

Research Article

A Soliton Hierarchy Associated with a Spectral Problem of 2nd Degree in a Spectral Parameter and Its Bi-Hamiltonian Structure

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Associated with $\widetilde{\text{so}}(3, R)$, a new matrix spectral problem of 2nd degree in a spectral parameter is proposed and its corresponding soliton hierarchy is generated within the zero curvature formulation. Bi-Hamiltonian structures of the presented soliton hierarchy are furnished by using the trace identity, and thus, all presented equations possess infinitely commuting many symmetries and conservation laws, which implies their Liouville integrability.

1. Introduction

Consider an evolution equation

$$u_t = K(u) = K(u, u_x, \dots), \quad (1)$$

where the field function $u(x, t)$ is in a linear space S and $K(u) = K(u, u_x, \dots)$ is a suitable C^∞ vector field. Our starting point is to determine isospectral deformations of a linear spatial spectral problem

$$\phi_x = U(u, \lambda) \phi, \quad (2)$$

where U is a square spectral matrix and λ is a spectral parameter [1–3]. With the linear spectral problem (2), let us associate an auxiliary temporal spectral problem

$$\phi_t = V(u, \lambda) \phi, \quad (3)$$

where V is a square matrix of the same order as U . The compatibility condition of (2) and (3) is the zero curvature equation

$$U_t - V_x + [U, V] = 0. \quad (4)$$

If an evolution equation (1) can be presented by such a zero curvature equation, we call it a soliton equation, and U and V a Lax pair of (1). In general, for a well-chosen spectral matrix U , we can obtain, from different particular choices of V , a hierarchy of soliton equations:

$$u_t = J \frac{\delta \mathcal{H}_n}{\delta u}, \quad (5)$$

where J is a Hamiltonian operator and \mathcal{H}_n 's are conserved functionals [1, 3, 4]. Many well-known examples of such soliton hierarchies are presented within the zero curvature formulation, which include the AKNS hierarchy [2, 5], the Kaup-Newell hierarchy [6, 7], the Wadati-Konno-Ichikawa hierarchy [8], the coupled Harry-Dym hierarchy [9], and the Dirac hierarchy [10].

Recently, the three-dimensional real special orthogonal Lie algebra $\text{so}(3, \mathbb{R})$ has been used to construct soliton hierarchies [11–13]. The Lie algebra $\text{so}(3, \mathbb{R})$ consists of 3×3 trace-free, skew-symmetric matrices and has the basis

$$e_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
(6)

with the commutator relations:

$$\begin{aligned} [e_1, e_2] &= e_3, \\ [e_2, e_3] &= e_1, \\ [e_3, e_1] &= e_2. \end{aligned}$$
(7)

The derived algebra $[\text{so}(3, \mathbb{R}), \text{so}(3, \mathbb{R})]$ is $\text{so}(3, \mathbb{R})$ itself. This is one of only two three-dimensional real Lie algebras with a three-dimensional derived algebra. The other one is $\text{sl}(2, \mathbb{R})$, which has been widely used in studying soliton equations [2, 6, 8, 12–15]. The corresponding matrix loop algebra is defined by

$$\begin{aligned} \widetilde{\text{so}}(3, \mathbb{R}) \\ = \left\{ \sum_{i \geq 0} M_i \lambda^{n-i} \mid M_i \in \text{so}(3, \mathbb{R}), i \geq 0, n \in \mathbb{Z} \right\}, \end{aligned}$$
(8)

that is, the space of all Laurent series in λ with a finite number of nonzero terms of positive powers of λ and coefficient matrices in $\text{so}(3, \mathbb{R})$. Particular examples of this matrix loop algebra $\widetilde{\text{so}}(3, \mathbb{R})$ contain the following linear combinations:

$$\lambda^m e_1 + \lambda^n e_2 + \lambda^l e_3$$
(9)

with arbitrary integers m, n, l .

