

# FOUR-COMPONENT LIOUVILLE INTEGRABLE MODELS AND THEIR BI-HAMILTONIAN FORMULATIONS

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*Abstract.* We aim at presenting Liouville integrable Hamiltonian models with four dependent variables from a specific matrix eigenvalue problem. The Liouville integrability of the resulting models is exhibited through formulating their bi-Hamiltonian formulations. The basic tools are the Lax pair approach and the trace identity. Two illustrative examples consist of novel four-component coupled integrable models of second-order and third-order.

*Key words:* Matrix eigenvalue problem, Lax pair, Zero curvature equation,  
Integrable model, bi-Hamiltonian formulation.

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## 1. INTRODUCTION

In soliton theory, Lax pairs play an important role in exploring integrable models [1, 2]. The concept of a Lax pair involves the formulation of a linear eigenvalue problem associated with a given nonlinear partial differential equation [3]. By formulating an appropriate Lax pair, we can generate a compatible set of model equations that possess remarkable integrable properties, for example, infinitely many symmetries and conserved quantities, and exhibit soliton solutions, making them amenable to analytical techniques and providing insights into their dynamics.

To construct integrable models by Lax pairs, it is crucial to formulate an appropriate spatial spectral matrix, being the spatial part of an infinite sequence of Lax pairs. Let us denote a  $q$ -dimensional column potential vector by  $p = (p_1, \dots, p_q)^T$  and assume that  $k$  stands for the spectral parameter.

To begin with, let us take a loop matrix algebra  $\tilde{g}$  with the loop parameter  $k$  and define a spatial spectral matrix as follows:

$$\mathcal{M} = \mathcal{M}(p, k) = p_1 E_1(k) + \dots + p_q E_q(k) + E_0(k), \quad (1)$$

where the elements  $E_1, \dots, E_q$  are linearly independent and the element  $E_0$  is pseudo-regular. The pseudo-regular conditions here consist of

$$\text{Im ad}_{E_0} \oplus \text{Ker ad}_{E_0} = \tilde{g}, \quad [\text{Ker ad}_{E_0}, \text{Ker ad}_{E_0}] = 0,$$

where  $\text{ad}_{E_0}$  denotes the adjoint action of  $E_0$  on  $\tilde{g}$ . These characteristic conditions ensure that the stationary zero curvature equation

$$\mathcal{Z}_x = i[\mathcal{M}, \mathcal{Z}] \quad (2)$$

has a solution among Laurent series matrices  $\mathcal{Z} = \sum_{n \geq 0} k^{-n} \mathcal{Z}^{[n]}$  in the underlying loop algebra  $\tilde{g}$ .

Second, we form an infinite sequence of temporal spectral matrices

$$\mathcal{N}^{[m]} = (k^m \mathcal{Z})_+ + \sigma_r = \sum_{n=0}^m k^{m-n} \mathcal{Z}^{[n]} + \Delta_m, \quad \Delta_m \in \tilde{g}, \quad m \geq 0, \quad (3)$$

being the other parts of a sequence of Lax pairs, so that we can generate an integrable hierarchy of soliton equations:

$$p_{t_m} = X^{[m]} = X^{[m]}(p), \quad m \geq 0, \quad (4)$$

via the zero curvature equations:

$$\mathcal{M}_{t_m} - \mathcal{N}_x^{[m]} + i[\mathcal{M}, \mathcal{N}^{[m]}] = 0, \quad m \geq 0. \quad (5)$$

They represent the conditions that guarantee the solvability of the spatial and temporal matrix eigenvalue problems:

$$-i\varphi_x = \mathcal{M}\varphi, \quad -i\varphi_{t_m} = \mathcal{N}^{[m]}\varphi, \quad m \geq 0. \quad (6)$$

Finally, Hamiltonian formulations and the associated Liouville integrability of a soliton hierarchy (4) can be explored by using the so-called trace identity (see [4, 5] for details):

$$\frac{\delta}{\delta p} \int \text{tr}(\mathcal{Z} \frac{\partial \mathcal{M}}{\partial k}) dx = k^{-\gamma} \frac{\partial}{\partial k} k^\gamma \text{tr}(\mathcal{Z} \frac{\partial \mathcal{M}}{\partial p}), \quad (7)$$

where  $\frac{\delta}{\delta p}$  denotes the variational derivative with respect to  $p$ ,  $\text{tr}$  stands for the trace of a matrix, and  $\gamma$  is a constant, being independent of the spectral parameter  $k$ .

