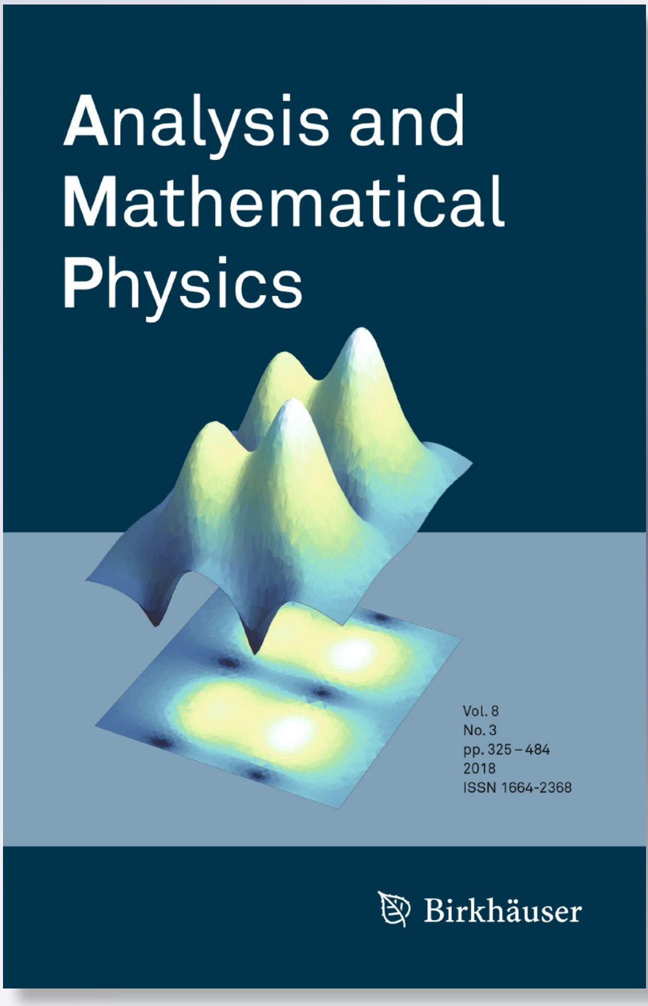
Lump and lump-soliton solutions to the $ $(2+1)$$ (2+1) -dimensional Ito equation

Jin-Yun Yang, Wen-Xiu Ma & Zhenyun Qin

Analysis and Mathematical Physics

ISSN 1664-2368
Volume 8
Number 3

Anal.Math.Phys. (2018) 8:427-436
DOI 10.1007/s13324-017-0181-9



Anal.Math.Phys. (2018) 8:427–436
https://doi.org/10.1007/s13324-017-0181-9

Lump and lump-soliton solutions to the (2 + 1)-dimensional Ito equation

Jin-Yun Yang[1] · **Wen-Xiu Ma**[2,3,4,6,7] · **Zhenyun Qin**[5]

Received: 1 April 2017 / Revised: 4 June 2017 / Accepted: 6 June 2017 / Published online: 17 June 2017

Abstract Based on the Hirota bilinear form of the $(2+1)$-dimensional Ito equation, one class of lump solutions and two classes of interaction solutions between lumps and line solitons are generated through analysis and symbolic computations with Maple. Analyticity is naturally guaranteed for the presented lump and interaction solutions, and the interaction solutions reduce to lumps (or line solitons) while the hyperbolic-cosine (or the quadratic function) disappears. Three-dimensional plots and contour plots are made for two specific examples of the resulting interaction solutions.

