



Lump-Wave Structures in an Extended KP-like Model with Spatially Balanced Nonlinearity and Dispersion

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Abstract

This study investigates lump-wave structures arising from the interaction between nonlinear and dispersive effects in an extended KP-like nonlinear model with spatially balanced nonlinearity and dispersion in $(2 + 1)$ dimensions. Using generalized bilinear derivatives associated with the prime number three, a generalized bilinear form is first proposed, from which a nonlinear model equation with spatially balanced nonlinearity and dispersion is derived. By employing symbolic computation in Maple, positive quadratic wave solutions are constructed, giving rise to localized lump-wave structures. It is shown that the stationary points of the quadratic waves lie on a straight line in the spatial plane and propagate with constant velocity. Along the trajectory of these stationary points, the lump waves maintain constant amplitude. The novelty of this work lies in the application of generalized bilinear derivatives associated with the prime number three. The results demonstrate that the formation of lump waves is fundamentally governed by the combined effects of nonlinearity and dispersion within the model.

Keywords Bilinear form · Lump wave · Dispersion · Nonlinearity · Symbolic computation

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1 Introduction

The pursuit of exact closed-form solutions in mathematical physics and engineering remains of fundamental importance, as such solutions offer deeper analytical insight into complex wave phenomena than results limited to existence or uniqueness theorems. Nevertheless, deriving explicit solutions is typically a formidable task. As a result, considerable effort has been devoted to developing systematic methods for constructing explicit solution formulae or identifying conditions under which such solutions exist.

Within soliton theory and the study of integrable systems, particular attention is given to three principal classes of nonlinear wave solutions: solitons, rogue waves, and lump waves. These structures are distinguished by their localization properties: solitons are exponentially localized and form stable multiwave patterns; rogue waves are transient, large-amplitude events that appear and disappear abruptly; and lump waves are rationally localized structures that decay algebraically in all spatial directions at fixed time. Such dispersive patterns emerge from the subtle interplay between nonlinearity and dispersion, and their analytical or numerical realization remains a central theme in nonlinear wave dynamics.

Two cornerstone approaches for obtaining exact solutions of integrable nonlinear partial differential equations are the inverse scattering transform (IST) [1] and Hirota's bilinear method [2]. The IST provides a spectral framework for solving Cauchy problems via associated Lax pairs, yielding profound insights into soliton interactions and long-time asymptotics of dispersive systems [4]. In contrast, Hirota's bilinear method offers a direct algebraic formalism that facilitates the systematic derivation of multi-soliton and rational solutions, proving especially effective in higher-dimensional integrable models [5]–[9].

Let P denote a polynomial in M variables. In general, a Hirota bilinear equation takes the form:

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M})f \cdot f = 0, \quad (1.1)$$

where D_{x_i} represents Hirota's bilinear operator [2], defined by

$$D_{x_i}^m f \cdot f = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x'_i} \right)^m f(x_1, x_2, \dots, x_M) f(x'_1, x'_2, \dots, x'_M) \Big|_{x'=x},$$

for $1 \leq i \leq M$, $m \geq 0$, with $x = (x_1, x_2, \dots, x_M)$ and $x' = (x'_1, x'_2, \dots, x'_M)$. The bilinear framework admits N -soliton solutions expressed as exponential superpositions (see, e.g., [5, 10]):

$$f = \sum_{v=0,1} \exp\left(\sum_{i=1}^N v_i \zeta_i + \sum_{i < j} v_i v_j d_{ij}\right), \quad (1.2)$$

where the summation runs over all combinations of $v_i \in \{0, 1\}$. The linear phases ζ_i and phase shifts d_{ij} are given by

$$\zeta_i = k_{1,i}x_1 + k_{2,i}x_2 + \dots + k_{M,i}x_M + \zeta_{i,0}, \quad 1 \leq i \leq N, \quad (1.3)$$

and

$$\exp(d_{ij}) = -\frac{P(\mathbf{k}_i - \mathbf{k}_j)}{P(\mathbf{k}_i + \mathbf{k}_j)}, \quad 1 \leq i < j \leq N, \tag{1.4}$$

subject to the dispersion relations

$$P(\mathbf{k}_i) = 0, \quad \mathbf{k}_i = (k_{1,i}, k_{2,i}, \dots, k_{M,i}), \quad 1 \leq i \leq N. \tag{1.5}$$

Let f be defined by (1.2). Assume that $\hat{\xi}$ indicates that ξ is omitted. Then, one obtains a recursive relation for computing the bilinear expression (see, e.g., [10] for details):

$$\begin{aligned} & P(D_{x_1}, \dots, D_{x_M}) f \cdot f \\ &= (-1)^{\frac{1}{2}N(N-1)} \frac{H(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_N)}{\prod_{1 \leq i < j \leq N} P(\mathbf{k}_i + \mathbf{k}_j)} e^{\zeta_1 + \zeta_2 + \dots + \zeta_N} \\ &+ \sum_{n=1}^{N-1} (-1)^{\frac{1}{2}(N-n)(N-n-1)} \sum_{1 \leq i_1 < \dots < i_n \leq N} \frac{H(\mathbf{k}_1, \dots, \hat{\mathbf{k}}_{i_1}, \dots, \hat{\mathbf{k}}_{i_n}, \dots, \mathbf{k}_N)}{\prod_{\substack{1 \leq i < j \leq N \\ i, j \notin \{i_1, \dots, i_n\}}} P(\mathbf{k}_i + \mathbf{k}_j)} \\ &\quad \times e^{\zeta_1 + \dots + \hat{\zeta}_{i_1} + \dots + \hat{\zeta}_{i_n} + \dots + \zeta_N} \\ &+ \sum_{n=1}^{N-1} \sum_{1 \leq i_1 < \dots < i_n \leq N} e^{2(\zeta_{i_1} + \dots + \zeta_{i_n} + \sum_{1 \leq r < s \leq n} d_{ir is})} P(D_{x_1}, \dots, D_{x_M}) \tilde{f} \cdot \tilde{f}, \tag{1.6} \end{aligned}$$