Abundant Liouville integrable hierarchies of soliton equations are computed through such a procedure, whose underlying loop algebras are generated from the special linear algebras (see, *e.g.*, [6–12]), and the special orthogonal algebras (see, *e.g.*, [13–15]). Bi-Hamiltonian formulations of soliton hierarchies [16] show the Liouville integrability of soliton equations in the corresponding hierarchies. Among integrable hierarchies with two dependent variables, the Heisenberg hierarchy [17], the Ablowitz-Kaup-Newell-Segur hierarchy [6], the Kaup-Newell hierarchy [18] and the Wadati-Konno-Ichikawa hierarchy [19] are the well-known integrable hierarchies.

Their spectral matrices read

$$\begin{bmatrix} k & p_1 \\ p_2 & -k \end{bmatrix}, \begin{bmatrix} kp_3 & kp_1 \\ kp_2 & -kp_3 \end{bmatrix}, \begin{bmatrix} k^2 & kp_1 \\ kp_2 & -k^2 \end{bmatrix}, \begin{bmatrix} k & kp_1 \\ kp_2 & -k \end{bmatrix}, \quad (8)$$

where  $p_1 p_2 + p_3^2 = 1$ , respectively.

This paper aims to present a Liouville integrable hierarchy of four-component bi-Hamiltonian equations through isospectral (*i.e.*  $k_{t_m} = 0$ ) Lax pairs. The corresponding bi-Hamiltonian formulations are established for the resulting hierarchy of soliton equations by an application of the trace identity. Two illustrative examples are composed of four-component coupled integrable nonlinear Schrödinger equations and modified Korteweg-de Vries equations. The last section consists of a conclusion and some concluding remarks.

## 2. AN INTEGRABLE HAMILTONIAN HIERARCHY WITH FOUR-DEPENDENT VARIABLES

Let  $\sigma_1$  and  $\sigma_2$  be two real numbers satisfying  $\sigma_1^2 = \sigma_2^2 = 1$ , *i.e.*  $\sigma_1, \sigma_2 \in \{1, -1\}$ . Motivated by a recent study on integrable models associated with a reduced spectral matrix [20], within the Lax pair formulation, we consider a matrix eigenvalue problem of the form:

$$-i\varphi_x = \mathcal{M}\varphi = \mathcal{M}(p, k)\varphi, \quad \mathcal{M} = \begin{bmatrix} k & r_1 & r_2 & r_2 & r_2 & r_1 & 0 \\ s_1 & 0 & 0 & 0 & 0 & 0 & \sigma_1 r_1 \\ s_2 & 0 & 0 & 0 & 0 & 0 & \sigma_2 r_2 \\ s_2 & 0 & 0 & 0 & 0 & 0 & \sigma_2 r_2 \\ s_2 & 0 & 0 & 0 & 0 & 0 & \sigma_2 r_2 \\ s_1 & 0 & 0 & 0 & 0 & 0 & \sigma_1 r_1 \\ 0 & \sigma_1 s_1 & \sigma_2 s_2 & \sigma_2 s_2 & \sigma_2 s_2 & \sigma_1 s_1 & -k \end{bmatrix}, \quad (9)$$

where  $p$  is the dependent variable consisting of four components:

$$p = p(x, t) = (r_1, r_2, s_1, s_2)^T. \quad (10)$$

This eigenvalue problem is associated with a  $B$ -type Lie algebra and it is different from the  $A$ -type matrix Ablowitz-Kaup-Newell-Segur eigenvalue problem (see, *e.g.*, [21] for cases of nonlocal reductions).