Keywords Bilinear form · Lump solution · Soliton solution

Mathematics Subject Classification 35Q51 · 35Q53 · 37K40

✉ Wen-Xiu Ma
mawx@cas.usf.edu

1 School of Mathematical and Physical Sciences, Xuzhou Institute of Technology, Xuzhou 221111, Jiangsu, People's Republic of China

2 Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

3 College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, People's Republic of China

4 International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

5 School of Mathematics and Key Lab for Nonlinear Mathematical Models and Methods, Fudan University, Shanghai 200433, People's Republic of China

6 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, People's Republic of China

7 College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, People's Republic of China

1 Introduction

Hirota bilinear forms are one of the integrablility characteristics of nonlinear partial differential equations [1] and associated bilinear equations can be solved by the Wronskian technique [2,3]. Solitons, positons and complexitons are typical solutions presented through the Wronskian formulation [4,5], and interaction solutions between two classes of such solutions describe more diverse nonlinear physical phenomena [3]. Moreover, upon taking long wave limits, lump solutions, rationally localized solutions in all directions in space, can be generated from solitons [6,7]. Hirota bilinear forms play a key role in generating the above mentioned solutions, and trial and error is a basic way to solve Hirota bilinear equations [1,8].

The KP equation of the following form:

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.1}$$

possesses the lump solution [9]:

$$u = 2(\ln f)_{xx},\ \ f = \left(a_1x + a_2y + \frac{a_1{a_2}^2 - a_1{a_6}^2 + 2\,a_2a_5a_6}{{a_1}^2 + {a_5}^2}t + a_4\right)^2 + \left(a_5x + a_6y + \frac{2a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2}{{a_1}^2 + {a_5}^2}t + a_8\right)^2 + \frac{3\left({a_1}^2 + {a_5}^2\right)^3}{(a_1a_6 - a_2a_5)^2}, \tag{1.2}$$

where the involved parameters a_i's are arbitrary but $a_1a_6 - a_2a_5 \neq 0$. This contains the following lump solution presented earlier [10]:

$$u = 4\frac{-\left[x + ay + \left(a^2 - b^2\right)t\right]^2 + b^2(y + 2at)^2 + 3/b^2}{\left\{\left[x + ay + \left(a^2 - b^2\right)t\right]^2 + b^2(y + 2at)^2 + 3/b^2\right\}^2}, \tag{1.3}$$

where a and b are real free parameters. There are many other soliton equations possessing lump solutions: the three-dimensional three-wave resonant interaction [11], the BKP equation [12,13], the Davey–Stewartson equation II [6], and the Ishimori-I equation [14]. General rational solutions to soliton equations have been generated within the Wronskian formulation, the Casoratian formulation and the Grammian or Pfaffian formulation [1,7], and typical physically significant examples include the KdV equation, the Boussinesq equation and the nolinear Schrödinger equation in $(1 + 1)$-dimensions, the KP and BKP equations in $(2 + 1)$-dimensions, and the Toda lattice equation in $(0+1)$-dimensions (see, e.g., [3,15–18]). Direct computations have been also made for general rational solutions to nonlinear equations (see, e.g., [19]), including generalized bilinear equations (see, e.g., [20–24]).

In this paper, we would like to consider lump solutions and interaction solutions between lumps and line solitons of the $(2 + 1)$-dimensional Ito equation, and present one class of lump solutions and two classes of interaction solutions through analysis and symbolic computations with Maple, which supplement the existing literature on lump and soliton solutions. We will begin with the Hirota bilinear form of the $(2+1)$-dimensional Ito equation, and test if quadratic functions and combined functions of quadratic functions and the hyperbolic cosine can solve the bilinear Ito equation. A few of concluding remarks will be given in the last section.

2 Lump and interaction solutions

The $(2+1)$-dimensional Ito equation reads [25,26]

$$\begin{aligned} P_{Ito}(u, v) :&= u_{tt} + u_{xxxt} + 6u_x u_t + 3uu_{xt} + 3u_{xx} v_t \\ &\quad + \alpha u_{yt} + \beta u_{xt} = 0, \end{aligned} \tag{2.1}$$

where $v_x = u$ and α, β are two constants, or equivalently,

$$v_{xtt} + v_{xxxxt} + 6v_{xx} v_{xt} + 3v_x v_{xxt} + 3v_{xxx} v_t + \alpha v_{xyt} + \beta v_{xxt} = 0.$$