where

$$\tilde{f} = \tilde{f}_{i_1 \dots i_n} = \sum_{\tilde{v}_{i_1 \dots i_n} = 0, 1} \exp\left(\sum_{\substack{1 \leq i \leq N \\ i \notin \{i_1, \dots, i_n\}}} v_i \tilde{\zeta}_i + \sum_{\substack{1 \leq i < j \leq N \\ i, j \notin \{i_1, \dots, i_n\}}} d_{ij} v_i v_j\right), \tag{1.7}$$

with

$$\tilde{\zeta}_i = \zeta_i + \sum_{r=1}^n d_{ii_r}, \quad \tilde{v}_{i_1 \dots i_n} = (v_1, \dots, \hat{v}_{i_1}, \dots, \hat{v}_{i_n}, \dots, v_N), \tag{1.8}$$

where each v_i in $\tilde{v}_{i_1 \dots i_n}$ takes values 0 or 1.

Based on the recursive relation (1.6), a Hirota bilinear equation admits an N -soliton solution if and only if all Hirota conditions are satisfied:

$$\begin{aligned} & H(\mathbf{k}_{i_1}, \dots, \mathbf{k}_{i_n}) := \\ & \sum_{\sigma = \pm 1} P\left(\sum_{r=1}^n \sigma_r \mathbf{k}_{i_r}\right) \prod_{1 \leq r < s \leq n} P(\sigma_r \mathbf{k}_{i_r} - \sigma_s \mathbf{k}_{i_s}) \sigma_r \sigma_s = 0, \tag{1.9} \end{aligned}$$

for $1 \leq n \leq N$ and $1 \leq i_1 < \dots < i_n \leq N$, where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ with $\sigma_r = \pm 1$. The special case $n = 1$ recovers the dispersion relations in (1.5).

For the (2+1)-dimensional case, let x, y denote spatial variables and t time. A general Hirota bilinear equation in $(2 + 1)$ -dimensions can be written as

$$P(D_x, D_y, D_t)f \cdot f = 0, \quad (1.10)$$

where P is a polynomial in the three Hirota derivatives. Using Bell polynomial theory, nonlinear PDEs for a scalar field u may be derived from such bilinear forms via logarithmic derivative transformations (see, e.g., [11]). Typical transformations include

$$u = \beta(\ln f)_{xx}, \quad u = \beta(\ln f)_{yy}, \quad u = \beta(\ln f)_{xy}, \quad u = \beta(\ln f)_x, \quad u = \beta(\ln f)_y, \quad (1.11)$$

where $\beta \neq 0$ is a constant. A crucial step in this approach is to verify that f satisfies the bilinear equation and that the corresponding field u , defined through the logarithmic transformations, satisfies the associated nonlinear PDE. Systematic algorithms for this verification have been developed in both the $(1 + 1)$ - and $(2 + 1)$ -dimensional settings (see, e.g., [10]).

Beyond soliton solutions, another significant class of explicit solutions comprises lump and rogue waves [12, 13]. Lump waves are rationally localized structures that decay algebraically in all spatial directions at fixed time [12, 14]. For example, the KPI equation admits diverse lump solutions, some of which are obtained as long-wave limits of multi-soliton configurations [15]. Lump-type solutions, rational and analytic in form, also appear in nonintegrable KP-, BKP-, KP-Boussinesq-type systems [16], and in breaking soliton equations [17], and even in linear higher-dimensional wave models via superposition principles [18].

The sum-of-squares ansatz, which incorporates a positive quadratic function into a bilinear framework, has proven highly effective for constructing lump-type solutions [19]. When paired with a logarithmic derivative transformation, this method yields explicit lump-wave solutions for a wide range of nonlinear PDEs. In this work, we apply the approach to a $(2 + 1)$ -dimensional extended KP-like model featuring spatially balanced nonlinear and dispersive terms. The resulting lump-wave structures emerge from a delicate interplay between nonlinearity and dispersion. Using symbolic computation, we derive closed-form lump solutions and analyze the stationary points of the associated quadratic form, providing insight into the mechanisms underlying their propagation and localization. A distinctive feature of this study is the application of generalized bilinear derivatives associated with the prime number three.

2 An Extended KP-like Model with Spatially Balanced Nonlinearity and Dispersion

To handle general differential equations, including terms of arbitrary order, Hirota's bilinear derivatives need to be extended. A broad class of such operators, called gener-

alized bilinear differential operators, was introduced in [20]. In the (2+1)-dimensional case with coordinates (x, y, t) , they are defined by

$$D_{p,x}^m D_{p,y}^n D_{p,t}^k f \cdot f = \left(\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} + \alpha_p \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'}\right)^k f(x, y, t) f(x', y', t')|_{x'=x, y'=y, t'=t}, \tag{2.1}$$

where the coefficients α_p^k are determined by

$$\alpha_p^k = (-1)^{r(k)} \text{ where } k \equiv r(k) \pmod p, 0 \leq r(k) < p. \tag{2.2}$$

These coefficients can be computed symbolically using computer algebra algorithms such as Maple and Mathematica.

For example, for $p = 3$, the sequence of coefficients is

$$\alpha_3 = -1, \alpha_3^2 = \alpha_3^3 = 1, \alpha_3^4 = -1, \alpha_3^5 = \alpha_3^6 = 1, \dots, \tag{2.3}$$

and for $p = 5$, it is

$$\begin{aligned} \alpha_5 = -1, \alpha_5^2 = 1, \alpha_5^3 = -1, \alpha_5^4 = \alpha_5^5 = 1, \alpha_5^6 = -1, \alpha_5^7 = 1, \\ \alpha_5^8 = -1, \alpha_5^9 = \alpha_5^{10} = 1, \dots \end{aligned} \tag{2.4}$$

Coefficients for other values of p can be computed similarly (see [20]).

