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A generalized Dirac soliton hierarchy and its bi-Hamiltonian structure

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ABSTRACT

By using symbolic computation software(Maple), a generalized Dirac soliton hierarchy is derived from a new matrix spectral problem associated with the Lie algebra $\mathrm{sl}(2,\mathbb{R})$. A bi-Hamiltonian structure yielding Liouville integrability is furnished by the trace identity.

1. Introduction

Recently, soliton theory has been widely studied and applied in many natural science branches such as condense matter physics, quantum field theory, fluid dynamics and nonlinear optics. One of the most important research fields in soliton theory is to seek for soliton hierarchies, their integrable properties and exact solutions [1–30]. Many articles have shown that matrix spectral problems associated with the three-dimensional real special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ are crucial keys to construct soliton hierarchies. Typical examples include the AKNS soliton hierarchy, the KN soliton hierarchy, the WKI soliton hierarchy, the KdV soliton hierarchy, the Benjamin–Ono soliton hierarchy, the coupled Harry-Dym soliton hierarchy and the Dirac soliton hierarchy. These soliton hierarchies often possess bi-Hamiltonian structures which guarantee the existence of hereditary recursion operators and Liouville integrability [7–30].

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In this letter, we will present a generalized Dirac soliton hierarchy and construct its bi-Hamiltonian structure. An important member in the hierarchy reads

$$\begin{cases} p_t = -\frac{1}{2}q_{xx} + p^2q + q^3 - 4\alpha p(pp_x + qq_x) - 2\alpha^2(p^4q + 2p^2q^3 + q^5), \\ q_t = \frac{1}{2}p_{xx} - pq^2 - p^3 - 4\alpha q(pp_x + qq_x) + 2\alpha^2(pq^4 + 2p^3q^2 + p^5), \end{cases} \tag{1.1}$$

where α is an arbitrary constant. Eq. (1.1) can be reduced to the classical Dirac equation with $\alpha = 0$. The remainder of this paper is organized as follows. In Section 2, we present a new matrix spectral problem associated with the three-dimensional real special linear Lie algebra $\mathrm{sl}(2,\mathbb{R})$ and then construct the generalized Dirac soliton hierarchy by using symbolic computation software (Maple). A bi-Hamiltonian structure yielding Liouville integrability is furnished by the trace identity in Section 3. The last section is devoted to conclusions and discussions.

2. Generalized Dirac soliton hierarchy

For the sake of readability, we give a brief description of the procedure for building a generalized Dirac soliton hierarchy associated with the Lie algebra $\mathrm{sl}(2,\mathbb{R})$.

Step 1: We need to construct an appropriate spectral matrix U to form a spatial matrix spectral problem $\phi_x = U\phi$.

Step 2: In order to obtain the recursion relations, we need to solve the stationary zero curvature equation $W_x = [U, W]$.

Step 3: We need to construct temporal matrix spectral problems $\phi_{t_n} = V^{[n]}\phi$ so that the zero curvature equations $U_{t_n} - V_x^{[n]} + \left[U, V^{[n]}\right] = 0$ will generate a soliton hierarchy.

It is well-known that $\mathrm{sl}(2,\mathbb{R})$ has basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus the classical Dirac spatial matrix spectral problem [29] is defined as

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{2.1}$$

where

$$\begin{aligned} U &= pe_1 + (\lambda + q)e_2 + (-\lambda + q)e_3 \\ &= \begin{bmatrix} p & \lambda + q \\ -\lambda + q & -p \end{bmatrix} \end{aligned}$$

and λ denotes the spectral parameter. The inverse scattering transformation, the Virasoro symmetry algebra and the binary nonlinearization of this classical Dirac soliton hierarchy have been studied in Refs. [31–34], respectively.