Soliton hierarchies possess many nice properties, for instance, Lax representations or zero curvature representations, Hamiltonian structures, infinitely many conservation laws, and infinitely many symmetries. So it is interesting to search for new soliton hierarchies associated with a particular Lie algebra. In this paper, we would like to construct a spectral problem of 2nd degree in a spectral parameter, based on the matrix loop algebra $\widetilde{\text{so}}(3, \mathbb{R})$. A hierarchy of commuting bi-Hamiltonian soliton equations will be generated from associated zero curvature equations. Bi-Hamiltonian structures will be furnished by using the trace identity, and thus, the resulting hierarchy possesses infinitely commuting symmetries and conservation laws. A conclusion will be given in the last section.

2. A Spectral Problem and Its Soliton Hierarchy

To obtain a soliton hierarchy, we introduce a new spectral problem

$$\phi_x = U(u, \lambda) \phi, \quad u = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$
(10)

where the spectral matrix U is defined by

$$\begin{aligned} U &= \lambda^2 q e_1 + \lambda e_2 + \lambda p e_3 \\ &= \begin{pmatrix} 0 & -\lambda p & -\lambda^2 q \\ \lambda p & 0 & -\lambda \\ \lambda^2 q & \lambda & 0 \end{pmatrix} \in \widetilde{\text{so}}(3, \mathbb{R}). \end{aligned}$$
(11)

Once a matrix spectral problem is chosen, we apply the generating procedure [1, 3] to work out a soliton hierarchy associated with the spectral problem. First, we solve the stationary zero curvature equation

$$W_x = [U, W].$$
(12)

If we assume W to be

$$W = a e_1 + b e_2 + c e_3 = \begin{pmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{pmatrix},$$
(13)

then (12) gives

$$\begin{aligned} a_x &= \lambda(c - pb), \\ b_x &= -\lambda^2 qc + \lambda pa, \\ c_x &= -\lambda a + \lambda^2 qb. \end{aligned}$$
(14)

Further, let a, b, c possess the Laurent expansions:

$$\begin{aligned} a &= \sum_{i \geq 0} a_i \lambda^{-2i}, \\ b &= \sum_{i \geq 0} b_i \lambda^{-2i-1}, \\ c &= \sum_{i \geq 0} c_i \lambda^{-2i-1}, \end{aligned}$$
(15)

and then, the system (14) leads to

$$\begin{aligned} a_{ix} &= c_i - pb_i, \\ b_{ix} &= pa_{i+1} - qc_{i+1}, \\ c_{ix} &= qb_{i+1} - a_{i+1}, \end{aligned}$$
(16)

$i \geq 0.$

Using the above recursion relations (16) and taking the initial data

$$\begin{aligned} a_0 &= 1, \\ b_0 &= \frac{1}{q}, \\ c_0 &= \frac{p}{q}, \end{aligned} \quad (17)$$

the sequences of $\{a_i, b_i, c_i \mid i \geq 1\}$ are determined uniquely provided that we take zero constants of integration. The first two sets are

$$\begin{aligned} a_1 &= -\frac{1+p^2}{2q^2}, \\ b_1 &= \frac{2qp_x - 2pq_x - p^2 - 1}{2q^3}, \\ c_1 &= \frac{2q_x - p - p^3}{2q^3}; \\ a_2 &= \frac{3 + 6p^2 + 3p^4 - 8qp_x}{8q^4}, \\ b_2 &= \frac{1}{8q^5} \left[3(1+p^2)(1+p^2 - 4qp_x + 4pq_x) - 24q_x^2 \right. \\ &\quad \left. + 8qq_{xx} \right], \\ c_2 &= \frac{1}{8q^5} \left[6p^3 + 3p^5 - 12p^2q_x + 12(2qp_x - 1)q_x \right. \\ &\quad \left. - 8q^2p_{xx} + p(3 - 24q_x^2 + 8qq_{xx}) \right]. \end{aligned} \quad (18)$$