To summarize, the resulting sign patterns for $p = 3, 5, 7$ are:

$$\begin{aligned} & -, +, +, -, +, +, -, +, +, \dots \quad (p = 3), \\ & -, +, -, +, +, -, +, -, +, +, -, +, -, +, +, \dots \quad (p = 5), \\ & -, +, -, +, -, +, +, -, +, -, +, -, +, -, +, -, +, -, +, +, \dots \quad (p = 7). \end{aligned}$$

It is also interesting to explore how such alternating sign sequences relate to group-theoretic structures. Note that for even p , the generalized bilinear derivatives reduce to the standard Hirota derivatives.

2.1 Generalized bilinear formulation

For $p = 3$, we introduce the following generalized bilinear KP-like model equation:

$$\begin{aligned} P_{\text{sbeKP-like}}(f) & := (\sigma_1 D_{3,x}^4 + \sigma_2 D_{3,y}^4 + \sigma_3 D_{3,x}^3 + \sigma_4 D_{3,y}^3 \\ & \quad + \rho_1 D_{3,t} D_{3,x} + \rho_2 D_{3,t} D_{3,y} + \rho_3 D_{3,x}^2 + \rho_4 D_{3,x} D_{3,y} + \rho_5 D_{3,y}^2) f \cdot f \\ & = 2[3\sigma_1 f_{xx}^2 + 3\sigma_2 f_{yy}^2 + \sigma_3 f f_{xxx} + \sigma_4 f f_{yyy} + \rho_1 (f_{tx} f - f_t f_x) \\ & \quad + \rho_2 (f_{ty} f - f_t f_y) \\ & \quad + \rho_3 (f_{xx} f - f_x^2) + \rho_4 (f_{xy} f - f_x f_y) + \rho_5 (f_{yy} f - f_y^2)] = 0, \end{aligned} \tag{2.5}$$

where $D_{3,x}$, $D_{3,y}$ and $D_{3,t}$ are the generalized bilinear derivatives, and σ_i for $1 \leq i \leq 4$ and ρ_i for $1 \leq i \leq 5$ are arbitrary constants. The model incorporates spatially balanced fourth- and third-order derivatives that generate nonlinear terms, as well as spatially balanced second-order derivatives that produce linear dispersion terms in the resulting nonlinear model. In the Bogoyavlensky-Konopelchenko-like equation, the second fourth-order term is $D_{3,x}^3 D_{3,y}$, (see, e.g., [21] for the $p = 2$ case). Although the equation (2.5) does not satisfy the Hirota conditions (1.9), this construction nevertheless yields a novel model that admits lump-wave solutions.

2.2 A nonlinear model

By redefining the dependent variables as

$$u = 2(\ln f)_{xy}, \quad v = 2(\ln f)_{xx}, \quad w = 2(\ln f)_{yy}, \quad r = 2(\ln f)_x, \quad s = 2(\ln f)_y, \quad (2.6)$$

the extended KP-like model can be expressed in nonlinear form as

$$\begin{aligned} &K_{\text{seKP-like}}(u, v, w, r, s) \\ &:= \frac{1}{2}\sigma_1[u_{xx}(6v + 3r^2) + u_x(6v_x + 12vr + 3r^3) + 3u(2v_xr + 2v^2 + 3vr^2)] \\ &\quad + \frac{1}{2}\sigma_2[u_{yy}(6w + 3s^2) + u_y(6w_y + 12ws + 3s^3) + 3u(2w_y s + 2w^2 + 3ws^2)] \\ &\quad + \frac{3}{2}\sigma_3\left[\frac{2}{3}u_{xxx} + u_{xx}r + 2u_xv + \frac{1}{2}u_xr^2 + u(v_x + vr)\right] \\ &\quad + \frac{3}{2}\sigma_4\left[\frac{2}{3}u_{yyy} + u_{yy}s + 2u_yw + \frac{1}{2}u_ys^2 + u(w_y + ws)\right] \\ &\quad + \rho_1u_{tx} + \rho_2u_{ty} + \rho_3u_{xx} + \rho_4u_{xy} + \rho_5u_{yy} = 0, \end{aligned} \quad (2.7)$$

subject to the compatibility condition

$$u_x = v_y, \quad u_y = w_x, \quad r_y = s_x = u. \quad (2.8)$$

This equation incorporates four sets of nonlinear terms and five dispersive contributions, all of which are spatially balanced. Despite its complexity, the model admits lump-wave solutions arising from the interplay between nonlinearity and dispersion.

Special reductions occur when only one pair of nonlinear coefficients and one pair of dispersive coefficients are nonzero: If $\sigma_1 = \sigma_2 = 1$ and $\rho_1 = -\rho_5 = 1$ and all others vanish, the model equation (2.7) reduces to

$$\begin{aligned} &\frac{1}{2}u_{xx}(6v + 3r^2) + \frac{1}{2}u_{yy}(6w + 3s^2) + \frac{1}{2}u_x(6v_x + 12vr + 3r^3) + \frac{1}{2}u_y(6w_y \\ &\quad + 12ws + 3s^3) + 3u(v_xr + w_y s + v^2 + w^2 + \frac{3}{2}vr^2 + \frac{3}{2}ws^2) \\ &\quad + u_{tx} - u_{yy} = 0. \end{aligned} \quad (2.9)$$

If $\sigma_1 = \sigma_2 = 1$ and $\rho_2 = -\rho_3 = 1$ and all others vanish, the model reduces to

$$\begin{aligned} & \frac{1}{2}u_{xx}(6v + 3r^2) + \frac{1}{2}u_{yy}(6w + 3s^2) + \frac{1}{2}u_x(6v_x + 12vr + 3r^3) + \frac{1}{2}u_y(6w_y \\ & + 12ws + 3s^3) + 3u(v_xr + w_ys + v^2 + w^2 + \frac{3}{2}vr^2 + \frac{3}{2}ws^2) \\ & + u_{ty} - u_{xx} = 0. \end{aligned} \quad (2.10)$$

In both cases, the compatibility condition (2.8) is satisfied, and the resulting reduced models admit lump-wave solutions.

