physical fields.

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