It is a $(2+1)$-dimensional generalization of the KdV equation from a perspective of Hirota bilinear forms, which possesses N-soliton solitons [25]. Its Hirota bilinear form is determined by (see, e.g., [25,27]):

$$\begin{aligned} B_{Ito}(f) :&= (D_t^2 + D_x^3 D_t + \alpha D_y D_t + \beta D_x D_t) f \cdot f \\ &= 2\Big[f_{tt} f - f_t^2 + f_{xxxt} f - 3 f_{xxt} f_x + 3 f_{xt} f_{xx} - f_{xxx} f_t \\ &\quad + \alpha(f_{yt} f - f_y f_t) + \beta(f_{xt} f - f_x f_t)\Big] = 0, \end{aligned} \tag{2.2}$$

under the transformations

$$\begin{aligned} u &= 2(\ln f)_{xx} = \frac{2(f_{xx} f - f_x^2)}{f^2}, \\ v &= 2(\ln f)_x = \frac{2 f_x}{f}. \end{aligned} \tag{2.3}$$

Such characteristic transformations have been adopted in Bell polynomial theories of soliton equations (see, e.g., [28,29]), and precisely, we have

$$P_{Ito}(u, v) = \left(\frac{B_{Ito}(f)}{f^2} \right)_{xx}.$$

Therefore, when f solves the bilinear Ito equation (2.2), $u = 2(\ln f)_{xx}$ and $v = 2(\ln f)_x$ will solve the $(2+1)$-dimensional Ito equation (2.1).

2.1 Lump solutions

Based on analysis and symbolic computations with Maple, we can show that the (2+1)-dimensional bilinear Ito equation (2.2) has a class of quadratic function solutions determined by

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9, \tag{2.4}$$

where

$$\beta a_1 + \alpha a_2 + a_3 = 0,\ \beta a_5 + \alpha a_6 + a_7 = 0,\ a_1a_3 + a_5a_7 = 0, \tag{2.5}$$

and the other parameters are arbitrary. By the transformations in (2.3), this yields a large class of lump solutions to the $(2 + 1)$-dimensional Ito equation (2.1).

We know that the condition

$$a_1a_6 - a_2a_5 \neq 0 \tag{2.6}$$

is necessary and sufficient for a solution f, defined by (2.4), to generate a lump solution in $(2 + 1)$-dimensions through (2.3). Under the condition (2.6), we can solve

$$f_x = 0,\ f_y = 0, \tag{2.7}$$

to get all critical points

$$\begin{aligned} x = x(t) &= \frac{(a_2a_7 - a_3a_6)t + (a_2a_8 - a_4a_6)}{a_1a_6 - a_2a_5}, \\ y = y(t) &= -\frac{(a_1a_7 - a_3a_5)t + (a_1a_8 - a_4a_5)}{a_1a_6 - a_2a_5}, \end{aligned} \tag{2.8}$$

where t is arbitrarily given. The function $f - a_9$, i.e., the sum of two squares, vanishes at this set of critical points. Thus, $f > 0$ if and only if $a_9 > 0$, which tells that u and v defined by (2.3) are analytical in $\mathbb{R}^3$ if and only if $a_9 > 0$. For any given time t, the point $(x(t), y(t))$ determined by (2.8) is also a critical point of the function $u = 2(\ln f)_{xx}$, and thus the lump solution u has a peak at this point $(x(t), y(t))$, since we have

$$\begin{aligned} u_{xx} &= -\frac{24({a_1}^2 + {a_5}^2)^2}{{a_9}^2} < 0,\ u_{xx}u_{yy} - {u_{xy}}^2 \\ &= \frac{192({a_1}^2 + {a_5}^2)^2(a_1a_6 - a_2a_5)^2}{{a_9}^4} > 0 \end{aligned}$$

at the point $(x(t), y(t))$.