2.3 Connection between bilinear and nonlinear forms

We can explicitly map the bilinear form to its nonlinear counterpart. More concretely, the bilinear form (2.5) and the nonlinear equation (2.7) are related through

$$K_{\text{seKP-like}}(u, v, w, r, s) = \left[\frac{P_{\text{sbcKP-like}}(f)}{f^2} \right]_{xy}, \quad (2.11)$$

which can be verified via symbolic computation. Consequently, any solution f of the bilinear equation generates the corresponding fields u, v, w, r, s solving the nonlinear model.

Let us now state the main results of this section in the following theorem.

Theorem 2.1 *Let a bilinear differential equation be given by (2.5), where $D_{3,x}$ is defined in (2.1) and (2.2). Then, under the transformation (2.6), the corresponding nonlinear differential equation is determined by (2.7), together with the compatibility condition (2.8). Moreover, the connection between the bilinear and nonlinear equations is established by (2.11).*

Finally, a natural question arises as to whether this generalized formulation exhibits integrability and supports lump-wave solutions, a characteristic feature of integrable systems. In the following section, we investigate lump-wave solutions that emerge from the interplay between nonlinearity and dispersion.

3 Lump-wave dynamics governed by nonlinearity and dispersion

We now focus on the explicit construction of lump-wave solutions for the extended KP-like nonlinear model (2.7), featuring spatially balanced nonlinearity and dispersion. This is achieved using its generalized bilinear form (2.5) in combination with symbolic computation. Particular attention is given to the interplay between nonlinear and dispersive terms, which collectively generate the lump-wave structures. Additionally, we analyze the stationary points of the associated quadratic function to gain insight into the dynamics and spatial localization of these waves.

3.1 Sum-of-squares approach

It is well known that a standard approach for constructing lump solutions in higher-dimensional nonlinear evolution equations is the sum-of-squares ansatz [19]. The key idea is to represent the dependent variable as a logarithmic derivative of a positive quadratic function. Specifically, we will take

$$f = \theta_1^2 + \theta_2^2 + a_9, \quad \theta_1 = a_1x + a_2y + a_3t + a_4, \quad \theta_2 = a_5x + a_6y + a_7t + a_8, \quad (3.1)$$

which ensures rational localization in both spatial directions provided that $a_1a_6 - a_2a_5 \neq 0$.

The standard procedure is to substitute (3.1) into the bilinear equation (2.5), which reduces the problem to an algebraic system for the parameters a_i . Using symbolic computation, this system can be solved to obtain explicit expressions for a_3 , a_7 , and a_9 in terms of the remaining coefficients, which can be chosen arbitrarily:

$$a_3 = -\frac{1}{(a_1\rho_1 + a_2\rho_2)^2 + (a_5\rho_1 + a_6\rho_2)^2} [a_1(a_1^2 + a_3^2)\rho_1\rho_3 + a_2(a_1^2 + a_3^2)\rho_1\rho_4 + (a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)\rho_1\rho_5 + (a_1^2a_2 + 2a_1a_5a_6 - a_2a_3^2)\rho_2\rho_3 + a_1(a_2^2 + a_6^2)\rho_2\rho_4 + a_2(a_2^2 + a_6^2)\rho_2\rho_5], \quad (3.2)$$

$$a_7 = -\frac{1}{(a_1\rho_1 + a_2\rho_2)^2 + (a_5\rho_1 + a_6\rho_2)^2} [a_5(a_1^2 + a_3^2)\rho_1\rho_3 + a_6(a_1^2 + a_3^2)\rho_1\rho_4 + (2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)\rho_1\rho_5 + (2a_1a_2a_5 - a_1^2a_6 + a_3^2a_6)\rho_2\rho_3 + a_5(a_2^2 + a_6^2)\rho_2\rho_4 + a_6(a_2^2 + a_6^2)\rho_2\rho_5], \quad (3.3)$$

and

$$a_9 = -\frac{3[\sigma_1(a_1^2 + a_3^2)^2 + \sigma_2(a_2^2 + a_6^2)^2][(a_1\rho_1 + a_2\rho_2)^2 + (a_5\rho_1 + a_6\rho_2)^2]}{(a_1a_6 - a_2a_5)^2(\rho_1^2\rho_5 - \rho_1\rho_2\rho_4 + \rho_2^2\rho_3)}. \quad (3.4)$$

These expressions encode the dispersion relations and the structural conditions for the lump waves. Specifically, a_3 and a_7 determine the temporal frequencies, expressed as second-order rational functions of the wave numbers, while a_9 characterizes the balance between the wave numbers and the dispersion coefficients. Similar dispersion relations appear in lump-wave solutions associated with the second flow of the KP hierarchy

and in generalized KP-type models (see, e.g., [22]-[26]). Interestingly, the coefficients σ_3 and σ_4 do not appear in the expressions for a_3 , a_5 and a_9 .

The well-posedness and localization of the lump waves require two essential non-degeneracy conditions, including a dispersion condition

$$\rho_1^2\rho_5 - \rho_1\rho_2\rho_4 + \rho_2^2\rho_3 \neq 0, \quad (3.5)$$

which implies

$$\rho_1^2 + \rho_2^2 \neq 0, \quad (3.6)$$

ensuring nontrivial temporal dynamics. The second is a determinant condition

$$a_1 a_6 - a_2 a_5 \neq 0, \quad (3.7)$$

which further guarantees

$$a_1^2 + a_5^2 \neq 0, \quad a_2^2 + a_6^2 \neq 0, \quad (3.8)$$

thereby ensuring that the fields u and v , defined via the logarithmic derivative transformations in (2.6), decay to zero as $x^2 + y^2 \rightarrow \infty$. This confirms the spatial localization of the lump wave solutions.