To present an associated four-component Liouville integrable hierarchy, we first solve the associated stationary zero curvature equation (2) by searching for a

particular Laurent series solution:

$$\mathcal{Z} = \begin{bmatrix} e & f_1 & f_2 & f_2 & f_2 & f_1 & 0 \\ g_1 & 0 & h & h & h & 0 & \sigma_1 f_1 \\ g_2 & -\sigma_1 \sigma_2 h & 0 & 0 & 0 & -\sigma_1 \sigma_2 h & \sigma_2 f_2 \\ g_2 & -\sigma_1 \sigma_2 h & 0 & 0 & 0 & -\sigma_1 \sigma_2 h & \sigma_2 f_2 \\ g_2 & -\sigma_1 \sigma_2 h & 0 & 0 & 0 & -\sigma_1 \sigma_2 h & \sigma_2 f_2 \\ g_1 & 0 & h & h & h & 0 & \sigma_1 f_1 \\ 0 & \sigma_1 g_1 & \sigma_2 g_2 & \sigma_2 g_2 & \sigma_2 g_2 & \sigma_1 g_1 & -e \end{bmatrix} = \sum_{n \geq 0} k^{-n} \mathcal{Z}^{[n]}, \quad (11)$$

where the basic objects are assumed to be expanded in Laurent series of the spectral parameter  $k$ :

$$e = \sum_{n \geq 0} k^{-n} e^{[n]}, \quad f_j = \sum_{n \geq 0} k^{-n} f_j^{[n]}, \quad g_j = \sum_{n \geq 0} k^{-n} g_j^{[n]}, \quad h = \sum_{n \geq 0} k^{-n} h^{[n]}, \quad i = 1, 2. \quad (12)$$

It is clear that the associated stationary zero curvature equation yields the initial requirements:

$$e_x^{[0]} = 0, \quad f_1^{[0]} = f_2^{[0]} = g_1^{[0]} = g_2^{[0]} = 0, \quad h_x^{[0]} = 0, \quad (13)$$

and the recursion relations for defining the Laurent series solution:

$$f_1^{[n+1]} = -i f_{1,x}^{[n]} + r_1 e^{[n]} + 3\sigma_1 \sigma_2 r_2 h^{[n]}, \quad f_2^{[n+1]} = -i f_{2,x}^{[n]} + r_2 e^{[n]} - 2r_1 h^{[n]}, \quad (14)$$

$$g_1^{[n+1]} = i g_{1,x}^{[n]} + s_1 e^{[n]} - 3s_2 h^{[n]}, \quad g_2^{[n+1]} = i g_{2,x}^{[n]} + s_2 e^{[n]} + 2\sigma_1 \sigma_2 s_1 h^{[n]}, \quad (15)$$

$$\begin{aligned} e_x^{[n+1]} &= -i(2s_1 f_1^{[n+1]} + 3s_2 f_2^{[n+1]} r_1 g_1^{[n+1]} - 3r_2 g_2^{[n+1]}) \\ &= -(2s_1 f_{1,x}^{[n]} + 3s_2 f_{2,x}^{[n]} + 2r_1 g_{1,x}^{[n]} 3r_2 g_{2,x}^{[n]}), \end{aligned} \quad (16)$$

and

$$h_x^{[n+1]} = i(s_1 f_2^{[n+1]} - \sigma_1 \sigma_2 s_2 f_1^{[n+1]} + \sigma_1 \sigma_2 r_1 g_2^{[n+1]} - r_2 g_1^{[n+1]}), \quad (17)$$

where  $n \geq 0$ . To have a unique Laurent series solution, we go with the initial data,

$$e^{[0]} = 1, \quad h^{[0]} = 0, \quad (18)$$

and take the constants of integration to be zero,

$$e^{[n]}|_{u=0} = 0, \quad h^{[n]}|_{u=0} = 0, \quad n \geq 1. \quad (19)$$

Under these conditions, one can obtain that

$$\begin{aligned} f_1^{[1]} &= r_1, \quad f_2^{[1]} = r_2, \quad g_1^{[1]} = s_1, \quad g_2^{[1]} = s_2, \quad e^{[1]} = 0, \quad h^{[1]} = 0; \\ \begin{cases} f_1^{[2]} = -ir_{1,x}, \quad f_2^{[2]} = -ir_{2,x}, \quad g_1^{[2]} = is_{1,x}, \quad g_2^{[2]} = is_{2,x}, \\ e^{[2]} = -2r_1 s_1 - 3r_2 s_2, \quad h^{[2]} = -\sigma_1 \sigma_2 r_1 s_2 + r_2 s_1; \end{cases} \end{aligned}$$