It is direct to see, through taking two special cases of (2.4), that the resulting lump solutions above cover the two classes of lump solutions presented by symbolic computations in [27]:

$$u_1 = \frac{4\,a_5{}^2}{f_1} - \frac{8\,a_5{}^2\left(a_5x - \frac{\beta\,a_5}{\alpha}y + a_8\right)^2}{f_1{}^2},\ \ v_1 = \frac{4\,a_5\left(a_5x - \frac{\beta\,a_5}{\alpha}y + a_8\right)}{f_1},$$

with

$$f_1 = \left(-\frac{a_3}{\alpha}y + a_3t + a_4\right)^2 + \left(a_5x - \frac{\beta\,a_5}{\alpha}y + a_8\right)^2 + a_9;$$

and

$$\begin{cases} u_2 = \dfrac{4\,a_1{}^2\left(a_3{}^2 + a_7{}^2\right)}{a_7{}^2\,f_2} - \dfrac{8\,a_1{}^2\left[a_1\left(a_3{}^2 + a_7{}^2\right)(\alpha\,x - \beta\,y) - \alpha\,a_7\,(a_3a_8 - a_4a_7)\right]^2}{\alpha^2\,a_7{}^4\,f_2{}^2}, \\ v_2 = \dfrac{4\,a_1\left[a_1\left(a_3{}^2 + a_7{}^2\right)(\alpha\,x - \beta\,y) - \alpha\,a_7(a_3a_8 - a_4a_7)\right]}{\alpha\,a_7\,f_2}, \end{cases}$$

with

$$f_2 = \left(a_1x - \frac{\beta\,a_1 + a_3}{\alpha}y + a_3t + a_4\right)^2 + \left(-\frac{a_1a_3}{a_7}x + \frac{\beta\,a_1a_3 - a_7{}^2}{\alpha\,a_7}y + a_7t + a_8\right)^2 + a_9.$$

2.2 Interaction solutions

Basic approaches to soliton solutions and dromion-type solutions include the Hirota perturbation technique and symmetry reductions and constraints (see, e.g., [30–33]). The following analysis aims to compute interaction solutions between lumps and line solitons to the (2 + 1)-dimensional Ito equation (2.1) through testing if combined functions of quadratic functions and the hyperbolic cosine can solve the (2 + 1)-dimensional Ito equation (2.2). Interaction solutions describe more diverse nonlinear phenomena in nature.

By the computer algebra system Maple, we first search for combined solutions to the bilinear Ito equation (2.2). We start from an ansatz

$$f = \xi_1{}^2 + \xi_2{}^2 + \cosh\xi_3 + a_{13}, \tag{2.9}$$

where three linear wave variables are defined by

$$\begin{cases} \xi_1 = a_1x + a_2y + a_3t + a_4, \\ \xi_2 = a_5x + a_6y + a_7t + a_8, \\ \xi_3 = a_9x + a_{10}y + a_{11}t + a_{12}, \end{cases} \tag{2.10}$$

with the parameters a_i's being real constants to be determined. The ansatz leads to a class of one-soliton solutions when $\xi_1 = \xi_2 = 0$, while it generates a class of lump solutions when $\xi_3 = 0$. Therefore, the combined solutons are called interaction solutions. We take two cases into consideration, where imposing $a_9 = 0$ or $a_{11} = 0$. Combined with Maple symbolic computations, a direct analysis gives rise to the following two classes of solutions for the parameters a_i's:

$$\begin{cases} \beta a_1 + \alpha a_2 + a_3 = 0, \\ \beta a_5 + \alpha a_6 + a_7 = 0, \\ a_1 a_3 + a_5 a_7 = 0, \\ a_9 = 0, \ \alpha a_{10} + a_{11} = 0, \end{cases} \tag{2.11}$$