The positivity of f , and thus analyticity of the resulting lump waves u, v, w, r, s is guaranteed by the following necessary and sufficient condition that couples the nonlinearity and dispersion parameters:

$$\frac{\sigma_1(a_1^2 + a_5^2)^2 + \sigma_2(a_2^2 + a_6^2)^2}{\rho_1^2 \rho_5 - \rho_1 \rho_2 \rho_4 + \rho_2^2 \rho_3} < 0. \quad (3.9)$$

This inequality holds under either of the following two scenarios:

$$\sigma_1 > 0, \quad \sigma_2 > 0, \quad \rho_1^2 \rho_5 - \rho_1 \rho_2 \rho_4 + \rho_2^2 \rho_3 < 0, \quad (3.10)$$

or

$$\sigma_1 < 0, \quad \sigma_2 < 0, \quad \rho_1^2 \rho_5 - \rho_1 \rho_2 \rho_4 + \rho_2^2 \rho_3 > 0. \quad (3.11)$$

Two reduced models, given in (2.9) and (2.10), satisfy the first scenario (3.10) and therefore support lump-wave structures. The dispersion coefficient ρ_4 contributes only when $\rho_1 \rho_2 \neq 0$. Condition (3.9) ensures that $a_9 > 0$, as given in (3.4), which keeps f , defined in (3.1), strictly positive. Consequently, the fields u, v, w, r, s remain analytic across the entire (x, y, t) -domain. Essentially, this condition imposes a constraint on the nonlinear and dispersive coefficients, and highlights that lump-wave formation arises from the interplay of nonlinear and linear dispersive terms.

In summary, constructing lump-wave solutions via the logarithmic derivative transformations requires two essential conditions: the determinant condition (3.7), which guarantees spatial localization in all directions, and the positivity condition (3.9), which ensures the well-posedness of u, v, w, r, s across the spatial-temporal domain. The sufficient conditions (3.10) and (3.11) further establish the existence of lump waves. Under these requirements, the resulting rational functions u, v, w, r, s indeed correspond to fully localized lump-wave solutions.

We can also present the main results obtained so far in the following theorem.

Theorem 3.1 *Let the quadratic function f be defined by (3.1), with a_3, a_7, a_9 given by (3.2), (3.3) and (3.4), respectively. If f satisfies the determinant condition (3.7) and the positivity condition (3.9), then the transformation (2.6) yields lump-wave solutions to the nonlinear model equation (2.7).*

A consequence of this theorem is given below.

Corollary 3.1 *If the nonlinear and dispersive coefficients satisfy (3.10) or (3.11), then (2.6), together with (3.1), generates lump-wave solutions to nonlinear model equation (2.7).*

It is therefore clear that the lump waves obtained correspond to the interplay of nonlinearity and dispersion.

3.2 Trajectory of stationary points

The stationary points of the quadratic function f can be computed directly and used to characterize the dynamical behavior of the lump waves. Setting $f_x = f_y = 0$ gives

$$a_1\theta_1 + a_5\theta_2 = 0, \quad a_2\theta_1 + a_6\theta_2 = 0, \quad (3.12)$$

which, under the non-degeneracy condition (3.7), reduces to

$$\theta_1 = 0, \quad \theta_2 = 0, \quad (3.13)$$

with θ_1, θ_2 defined by (3.1). Solving this system explicitly yields linear trajectories for $x(t)$ and $y(t)$, representing the stationary points of f :

$$\begin{aligned} x(t) = & \frac{[(a_1^2 + a_5^2)\rho_3 - (a_2^2 + a_6^2)\rho_5]\rho_1 + [2(a_1a_2 + a_5a_6)\rho_3 + (a_2^2 + a_6^2)\rho_4]\rho_2}{(a_1\rho_1 + a_2\rho_2)^2 + (a_5\rho_1 + a_6\rho_2)^2} t \\ & + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \end{aligned} \quad (3.14)$$

$$\begin{aligned} y(t) = & \frac{[(a_1^2 + a_5^2)\rho_4 + 2(a_1a_2 + a_5a_6)\rho_5]\rho_1 - [(a_1^2 + a_5^2)\rho_3 - (a_2^2 + a_6^2)\rho_5]\rho_2}{(a_1\rho_1 + a_2\rho_2)^2 + (a_5\rho_1 + a_6\rho_2)^2} t \\ & - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}. \end{aligned} \quad (3.15)$$

These expressions describe straight-line trajectories in the (x, y) -plane along which the lump waves maintain constant values, while retaining rational localization elsewhere.

We present the main results in the following theorem.

Theorem 3.2 *The stationary points of f are given by (3.14) and (3.15), forming straight-line trajectories with constant velocities. The lump waves generated by (2.6) attain constant amplitudes along these trajectories while remaining rationally localized elsewhere.*

Figures 1, 2, 3, 4, 5, 6, 7, 8, 9 illustrate the lump wave $u = 2(\ln f)_{xy}$ in both 3d and 2d representations for the parameter values specified below:

$$\sigma_1 = -1, \quad \sigma_2 = 2, \quad \rho_1 = 1, \quad \rho_2 = -2, \quad \rho_3 = 1, \quad \rho_4 = 3, \quad \rho_5 = -2, \quad (3.16)$$