From (16), we have

$$\begin{aligned} \lambda(\lambda^{2m+1}W)_{+x} - [U, \lambda(\lambda^{2m+1}W)_+] \\ = \lambda(pa_{m+1} - qc_{m+1})e_2 + \lambda(qb_{m+1} - a_{m+1})e_3, \quad (19) \\ m \geq 0, \end{aligned}$$

where p_+ denotes the polynomial part of p . However, a direct calculation gives

$$U_t = \lambda p_t e_3 + \lambda^2 q_t e_1. \quad (20)$$

So in order to work out a soliton hierarchy, we should introduce modification terms Δ_m , $m \geq 0$. Suppose

$$\Delta_m = \lambda^2 \delta_{1m} e_1 + \lambda \delta_{2m} e_2 + \lambda \delta_{3m} e_3, \quad m \geq 0; \quad (21)$$

we have

$$\begin{aligned} \Delta_{mx} - [U, \Delta_m] &= (\lambda^2 \delta_{3m} - \lambda^2 p \delta_{2m} + \lambda^2 \delta_{1mx}) e_1 \\ &\quad + (\lambda^3 p \delta_{1m} - \lambda^3 q \delta_{3m} + \lambda \delta_{2mx}) e_2 \\ &\quad + (\lambda^3 q \delta_{2m} - \lambda^3 \delta_{1m} + \lambda \delta_{3mx}) e_3. \end{aligned} \quad (22)$$

Based on (19)–(22), δ_{1m} , δ_{2m} , and δ_{3m} should satisfy

$$\begin{aligned} p \delta_{1m} - q \delta_{3m} &= 0, \\ q \delta_{2m} - \delta_{1m} &= 0, \\ \delta_{2mx} + (pa_{m+1} - qc_{m+1}) &= 0, \\ m &\geq 0. \end{aligned} \quad (23)$$

The third equation in (23) gives $\delta_{2mx} = qc_{m+1} - pa_{m+1} = -b_{mx}$, so $\delta_{2m} = -b_m$, $\delta_{1m} = -qb_m$, and $\delta_{3m} = -pb_m$; that is,

$$\begin{aligned} \Delta_m &= \lambda^2 \delta_{1m} e_1 + \lambda \delta_{2m} e_2 + \lambda \delta_{3m} e_3 \\ &= \begin{pmatrix} 0 & \lambda pb_m & \lambda^2 qb_m \\ -\lambda pb_m & 0 & \lambda b_m \\ -\lambda^2 qb_m & -\lambda b_m & 0 \end{pmatrix}, \quad m \geq 0. \end{aligned} \quad (24)$$

As usual, we define

$$V^{[m]} = \lambda(\lambda^{2m+1}W)_+ + \Delta_m, \quad m \geq 0, \quad (25)$$

and then the corresponding zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0, \quad (26)$$

give rise to a soliton hierarchy

$$u_{t_m} = K_m = \begin{pmatrix} p \\ q \end{pmatrix}_{t_m} = \begin{pmatrix} c_{mx} - (pb_m)_x \\ -(qb_m)_x \end{pmatrix}, \quad m \geq 0. \quad (27)$$

Remark 1. The Lax pair for the hierarchy (27) is given by (2) and (3), where U and V are determined by (11) and (25). This implies that the hierarchy (27) is integrable in the Lax sense.

The first nonlinear system in this soliton hierarchy (27) is as follows:

$$u_{t_1} = \begin{pmatrix} p \\ q \end{pmatrix}_{t_1} = \begin{pmatrix} \frac{1}{q^4} \left[(1+p^2)(qq_{xx} - 3q_x^2) - q^2(p_x^2 + pp_{xx}) + 4pqp_xq_x \right] \\ \frac{1}{q^3} \left[(2qp_x - 1)q_x + pq(p_x + q_{xx}) - p^2q_x - q^2p_{xx} - 2pq_x^2 \right] \end{pmatrix}. \quad (28)$$

3. Bi-Hamiltonian Structures and Liouville Integrability

In this subsection, we will show that the soliton hierarchy (27) is Liouville integrable [16–18]. First, let us establish bi-Hamiltonian structures for the hierarchy (27) by using the trace identity [11, 19] or the variational identity [20, 21]:

$$\begin{aligned} \frac{\delta}{\delta u} \int \text{tr} \left(\frac{\partial U}{\partial \lambda} W \right) dx &= \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr} \left(\frac{\partial U}{\partial u} W \right), \\ \gamma &= -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(W^2)|. \end{aligned} \quad (29)$$