Now we present a new spatial matrix spectral problem (2.1) with

$$\begin{aligned} U &= pe_1 + (\lambda + q + h)e_2 + (-\lambda + q - h)e_3 \\ &= \begin{bmatrix} p & \lambda + q + h \\ -\lambda + q - h & -p \end{bmatrix}, \quad h = \alpha(p^2 + q^2). \end{aligned}$$

When $\alpha = 0$, it degenerates into the classical result, namely, the Dirac spatial matrix spectral problem.

Next we solve the stationary zero curvature equation $W_x = [U, W]$ by setting

$$W = \begin{bmatrix} c & a+b \\ a-b & -c \end{bmatrix}. \tag{2.2}$$

Thus we have

$$\begin{cases} a_x = -2\lambda c + 2pb - 2hc, \\ b_x = 2pa - 2qc, \\ c_x = 2\lambda a - 2qb + 2ha. \end{cases} \tag{2.3}$$

To consider furthermore, substituting $a = \sum_{k=0}^{\infty} a_k \lambda^{-k}$, $b = \sum_{k=0}^{\infty} b_k \lambda^{-k}$, $c = \sum_{k=0}^{\infty} c_k \lambda^{-k}$ into Eqs. (2.3), we can obtain the recursion relations

$$\begin{cases} a_{k+1} = \dfrac{1}{2} c_{kx} + q b_k - h a_k, \\ b_{kx} = 2p a_k - 2q c_k, \qquad\qquad k \geq 0. \\ c_{k+1} = -\dfrac{1}{2} a_{kx} + p b_k - h c_k, \end{cases} \tag{2.4}$$

To guarantee the uniqueness of $\{a_k, b_k, c_k\}$, we also need to impose the integration conditions

$$a_k|_{u=0} = b_k|_{u=0} = c_k|_{u=0} = 0, \quad k \geq 1.$$

Therefore, we can compute $\{a_k, b_k, c_k\}$ recursively from the given initial value $\{a_0 = 0, b_0 = 1, c_0 = 0\}$ by using symbolic computation software (Maple). We list the first three sets as follows:

$$\begin{aligned}
&a_1 = q, \qquad b_1 = 0, \qquad c_1 = p; \\
&a_2 = \frac{1}{2} p_x - \alpha(p^2+q^2) q, \\
&b_2 = \frac{1}{2}(p^2+q^2), \\
&c_2 = -\frac{1}{2} q_x - \alpha(p^2+q^2) p; \\
&a_3 = -\frac{1}{4} q_{xx} + \frac{1}{2} q(p^2+q^2) - \alpha[p(pp_x+qq_x) + p_x(p^2+q^2)] + \alpha^2(p^2+q^2)^2 q, \\
&b_3 = \frac{1}{2} p_x q - \frac{1}{2} p q_x - \alpha(p^2+q^2)^2, \\
&c_3 = -\frac{1}{4} p_{xx} + \frac{1}{2} p(p^2+q^2) + \alpha[q(pp_x+qq_x) + q_x(p^2+q^2)] + \alpha^2(p^2+q^2)^2 p.
\end{aligned}$$

In the third step, we firstly introduce corresponding temporal matrix spectral problems

$$\phi_{t_m} = V^{[m]}\phi, \quad V^{[m]} = (\lambda^m W)_+ + \begin{bmatrix} 0 & 2\alpha b_{m+1} \\ -2\alpha b_{m+1} & 0 \end{bmatrix}, \qquad m \geq 0, \tag{2.5}$$

where P_+ denotes the polynomial part of P in λ. Thus the compatibility conditions of the spatial matrix spectral problem and the temporal matrix spectral problems, namely, the zero curvature equations $U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0$ give

$$\begin{aligned} p_{t_m} &= 2a_{m+1} + 4\alpha q b_{m+1}, \\ q_{t_m} &= -2c_{m+1} - 4\alpha p b_{m+1}. \end{aligned} \tag{2.6}$$

So we have obtained a generalized Dirac soliton hierarchy

$$u_{t_m} = \begin{bmatrix} p \\ q \end{bmatrix}_{t_m} = K_m = 2R \begin{bmatrix} a_{m+1} \\ c_{m+1} \end{bmatrix}, \quad m \geq 0 \tag{2.7}$$

where operator matrix R is defined as $R = \begin{bmatrix} 1+4\alpha q\partial^{-1}p & -4\alpha q\partial^{-1}q \\ -4\alpha p\partial^{-1}p & -1+4\alpha p\partial^{-1}q \end{bmatrix}$. The nonlinear example with $m = 2$ is just Eq. (1.1).