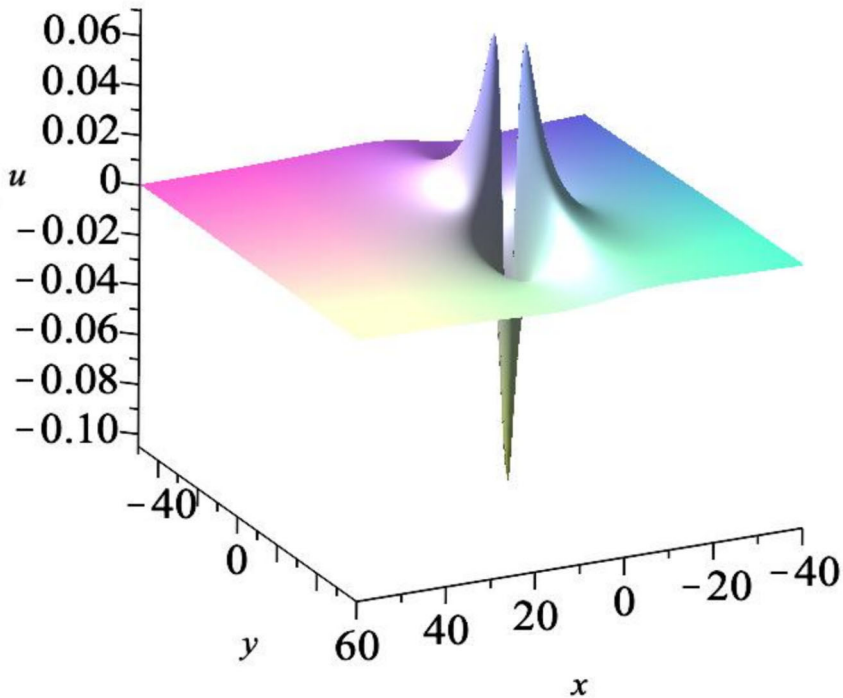


Fig. 1 3d-plot of $u = u(x, y, t)$ at $t = 0$, corresponding to the parameters in (3.16) and (3.17).

and

$$a_1 = 1, a_2 = 1, a_4 = -2, a_5 = 3, a_6 = -1, a_8 = 1. \quad (3.17)$$

4 Concluding remarks

We introduce and investigate a novel $(2 + 1)$ -dimensional extended KP-type nonlinear model featuring spatially balanced nonlinearities and dispersive effects. Using symbolic computation within computer algebra systems, we construct explicit lump-wave solutions by defining a quadratic function f as in (3.1), where the parameters a_3 , a_7 , and a_9 are given by (3.2), (3.3), and (3.4), respectively. The transformation (2.6) then yields lump-wave solutions to the nonlinear model (2.7). Our approach employs generalized bilinear derivatives linked to the prime number three. Particular, our model represents the first example employing third-order bilinear derivatives in this context. Notably, the third-order bilinear derivatives yield a zero term only, in the case of the standard Hirota bilinear derivative. The resulting lump waves maintain constant amplitude along characteristic trajectories determined by the stationary points of the associated quadratic function, illustrating the complex interaction between the model's nonlinear and dispersive terms. Nonlinear dynamical models based on gener-

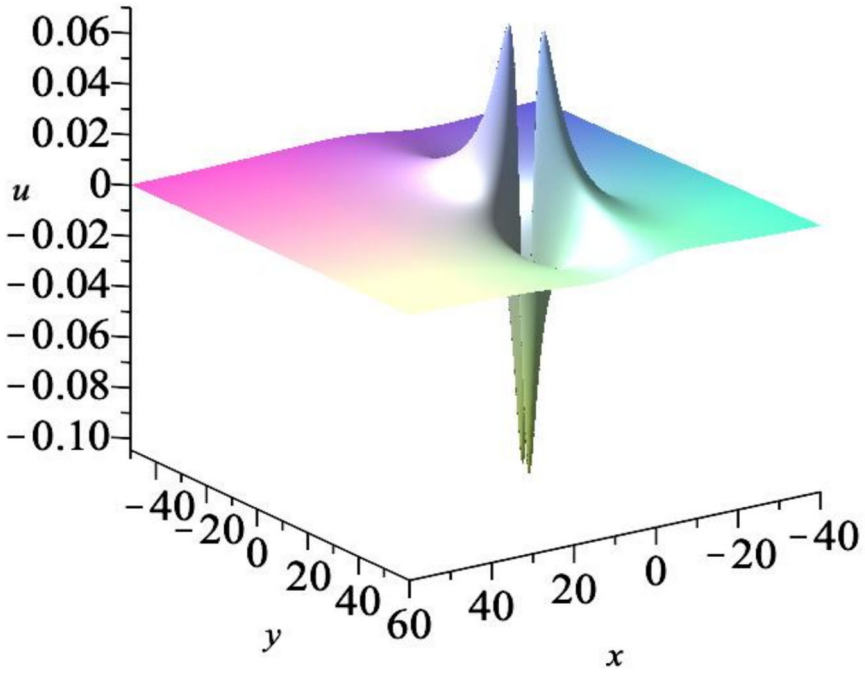


Fig. 2 3d-plot of $u = u(x, y, t)$ at $t = 3$, corresponding to the parameters in (3.16) and (3.17).

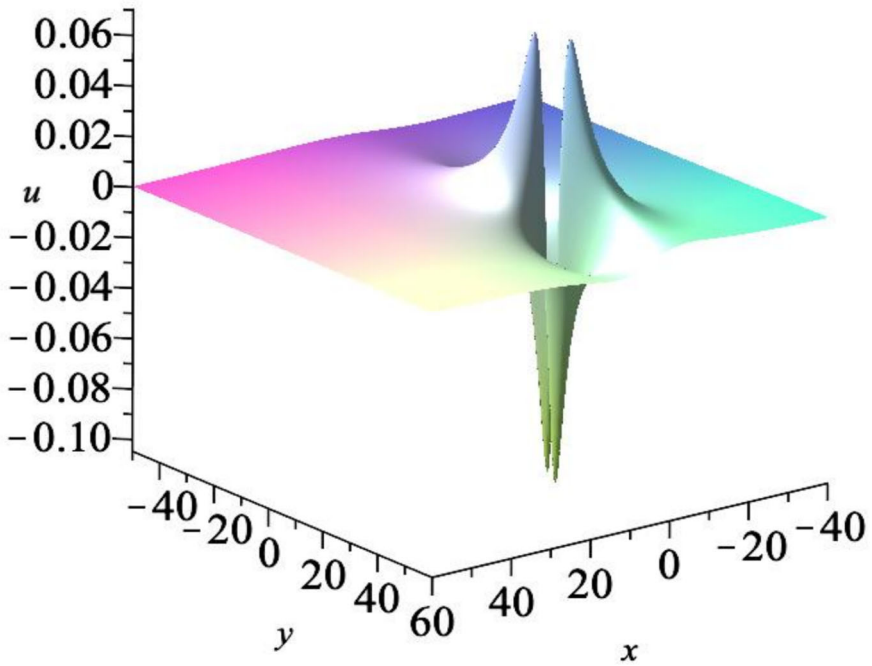


Fig. 3 3d-plot of $u = u(x, y, t)$ at $t = 6$, corresponding to the parameters in (3.16) and (3.17)