and

$$\begin{cases} \beta a_1 + \alpha a_2 + a_3 = 0, \\ \beta a_5 + \alpha a_6 + a_7 = 0, \\ a_1 a_3 + a_5 a_7 = 0, \\ \alpha a_{10} + \beta a_9 + {a_9}^3 = 0, \ a_{11} = 0, \end{cases} \tag{2.12}$$

where the other parameters are arbitrary. These sets of solutions for the parameters generate two classes of combined solutions f_1 and f_2 to the bilinear Ito equation (2.2), defined by (2.9) and (2.10) with (2.11) or (2.12), and then the resulting combined solutions present two classes of interaction solutions u_1 and u_2 to the $(2+1)$-dimensional Ito equation (2.1), under the transformations in (2.3). The analyticity of the interactions solutions is definitely guaranteed, if we require $a_{13} > -1$. These interaction solutions reduce to the soliton solutions [26] when the quadratic function is not involved, and the lump solutions [27] when the hyperbolic cosine is not involved. We point out that the presented interaction solutions do not approach zero in all directions in the independent variable space since a line soliton wave is involved, and they form a peak at finite times due to the existence of a lump wave.

To get two specific interaction solutions to the $(2 + 1)$-dimensional Ito equation (2.1), let us choose the following two special sets of the parameters:

$$\begin{cases} \alpha = 1, \ \beta = -1, \ a_1 = 1, \ a_2 = 2, \ a_3 = -1, \ a_4 = 1, \ a_5 = 2, \ a_6 = \frac{3}{2}, \\ a_7 = \frac{1}{2}, \ a_8 = 2, \ a_9 = 0, \ a_{10} = 2, \ a_{11} = -2, \ a_{12} = 5, \ a_{13} = 1, \end{cases} \tag{2.13}$$

and

$$\begin{cases} \alpha = 2, \ \beta = -2, \ a_1 = 2, \ a_2 = -1, \ a_3 = 6, \ a_4 = -1, \ a_5 = -2, \ a_6 = -5, \\ a_7 = 6, \ a_8 = -2, \ a_9 = -1, \ a_{10} = -\frac{1}{2}, \ a_{11} = 0, \ a_{12} = 3, \ a_{13} = 2. \end{cases} \tag{2.14}$$

Since $a_{13} > -1$ in these two cases, the analyticity is guaranteed for the two corresponding specific interaction solutions. It is easy to work out these two interaction solutions:

$$u_1 = \frac{20}{g_1} - \frac{2(10\,x + 10\,y + 10)^2}{{g_1}^2}, \quad v_1 = \frac{2(10\,x + 10\,y + 10)}{g_1}, \tag{2.15}$$

with

$$g_1 = (x + 2\,y - t + 1)^2 + \left(2\,x + \frac{3}{2}\,y + \frac{1}{2}\,t + 2\right)^2 + \cosh(-2\,y + 2\,t - 5) + 1, \tag{2.16}$$

and

$$u_2 = \frac{2\left[16 + \cosh(x + \frac{1}{2}\,y - 3)\right]}{g_2} - \frac{2\left[16\,x + 16y + 4 + \sinh(x + \frac{1}{2}\,y - 3)\right]^2}{{g_2}^2}, \tag{2.17}$$

$$v_2 = \frac{2\left[16\,x + 16\,y + 4 + \sinh(x + \frac{1}{2}\,y - 3)\right]}{g_2}, \tag{2.18}$$

with

$$g_2 = (2\,x - y + 6\,t - 1)^2 + (-2\,x - 5\,y + 6\,t - 2)^2 + \cosh\left(x + \frac{1}{2}\,y - 3\right) + 2. \tag{2.19}$$

Three 3-dimensional plots and contour plots of the solution u_1 at $t = 0, 2, 5$ and the solution u_2 at $t = 0, 1, 2$ are shown in Figs. 1 and 2, respectively.