It is easy to find that

$$\begin{aligned} \frac{\partial U}{\partial \lambda} &= \begin{pmatrix} 0 & -p & -2\lambda q \\ p & 0 & -1 \\ \lambda q & 1 & 0 \end{pmatrix}, \\ \frac{\partial U}{\partial p} &= \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \frac{\partial U}{\partial q} &= \begin{pmatrix} 0 & 0 & -\lambda^2 \\ 0 & 0 & 0 \\ \lambda^2 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (30)$$

Thus, we have

$$\begin{aligned} \text{tr} \left(W \frac{\partial U}{\partial \lambda} \right) &= -4\lambda q a - 2p c - 2b, \\ \text{tr} \left(W \frac{\partial U}{\partial p} \right) &= -2\lambda c, \\ \text{tr} \left(W \frac{\partial U}{\partial q} \right) &= -2\lambda^2 a. \end{aligned} \quad (31)$$

Plugging these quantities into the trace identity (29) gives

$$\frac{\delta}{\delta u} \int (-4\lambda q a - 2p c - 2b) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{pmatrix} -2\lambda c \\ -2\lambda^2 a \end{pmatrix}. \quad (32)$$

Balancing the coefficients of λ^{-2m-1} in the above equation leads to

$$\begin{aligned} \frac{\delta}{\delta u} \int (2q a_{m+1} + p c_m + b_m) dx &= (\gamma - 2m) \begin{pmatrix} c_m \\ a_{m+1} \end{pmatrix}, \\ m &\geq 0. \end{aligned} \quad (33)$$

The identity with $m = 1$ yields $\gamma = 0$ and thus we have

$$\frac{\delta}{\delta u} \mathcal{H}_m = \begin{pmatrix} c_m \\ a_{m+1} \end{pmatrix}, \quad m \geq 0, \quad (34)$$

with the Hamiltonian functions being defined by

$$\begin{aligned} \mathcal{H}_0 &= \int \frac{1 + p^2}{2q} dx, \\ \mathcal{H}_m &= - \int \frac{2q a_{m+1} + p c_m + b_m}{2m} dx, \quad m \geq 1. \end{aligned} \quad (35)$$

From (16), we have

$$\begin{aligned} b_m &= -\partial^{-1} q c_{m+1} + \partial^{-1} p a_{m+1} \\ &= -\partial^{-1} q (\partial a_{m+1} + p b_{m+1}) + \partial^{-1} p a_{m+1} \\ &= -\partial^{-1} q \partial a_{m+1} - \partial^{-1} p (c_{mx} + a_{m+1}) + \partial^{-1} p a_{m+1}, \\ m &\geq 0. \end{aligned} \quad (36)$$

Thus the hierarchy (27) can be written as

$$\begin{aligned} u_{t_m} = K_m &= \begin{pmatrix} c_{mx} - (p b_m)_x \\ -(q b_m)_x \end{pmatrix} \\ &= \begin{pmatrix} c_{mx} + \partial p \partial^{-1} q \partial a_{m+1} + \partial p \partial^{-1} p \partial c_m \\ \partial q \partial^{-1} q \partial a_{m+1} + \partial q \partial^{-1} p \partial c_m \end{pmatrix} \\ &= J \begin{pmatrix} c_m \\ a_{m+1} \end{pmatrix}, \quad m \geq 0, \end{aligned} \quad (37)$$

where

$$J = \begin{pmatrix} \partial + \partial p \partial^{-1} p \partial & \partial p \partial^{-1} q \partial \\ \partial q \partial^{-1} p \partial & \partial q \partial^{-1} q \partial \end{pmatrix}. \quad (38)$$

It can be verified that J is skew symmetry and satisfies the Jacobi identity, so it is a Hamiltonian operator. It now follows that the soliton hierarchy (27) has the Hamiltonian structures

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u}, \quad m \geq 0. \quad (39)$$