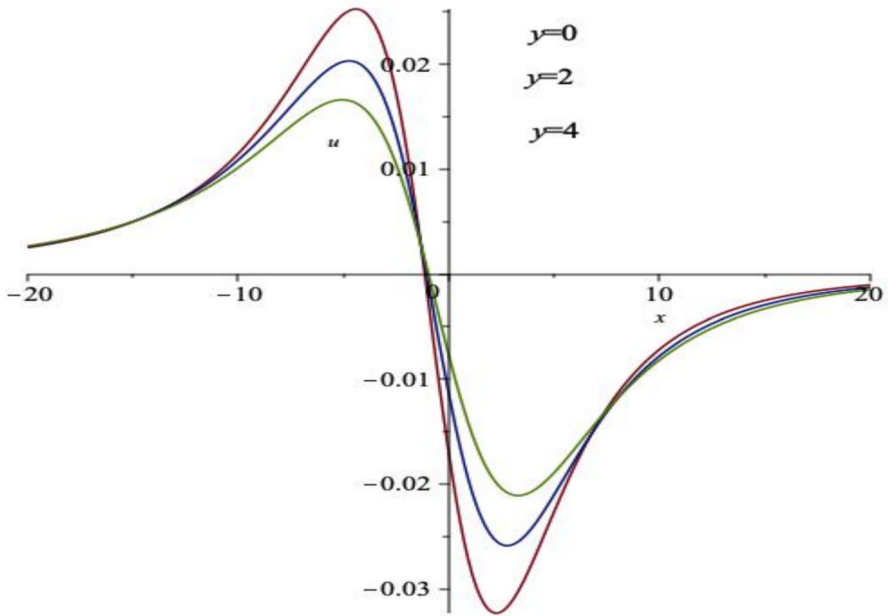


Fig. 4 x curves of $u = u(x, y, t)$ at $t = -5$ for $y = 0, 2, 4$, corresponding to the parameters in (3.16) and (3.17).

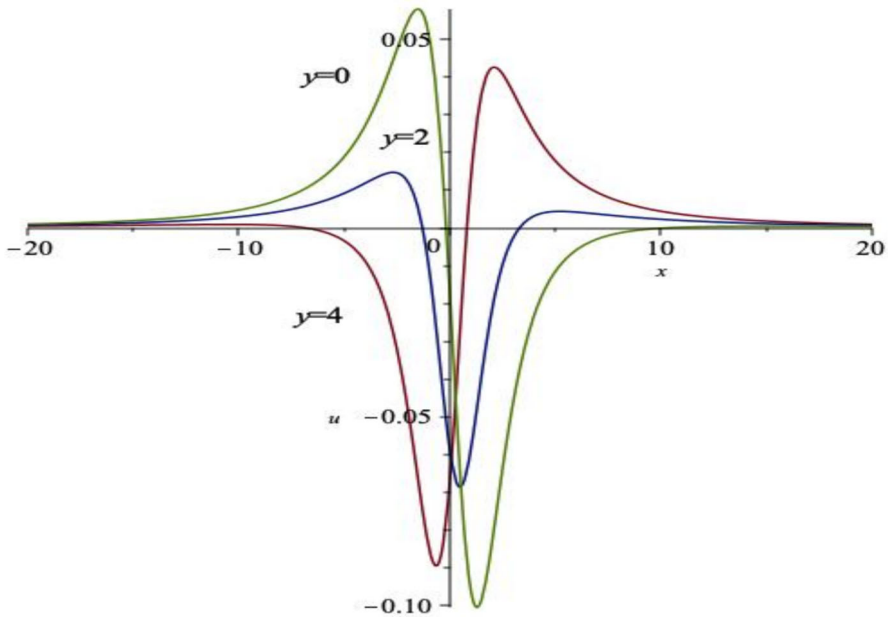


Fig. 5 x curves of $u = u(x, y, t)$ at $t = 0$ for $y = 0, 2, 4$, corresponding to the parameters in (3.16) and (3.17).

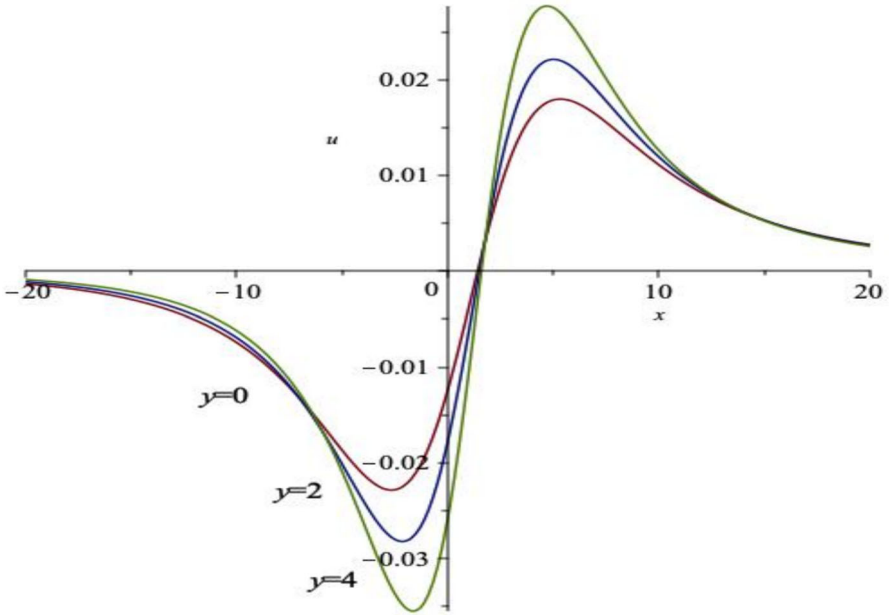


Fig. 6 x curves of $u = u(x, y, t)$ at $t = 5$ for $y = 0, 2, 4$, corresponding to the parameters in (3.16) and (3.17).