3 Concluding remarks

Based on the Hirota form of the $(2 + 1)$-dimensional Ito equation, we computed one class of lump solutions and two classes of interaction solutions between lumps and line solitons to the $(2 + 1)$-dimensional Ito equation explicitly by symbolic computations with Maple, and the resulting classes of interaction solutions provide supplements to the existing lump and soliton solutions in the literature.

We point out that we can also present a kind of interactions solutions between the lumps presented above and kinks as in [27]. In addition, if we replace the Hirota derivatives in (2.2) with generalized bilinear derivatives [34], all previous computations would be different in the case of the Ito-like equation defined by the generalized bilinear derivatives $D_{p,z}$ with $p = 3$:

$$\left(D_{3,t}^2 + D_{3,x}^3 D_{3,t} + \alpha D_{3,y} D_{3,t} + \beta D_{3,x} D_{3,t}\right) f \cdot f = 0,$$

but lump solutions derived from quadratic functions are not changed. It would also be interesting to find combined solutions to other generalized bilinear and tri-linear

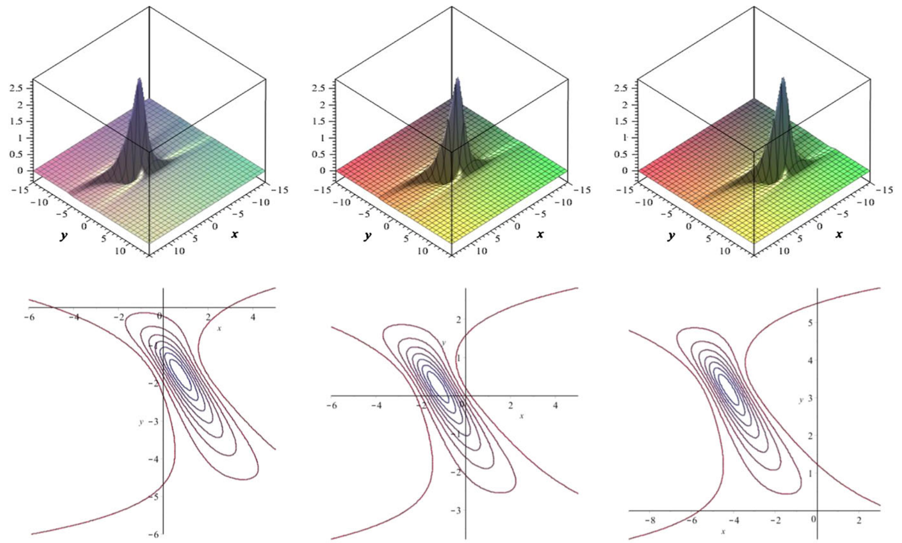


Fig. 1 Profiles of (2.15) with $t = 0, 2, 5$: 3d plots (*top*) and contour plots (*bottom*)

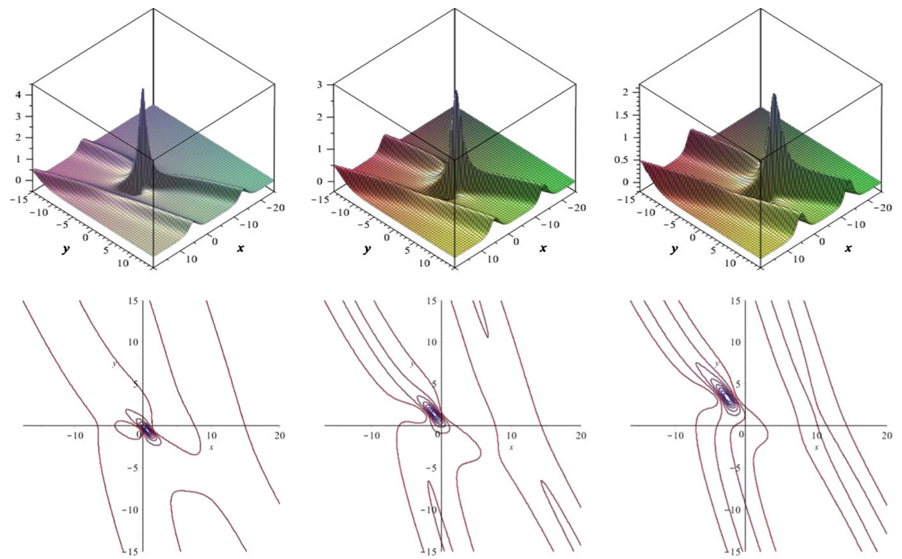


Fig. 2 Profiles of (2.17) with $t = 0, 1, 2$: 3d plots (*top*) and contour plots (*bottom*)