From the recursion relations (16), we can have

$$\begin{pmatrix} c_m \\ a_{m+1} \end{pmatrix} = \Psi \begin{pmatrix} c_{m-1} \\ a_m \end{pmatrix}, \quad m \geq 0, \quad (40)$$

where

$$\Psi = \begin{pmatrix} \frac{p}{q} \partial & \partial + \frac{p}{q} \\ -\partial^{-1} \frac{1}{q} \partial \frac{1}{q} \partial - \partial^{-1} \frac{p}{q} \partial \frac{p}{q} \partial & -\partial^{-1} \frac{1}{q} \partial \frac{1}{q} - \partial^{-1} \frac{p}{q} \partial^2 - \partial^{-1} \frac{p}{q} \partial \frac{p}{q} \end{pmatrix}. \quad (41)$$

It is easy to verify that $J\Psi = \Psi^* J$. Actually, we can show that the hierarchy (27) is bi-Hamiltonian:

$$u_{t_m} = K_m = J \frac{\delta \mathcal{H}_m}{\delta u} = M \frac{\delta \mathcal{H}_{m-1}}{\delta u}, \quad m \geq 1, \quad (42)$$

where the second Hamiltonian operator is

$$M = J\Psi = \begin{pmatrix} 0 & \partial^2 \\ -\partial^2 & -\partial \end{pmatrix}. \quad (43)$$

Here J and M constitute a Hamiltonian pair. Particularly,

$$\begin{aligned} \{\mathcal{H}_k, \mathcal{H}_l\}_J &= \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T J \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \quad k, l \geq 0, \\ \{\mathcal{H}_k, \mathcal{H}_l\}_M &= \int \left(\frac{\delta \mathcal{H}_k}{\delta u} \right)^T M \frac{\delta \mathcal{H}_l}{\delta u} dx = 0, \quad k, l \geq 0, \\ [K_k, K_l] &= K'_k(u) [K_l] - K'_l(u) [K_k] = 0, \\ &\quad k, l \geq 0. \end{aligned} \quad (44)$$

These commuting relations are also consequences of the Virasoro algebra of Lax operators [22, 23]. To sum up the above discussion, we obtain the following proposition and theorem.

Proposition 2. *The soliton hierarchy (27) has infinitely many common commuting symmetries $\{J(\delta H_m / \delta u)\}_{m=1}^{\infty}$ and infinitely many conserved functions $\{H_m\}_{m=0}^{\infty}$.*

Theorem 3. *The soliton equations in the soliton hierarchy (27) are all integrable in Liouville sense.*

4. Concluding Remarks

Based on the special orthogonal Lie algebra $\widetilde{\text{so}}(3, \mathbb{R})$, the AKNS spectral matrix, the KN spectral matrix, and the WKI spectral matrix were presented in [11, 16, 24], respectively. Those spectral matrices are

$$\begin{aligned} U(u, \lambda) &= \lambda e_1 + q e_2 + p e_3, \\ U(u, \lambda) &= \lambda^2 e_1 + \lambda q e_2 + \lambda p e_3, \\ U(u, \lambda) &= \lambda e_1 + \lambda q e_2 + \lambda p e_3, \\ U(u, \lambda) &= (\lambda + q) e_1 + p e_2 + e_3, \end{aligned} \quad (45)$$

where $u = (p, q)^T$ includes two dependent variables.

In our paper, the new spectral problem defined by (10) with (11), associated with $\widetilde{\text{so}}(3, \mathbb{R})$, has been proposed and its corresponding soliton hierarchy has been worked out. The new soliton hierarchy is bi-Hamiltonian, and so, the resulting equations possess infinitely many commuting symmetries and conserved functionals, which implies that they are Liouville integrable. Furthermore, we will further research the integrable couplings and Darboux transformation of the integrable hierarchy (27).

It is particularly interesting to explore other types of spectral problems of even higher degree in a spectral parameter, associated with $\text{so}(3, \mathbb{R})$, which generate soliton hierarchies. All those studies should provide insightful thoughts to classify multicomponent integrable systems.

Competing Interests

The authors declare that they have no competing interests.

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