$$\begin{cases} f_1^{[3]} = -r_{1,xx} - 2r_1^2 s_1 - 6r_1 r_2 s_2 + 3\sigma_1 \sigma_2 r_2^2 s_1, \\ f_2^{[3]} = -r_{2,xx} + 2\sigma_1 \sigma_2 r_1^2 s_2 - 4r_1 r_2 s_1 - 3r_2^2 s_2, \\ g_1^{[3]} = -s_{1,xx} - 2r_1 s_1^2 + 3\sigma_1 \sigma_2 r_1 s_2^2 - 6r_2 s_1 s_2, \\ g_2^{[3]} = -s_{2,xx} - 4r_1 s_1 s_2 + 2\sigma_1 \sigma_2 r_2 s_1^2 - 3r_2 s_2^2, \\ e^{[3]} = i(2r_{1,x}s_1 - 2r_1 s_{1,x} + 3r_{2,x}s_2 - 3r_2 s_{2,x}), \\ h^{[3]} = -i(\sigma_1 \sigma_2 r_1 s_{2,x} - r_2 s_{1,x} - \sigma_1 \sigma_2 r_{1,x}s_2 + r_{2,x}s_1); \end{cases}$$

and

$$\begin{cases} f_1^{[4]} = i(r_{1,xxx} + 6r_1 r_{1,x}s_1 + 9r_1 r_{2,x}s_2 - 9\sigma_1 \sigma_2 r_2 r_{2,x}s_1 + 9r_{1,x}r_2 s_2), \\ f_2^{[4]} = i(r_{2,xxx} + 6r_1 r_{2,x}s_1 - 6\sigma_1 \sigma_2 r_1 r_{1,x}s_2 + 6_{1,x}r_2 s_1 + 9r_2 r_{2,x}s_2), \\ g_1^{[4]} = -i(s_{1,xxx} + 6r_1 s_1 s_{1,x} - 9\sigma_1 \sigma_2 r_1 s_2 s_{2,x} + 9r_2 s_1 s_{2,x} + 9r_2 s_{1,x}s_2), \\ g_2^{[4]} = -i(s_{2,xxx} + 6r_1 s_1 s_{2,x} + 6_{1,x}s_1 s_2 - 6\sigma_1 \sigma_2 r_2 s_1 s_{1,x} + 9r_2 s_2 s_{2,x}), \\ e^{[4]} = 6r_1^2 s_1^2 - 9\sigma_1 \sigma_2 r_1^2 s_2^2 + 36r_1 r_2 s_1 s_2 - 9\sigma_1 \sigma_2 r_2^2 s_1^2 + \frac{27}{2}r_2^2 s_2^2 \\ \quad + 2r_1 s_{1,xx} + 2r_{1,x} s_1 + 3r_2 s_{2,xx} + 3r_{2,xx} s_2 - 2r_{1,x} s_{1,x} - 3r_{2,x} s_{2,x}, \\ h^{[4]} = -3(2r_1 s_1 + 3r_2 s_2)(\sigma_1 \sigma_2 r_1 s_2 r_2 s_1) + \sigma_1 \sigma_2 r_{1,xx} s_2 - r_{2,xx} s_1 \\ \quad - r_2 s_{1,xx} + \sigma_1 \sigma_2 r_1 s_{2,xx} - \sigma_1 \sigma_2 r_{1,x} s_{2,x} + r_{2,x} s_{1,x}. \end{cases}$$

Based on these computations, we can take  $\Delta_m = 0$ ,  $m \geq 0$ , to formulate

$$-i\varphi_{t_m} = \mathcal{N}^{[m]}\varphi = \mathcal{N}^{[m]}(p, k)\varphi, \quad \mathcal{N}^{[m]} = (k^m \mathcal{Z})_+ = \sum_{n=0}^m k^n \mathcal{Z}^{[m-n]}, \quad m \geq 0, \quad (20)$$

which are the temporal matrix eigenvalue problems within the Lax pair formulation. The conditions that guarantee the solvability of the spatial and temporal matrix eigenvalue problems in (9) and (20) are given by the zero curvature equations in (5). They generate a soliton hierarchy with four potentials:

$$p_{t_m} = X^{[m]} = X^{[m]}(p) = (i f_1^{[m+1]}, i f_2^{[m+1]}, -i g_1^{[m+1]}, -i g_2^{[m+1]})^T, \quad m \geq 0, \quad (21)$$

or more concretely,

$$r_{1,t_m} = i f_1^{[m+1]}, \quad r_{2,t_m} = i f_2^{[m+1]}, \quad s_{1,t_m} = -i g_1^{[m+1]}, \quad s_{2,t_m} = -i g_2^{[m+1]}, \quad m \geq 0. \quad (22)$$

As particular examples, this soliton hierarchy contains the coupled systems of

integrable nonlinear Schrödinger equations:

$$\begin{cases} ir_{1,t_2} = r_{1,xx} + 2r_1^2s_1 + 6r_1r_2s_2 - 3\sigma_1\sigma_2r_2^2s_1, \\ ir_{2,t_2} = r_{2,xx} - 2\sigma_1\sigma_2r_1^2s_2 + 4r_1r_2s_13r_2^2s_2, \\ is_{1,t_2} = -s_{1,xx} - 2r_1s_1^2 + 3\sigma_1\sigma_2r_1s_2^2 - 6r_2s_1s_2, \\ is_{2,t_2} = -s_{2,xx} - 4r_1s_1s_2 + 2\sigma_1\sigma_2r_2s_1^2 - 3r_2s_2^2, \end{cases} \quad (23)$$

and the coupled system of integrable modified Korteweg-de Vries equations:

$$\begin{cases} r_{1,t_3} = -r_{1,xxx} - 6r_1r_{1,x}s_1 - 9r_1r_{2,x}s_2 + 9\sigma_1\sigma_2r_2r_{2,x}s_1 - 9r_{1,x}r_2s_2, \\ r_{2,t_3} = -r_{2,xxx} - 6r_1r_{2,x}s_1 + 6\sigma_1\sigma_2r_1r_{1,x}s_2 - 6r_{,x}r_2s_1 - 9r_2r_{2,x}s_2, \\ s_{1,t_3} = -s_{1,xxx} - 6r_1s_1s_{1,x} + 9\sigma_1\sigma_2r_1s_2s_{2,x} - 9r_2s_1s_{2,x} - 9r_2s_{1,x}s_2, \\ s_{2,t_3} = -s_{2,xxx} - 6r_1s_1s_{2,x} - 6r_1s_{1,x}s_2 + 6\sigma_1\sigma_2r_2s_1s_{1,x} - 9r_2s_2s_{2,x}. \end{cases} \quad (24)$$

These two systems provide typical coupled integrable models, which extend the category of coupled integrable nonlinear Schrödinger equations and modified Korteweg-de Vries equations.

### 3. BI-HAMILTONIAN FORMULATIONS

To furnish bi-Hamiltonian formulations for the soliton hierarchy (22), one can take advantage of the so-called trace identity (7) to the spatial matrix eigenvalue problem (9). The trace identity uses the solution  $\mathcal{Z}$  defined by (11). One can then easily evaluate

$$\text{tr}(\mathcal{Z} \frac{\partial \mathcal{M}}{\partial k}) = 2a, \quad \text{tr}(\mathcal{Z} \frac{\partial \mathcal{M}}{\partial p}) = (4g_1, 6g_2, 4f_1, 6f_2)^T, \quad (25)$$

and consequently, the trace identity gives

$$\frac{\delta}{\delta p} \int k^{-(n+1)} e^{[n+1]} dx = k^{-\gamma} \frac{\partial}{\partial k} k\gamma - n(2g_1^{[n]}, 3g_2^{[n]}, 2f_1^{[n]}, 3f_2^{[n]})^T, \quad n \geq 0. \quad (26)$$

A check with  $n = 2$  leads to  $\gamma = 0$ , and as a consequence, one obtains

$$\frac{\delta}{\delta p} \mathcal{H}^{[n]} = (2g_1^{[n+1]}, 3g_2^{[n+1]}, 2f_1^{[n+1]}, 3f_2^{[n+1]})^T, \quad n \geq 0, \quad (27)$$

where the following Hamiltonian functionals are computed in the following way:

$$\mathcal{H}^{[n]} = - \int \frac{e^{[n+2]}}{n+1} dx, \quad n \geq 0. \quad (28)$$