differential equations, generated from using generalized bilinear derivatives [34]. This kind of mixed solutions is different from resonant solutions by the linear superposition principle [35,36] and could describe complicated nonlinear physical phenomena.

Acknowledgements The work was supported in part by a university grant XKY2016112 from Xuzhou Institute of Technology, NSFC under the Grants 11371326, 11301331, and 11371086, NSF under the

Grant DMS-1664561, and the Distinguished Professorships by Shanghai University of Electric Power and Shanghai Second Polytechnic University.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Hirota, R.: The Direct Method in Soliton Theory. Cambridge University Press, New York (2004)
2. Freeman, N.C., Nimmo, J.J.C.: Soliton solutions of the Korteweg–de Vries and Kadomtsev–Petviashvili equations: the Wronskian technique. Phys. Lett. A **95**(1), 1–3 (1983)
3. Ma, W.X., You, Y.: Solving the Korteweg–de Vries equation by its bilinear form: Wronskian solutions. Trans. Am. Math. Soc. **357**(5), 1753–1778 (2005)
4. Ma, W.X.: Wronskian solutions to integrable equations. Discrete Contin. Dyn. Syst. **Suppl**, 506–515 (2009)
5. Wazwaz, A.-M., El-Tantawy, S.A.: New (3 + 1)-dimensional equations of Burgers type and Sharma–Tasso–Olver type: multiple-soliton solutions. Nonlinear Dyn. **87**(4), 2457–2461 (2017)
6. Satsuma, J., Ablowitz, M.J.: Two-dimensional lumps in nonlinear dispersive systems. J. Math. Phys. **20**(7), 1496–1503 (1979)
7. Ablowitz, M.J., Clarkson, P.A.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
8. Caudrey, P.J.: Memories of Hirota's method: application to the reduced Maxwell–Bloch system in the early 1970s. Philos. Trans. R. Soc. A **369**(1939), 1215–1227 (2011)
9. Ma, W.X.: Lump solutions to the Kadomtsev–Petviashvili equation. Phys. Lett. A **379**(36), 1975–1978 (2015)
10. Manakov, S.V., Zakharov, V.E., Bordag, L.A., Matveev, V.B.: Two-dimensional solitons of the Kadomtsev–Petviashvili equation and their interaction. Phys. Lett. A **63**(3), 205–206 (1977)
11. Kaup, D.J.: The lump solutions and the Bäcklund transformation for the three-dimensional three-wave resonant interaction. J. Math. Phys. **22**(6), 1176–1181 (1981)
12. Gilson, C.R., Nimmo, J.J.C.: Lump solutions of the BKP equation. Phys. Lett. A **147**(8–9), 472–476 (1990)
13. Yang, J.Y., Ma, W.X.: Lump solutions of the BKP equation by symbolic computation. Int. J. Mod. Phys. B **30**(28–29), 1640028 (2016)
14. Imai, K.: Dromion and lump solutions of the Ishimori-I equation. Prog. Theor. Phys. **98**(5), 1013–1023 (1997)
15. Ma, W.X., You, Y.: Rational solutions of the Toda lattice equation in Casoratian form. Chaos Solitons Fract. **22**(2), 395–406 (2004)
16. Ankiewicz, A., Kedziora, D.J., Akhmediev, N.: Rogue wave triplets. Phys. Lett. A **375**(28–29), 2782–2785 (2011)
17. Gaillard, P.: Rational solutions to the KPI equation and multi rogue waves. Ann. Phys. **367**, 1–5 (2016)
18. Chakravarty, S., Kodama, Y.: Line-soliton solutions of the KP equation. In: Nonlinear and Modern Mathematical Physics, AIP Conference Proceedings, 1212, pp. 312–341. American Institute of Physics, Melville, NY (2010)
19. Aslan, İ.: Rational and multi-wave solutions to some nonlinear physical models. Rom. J. Phys. **58**(7–8), 893–903 (2013)
20. Zhang, Y., Ma, W.X.: Rational solutions to a KdV-like equation. Appl. Math. Comput. **256**, 252–256 (2015)
21. Zhang, Y.F., Ma, W.X.: A study on rational solutions to a KP-like equation. Z. Naturforsch. A **70**(4), 263–268 (2015)
22. Zhang, Y., Dong, H.H., Zhang, X.E., Yang, H.W.: Rational solutions and lump solutions to the generalized (3 + 1)-dimensional shallow water-like equation. Comput. Math. Appl. **73**(2), 246–252 (2017)
23. Ma, W.X., Qin, Z.Y., Lü, X.: Lump solutions to dimensionally reduced p-gKP and p-gBKP equations. Nonlinear Dyn. **84**(2), 923–931 (2016)
24. Yu, J.P., Sun, Y.L.: Lump solutions to dimensionally reduced Kadomtsev–Petviashvili-like equations. Nonlinear Dyn. **87**(2), 1405–1412 (2017)