3. Bi-Hamiltonian structure

In this section, we consider Hamiltonian structures by using the trace identity [11,14,16,17,19,20,23,24]

$$\frac{\delta}{\delta u} \int \mathrm{tr}\left(\frac{\partial U}{\partial \lambda} W\right) \mathrm{d}x = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \mathrm{tr}\left(\frac{\partial U}{\partial u} W\right), \qquad \gamma = -\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \left|\mathrm{tr}\left(W^2\right)\right|. \tag{3.1}$$

It is direct to see

$$\frac{\partial U}{\partial \lambda} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathrm{tr}\left(W \frac{\partial U}{\partial \lambda}\right) = -2b,$$

$$\frac{\partial U}{\partial p} = \begin{bmatrix} 1 & 2\alpha p \\ -2\alpha p & -1 \end{bmatrix}, \qquad \mathrm{tr}\left(W \frac{\partial U}{\partial p}\right) = 2c - 4\alpha p b,$$

$$\frac{\partial U}{\partial q} = \begin{bmatrix} 0 & 1+2\alpha q \\ 1-2\alpha q & 0 \end{bmatrix}, \qquad \mathrm{tr}\left(W \frac{\partial U}{\partial q}\right) = 2a - 4\alpha q b.$$

Thus Eq. (3.1) becomes

$$\frac{\delta}{\delta u} \int -b\mathrm{d}x = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{bmatrix} c - 2\alpha p b \\ a - 2\alpha q b \end{bmatrix}.$$

Balancing coefficients of each power of λ in the above equality, we have

$$\frac{\delta}{\delta u} \int -b_{m+1}\mathrm{d}x = (\gamma - m) \begin{bmatrix} c_m - 2\alpha p b_m \\ a_m - 2\alpha q b_m \end{bmatrix}.$$

Checking a particular case with $m = 1$ yields $\gamma = 0$, and so we obtain

$$\frac{\delta}{\delta u} \int \left(\frac{b_{m+2}}{m+1}\right) \mathrm{d}x = \begin{bmatrix} c_{m+1} - 2\alpha p b_{m+1} \\ a_{m+1} - 2\alpha q b_{m+1} \end{bmatrix}, \qquad m \geq 0. \tag{3.2}$$

Consequently the generalized Dirac soliton hierarchy (2.7) has a Hamiltonian structure

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 1, \tag{3.3}$$

with the Hamiltonian operator

$$J = \begin{bmatrix} -8\alpha q\partial^{-1}q & 1+8\alpha q\partial^{-1}p \\ -1+8\alpha p\partial^{-1}q & -8\alpha p\partial^{-1}p \end{bmatrix}$$

and the Hamiltonian functional

$$\mathcal{H}_m = \int \left(\frac{2b_{m+2}}{m+1}\right) \mathrm{d}x.$$

By further calculation, we have corresponding bi-Hamiltonian structure

$$u_{t_m} = K_m = J\frac{\delta \mathcal{H}_m}{\delta u} = M\frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1,$$