This allows us to present the Hamiltonian formulations for the soliton hierarchy (22):

$$p_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta p}, \quad J_1 = \begin{bmatrix} 0 & \frac{1}{2}i & 0 \\ -\frac{1}{2}i & 0 & \frac{1}{3}i \\ 0 & -\frac{1}{3}i & 0 \end{bmatrix}, \quad m \geq 0, \quad (29)$$

where  $J_1$  is Hamiltonian and  $\mathcal{H}^{[m]}$  are the functionals given by (28). We point out that an important property that the Hamiltonian formulations exhibit is an interrelation  $S = J_1 \frac{\delta \mathcal{H}}{\delta p}$  between a symmetry  $S$  and a conserved functional  $\mathcal{H}$  of the same model.

On one hand, those vector fields  $X^{[n]}$  satisfy a characteristic commuting property:

$$[[X^{[n_1]}, X^{[n_2]}]] = X^{[n_1]'}(p)[X^{[n_2]}] - X^{[n_2]'}(p)[X^{[n_1]}] = 0, \quad n_1, n_2 \geq 0, \quad (30)$$

which can be seen from an algebra of Lax operators:

$$[[\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}]] = \mathcal{N}^{[n_1]'}(p)[X^{[n_2]}] - \mathcal{N}^{[n_2]'}(p)[X^{[n_1]}] + [\mathcal{N}^{[n_1]}, \mathcal{N}^{[n_2]}] = 0, \quad n_1, n_2 \geq 0. \quad (31)$$

One can directly check this by analyzing the relation between the isospectral zero curvature equations (see [22] for details).

On the other hand, from the recursion relation  $X^{[m+1]} = \Phi X^{[m]}$ , we can work out a hereditary recursion operator  $\Phi = (\Phi_{jk})_{4 \times 4}$  [23] for the soliton hierarchy (22), and it reads as follows:

$$\begin{cases} \Phi_{11} = i(-\partial_x - 2r_1 \partial^{-1} s_1 - 3r_2 \partial^{-1} s_2), \quad \Phi_{12} = i(-3r_1 \partial^{-1} s_2 + 3\sigma_1 \sigma_2 r_2 \partial^{-1} s_1), \\ \Phi_{13} = i(-2r_1 \partial^{-1} r_1 + 3\sigma_1 \sigma_2 r_2 \partial^{-1} s_2), \quad \Phi_{14} = i(-3r_1 \partial^{-1} r_2 - 3r_2 \partial^{-1} r_1); \end{cases} \quad (32)$$

$$\begin{cases} \Phi_{21} = i(-2r_2 \partial^{-1} s_1 + 2\sigma_1 \sigma_2 r_1 \partial^{-1} s_2), \quad \Phi_{22} = i(-\partial_x - 3r_2 \partial^{-1} s_2 - 2r_1 \partial^{-1} s_1), \\ \Phi_{23} = i(-2r_2 \partial^{-1} r_1 - 2r_1 \partial^{-1} r_2), \quad \Phi_{24} = i(3r_2 \partial^{-1} r_2 + 2\sigma_1 \sigma_2 r_1 \partial^{-1} r_1); \end{cases} \quad (33)$$

$$\begin{cases} \Phi_{31} = i(2s_1 \partial^{-1} s_1 - 3\sigma_1 \sigma_2 s_2 \partial^{-1} s_2), \quad \Phi_{32} = i(3s_1 \partial^{-1} s_2 + 3s_2 \partial^{-1} s_1), \\ \Phi_{33} = i(\partial_x + 2s_1 \partial^{-1} r_1 + 3s_2 \partial^{-1} r_2), \quad \Phi_{34} = i(3s_1 \partial^{-1} r_2 - 3\sigma_1 \sigma_2 s_2 \partial^{-1} r_1); \end{cases} \quad (34)$$

$$\begin{cases} \Phi_{41} = i(2s_2 \partial^{-1} s_1 + 2s_1 \partial^{-1} s_2), \quad \Phi_{42} = i(3s_2 \partial^{-1} s_2 - 2\sigma_1 \sigma_2 s_1 \partial^{-1} s_1), \\ \Phi_{43} = i(2s_2 \partial^{-1} r_1 - 2\sigma_1 \sigma_2 s_1 \partial^{-1} r_2), \quad \Phi_{44} = i(\partial_x + 3s_2 \partial^{-1} r_2 + 2s_1 \partial^{-1} r_1). \end{cases} \quad (35)$$