25. Ito, M.: An extension of nonlinear evolution equations of the K-dV (mK-dV) type to higher orders. J. Phys. Soc. Jpn. **49**(2), 771–778 (1980)
26. Wazwaz, A.-M.: Multiple-soliton solutions for the generalized (1 + 1)-dimensional and the generalized (2 + 1)-dimensional Ito equations. Appl. Math. Comput. **202**, 840–849 (2008)
27. Tang, Y.N., Tao, S.Q., Qing, G.: Lump solitons and the interaction phenomena of them for two classes of nonlinear evolution equations. Comput. Math. Appl. **72**(9), 2334–2342 (2016)
28. Gilson, C., Lambert, F., Nimmo, J., Willox, R.: On the combinatorics of the Hirota D-operators. Proc. R. Soc. Lond. Ser. A **452**(1945), 223–234 (1996)
29. Ma, W.X.: Bilinear equations, Bell polynomials and linear superposition principle. J. Phys. Conf. Ser. **411**, 012021 (2013)
30. Dorizzi, B., Grammaticos, B., Ramani, A., Winternitz, P.: Are all the equations of the Kadomtsev–Petviashvili hierarchy integrable? J. Math. Phys. **27**(12), 2848–2852 (1986)
31. Konopelchenko, B., Strampp, W.: The AKNS hierarchy as symmetry constraint of the KP hierarchy. Inverse Probl. **7**(2), L17–L24 (1991)
32. Li, X.Y., Zhao, Q.L., Li, Y.X., Dong, H.H.: Binary Bargmann symmetry constraint associated with 3×3 discrete matrix spectral problem. J. Nonlinear Sci. Appl. **8**(5), 496–506 (2015)
33. Dong, H.H., Zhang, Y., Zhang, X.E.: The new integrable symplectic map and the symmetry of integrable nonlinear lattice equation. Commun. Nonlinear Sci. Numer. Simul. **36**, 354–365 (2016)
34. Ma, W.X.: Generalized bilinear differential equations. Stud. Nonlinear Sci. **2**(4), 140–144 (2011)
35. Ma, W.X.: Bilinear equations and resonant solutions characterized by Bell polynomials. Rep. Math. Phys. **72**(1), 41–56 (2013)
36. Ma, W.X.: Trilinear equations, Bell polynomials, and resonant solutions. Front. Math. China **8**(5), 1139–1156 (2013)