where the second Hamiltonian operator $M = (M_{ij})_{2\times 2}$ is defined as

$$\begin{cases} M_{11} = \frac{1}{2}\partial - 2\alpha\partial p\partial^{-1}q + 2q\partial^{-1}q[4\alpha^2\partial q\partial^{-1}q - 1 - 2\alpha^2(p^2+q^2)] \\ \quad + 2\alpha q\partial^{-1}p\partial(1 - 4\alpha p\partial^{-1}q) + 4\alpha^2(p^2+q^2)q\partial^{-1}q \\ \quad + 8\alpha^2 q\partial^{-1}q(p^2+q^2)(1-4\alpha p\partial^{-1}q) - 16\alpha^2 q\partial^{-1}q\partial q\partial^{-1}q + 32\alpha^3 q\partial^{-1}p(p^2+q^2)q\partial^{-1}q, \\ M_{12} = 2\alpha\partial p\partial^{-1}p - 2q\partial^{-1}p[4\alpha^2\partial p\partial^{-1}p - 1 - 2\alpha^2(p^2+q^2)] \\ \quad + 2\alpha q\partial^{-1}q\partial(1 + 4\alpha q\partial^{-1}p) - \alpha(p^2+q^2)(1+4\alpha q\partial^{-1}p) \\ \quad - 8\alpha^2 q\partial^{-1}p(p^2+q^2)(1+4\alpha q\partial^{-1}p) + 16\alpha^2 q\partial^{-1}p\partial p\partial^{-1}p + 32\alpha^3 q\partial^{-1}q(p^2+q^2)p\partial^{-1}p, \\ M_{21} = -2\alpha\partial q\partial^{-1}q - 2p\partial^{-1}q[4\alpha^2\partial q\partial^{-1}q - 1 - 2\alpha^2(p^2+q^2)] \\ \quad - 2\alpha p\partial^{-1}p\partial(1 - 4\alpha p\partial^{-1}q) + \alpha(p^2+q^2)(1-4\alpha p\partial^{-1}q) \\ \quad - 8\alpha^2 p\partial^{-1}q(p^2+q^2)(1-4\alpha p\partial^{-1}q) + 16\alpha^2 p\partial^{-1}q\partial q\partial^{-1}q - 32\alpha^3 p\partial^{-1}p(p^2+q^2)q\partial^{-1}q, \\ M_{22} = \frac{1}{2}\partial + 2\alpha\partial q\partial^{-1}p + 2p\partial^{-1}p[4\alpha^2\partial p\partial^{-1}p - 1 - 2\alpha^2(p^2+q^2)] \\ \quad - 2\alpha p\partial^{-1}q\partial(1+4\alpha q\partial^{-1}p) + 4\alpha^2(p^2+q^2)p\partial^{-1}p \\ \quad + 8\alpha^2 p\partial^{-1}p(p^2+q^2)(1+4\alpha q\partial^{-1}p) - 16\alpha^2 p\partial^{-1}p\partial p\partial^{-1}p - 32\alpha^3 p\partial^{-1}q(p^2+q^2)p\partial^{-1}p. \end{cases}$$

4. Conclusions and discussions

In this paper, we have introduced a new 2×2 matrix spectral problem associated with $\mathrm{sl}(2,\mathbb{R})$ and get a new generalized Dirac soliton hierarchy (2.7) with a bi-Hamiltonian structure. These results are reduced to the classical results when $\alpha = 0$. We used the Maple software to deal with some complicated symbolic computations. In fact, the function h has a more general expression $h = \sum_{j=0}^{N} \alpha_j (p^2+q^2)_{jx}$.

We also can construct an $\mathrm{so}(3,\mathbb{R})$ counterpart of the generalized Dirac soliton hierarchy (2.7) by using the following spatial matrix spectral problem [29]

$$\phi_x = U\phi = U(u,\lambda)\phi, \qquad u = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix},$$

with

$$U = \begin{bmatrix} 0 & \lambda - q + h & -p \\ -\lambda + q - h & 0 & -\lambda - q - h \\ p & \lambda + q + h & 0 \end{bmatrix}, \quad h = \alpha(p^2 + 2q^2).$$

For convenience, we omit the calculation process.

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