We remark that although the above recursion operator is nonlocal, all isospectral (*i.e.*  $k_{t_m} = 0$ ) flows are local. This is a common characteristic of integrable hierarchies. Further, with some analysis, one can see that  $J_1$  and  $J_2 = \Phi J_1$  constitute a Hamiltonian

pair, and thus, the soliton hierarchy (22) possesses the bi-Hamiltonian formulations [16]:

$$p_{t_m} = X^{[m]} = J_1 \frac{\delta \mathcal{H}^{[m]}}{\delta p} = J_2 \frac{\delta \mathcal{H}^{[m-1]}}{\delta p}, \quad m \geq 1. \quad (36)$$

It then follows that the associated Hamiltonian functionals commute under the corresponding two Poisson brackets [4]:

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_1} = \int \left( \frac{\delta \mathcal{H}^{[n_1]}}{\delta p} \right)^T J_1 \frac{\delta \mathcal{H}^{[n_2]}}{\delta p} dx = 0, \quad n_1, n_2 \geq 0, \quad (37)$$

and

$$\{\mathcal{H}^{[n_1]}, \mathcal{H}^{[n_2]}\}_{J_2} = \int \left( \frac{\delta \mathcal{H}^{[n_1]}}{\delta p} \right)^T J_2 \frac{\delta \mathcal{H}^{[n_2]}}{\delta p} dx = 0, \quad n_1, n_2 \geq 0. \quad (38)$$

To summarize, each model in the soliton hierarchy (22) possesses a bi-Hamiltonian formulation, and thus, it is Liouville integrable, which means that it has infinitely many commuting symmetries  $\{X^{[n]}\}_{n=0}^{\infty}$  and conserved functionals  $\{\mathcal{H}^{[n]}\}_{n=0}^{\infty}$ . Particularly (see also, [24–26]), the equations in (23) and (24) present two specific examples of nonlinear coupled Liouville integrable models, which possess bi-Hamiltonian formulations.

#### 4. CONCLUDING REMARKS

A Liouville integrable hierarchy of bi-Hamiltonian equations with four dependent variables has been generated from a specific special matrix eigenvalue problem, through a sequence of Lax pairs. The key point is to determine a particular Laurent series solution to the associated stationary zero curvature equation. The obtained integrable models have been shown to possess bi-Hamiltonian formulations, upon applying the trace identity to the underlying matrix isospectral eigenvalue problem.

One can, of course, enlarge the considered spatial matrix eigenvalue problem by introducing more copies of  $r_1$  and  $r_2$ . It is also possible to generate larger integrable models *via* involving more dependent variables in a spatial spectral matrix (see, *e.g.*, [27]). Moreover, various higher-order integrable models and local integrable reductions of the obtained hierarchy could be computed (see, [28–30] for examples in the case of the matrix Ablowitz-Kaup-Newell-Segur eigenvalue problem).

It should be of much interest to study structures of soliton solutions to the obtained integrable models by powerful and effective methods in soliton theory, including the Zakharov-Shabat dressing method [31], the Riemann-Hilbert technique [32], the determinant approach [33] and the Darboux transformation [34, 35]. Other important solutions, such as lump, kink, breather and rogue wave solutions, including their interaction solutions (see, *e.g.*, [36–38, 40]), can be computed from specific

wave number reductions of solitons. Nonlocal reduced integrable equations can also be presented by considering nonlocal group reductions of matrix eigenvalue problems under similarity transformations (see, *e.g.*, [41, 42]).

The structures of integrable models are complex and multifaceted, and the study of these models, requiring a deep understanding of many different areas of mathematics and physics, has led to the discovery of various types of soliton solutions and other nonlinear coherent structures, and has the potential to significantly expand our understanding of the natural world.

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