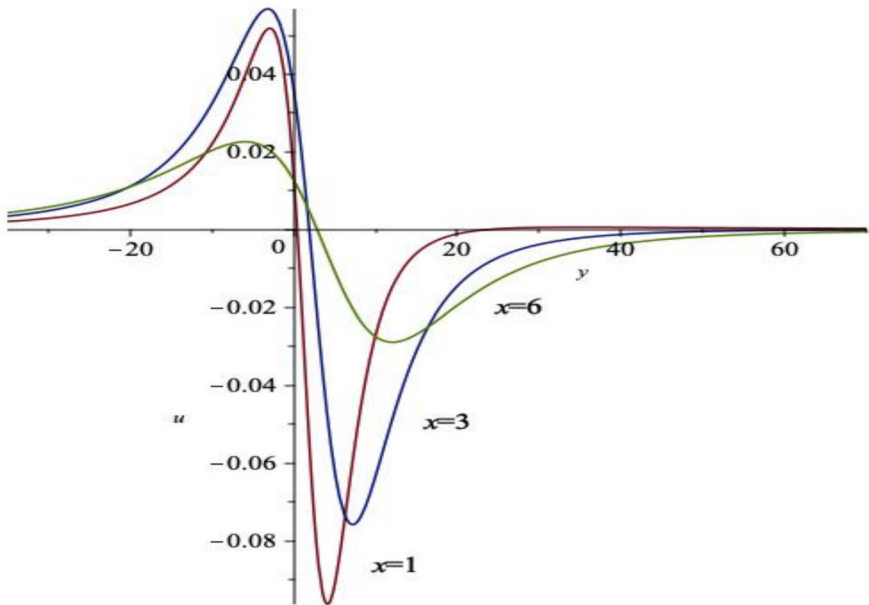


Fig. 7 y curves of $u = u(x, y, t)$ at $t = 0$ for $x = 1, 3, 6$, corresponding to the parameters in (3.16) and (3.17)

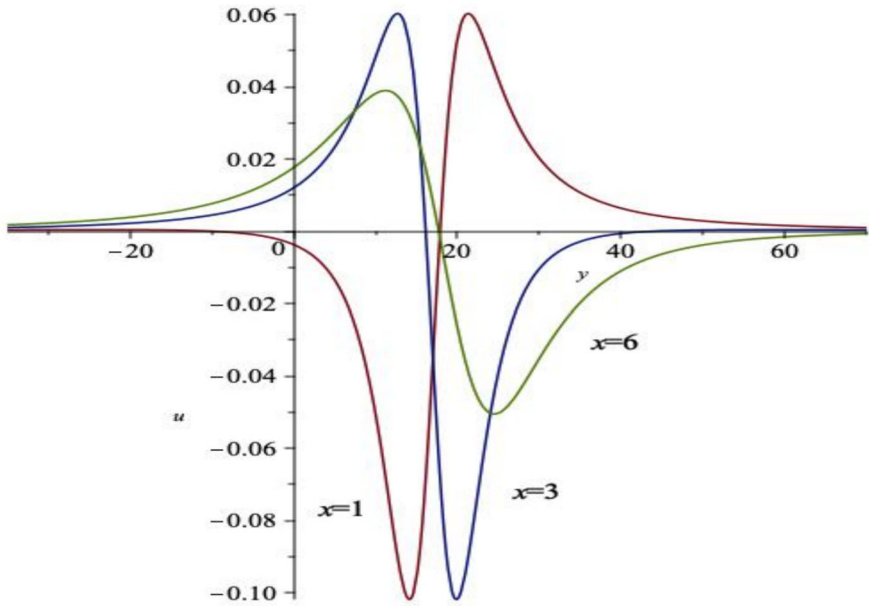


Fig. 8 y curves of $u = u(x, y, t)$ at $t = 5$ for $x = 1, 3, 6$ corresponding to the parameters in (3.16) and (3.17)

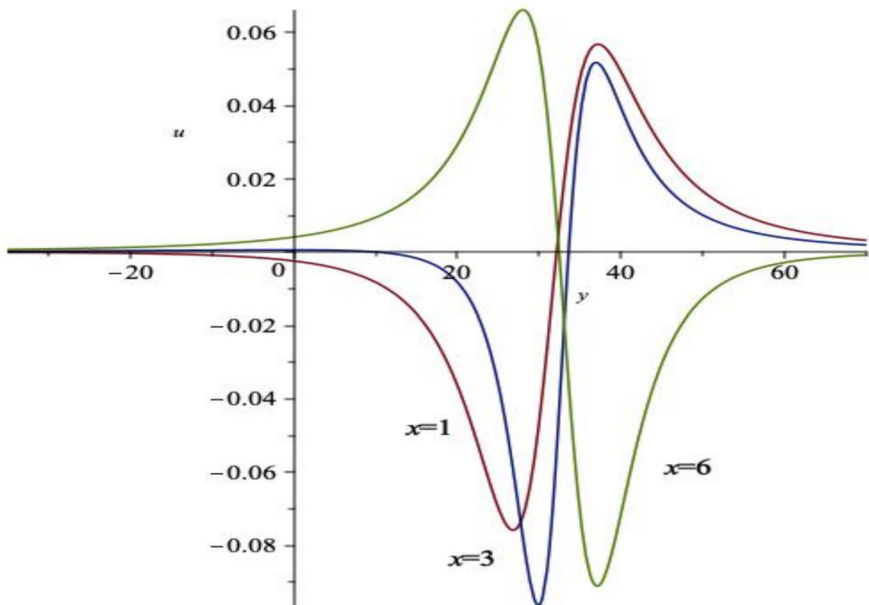


Fig. 9 y curves of $u = u(x, y, t)$ at $t = 10$ for $x = 1, 3, 6$, corresponding to the parameters in (3.16) and (3.17).

alized bilinear derivatives are still relatively new, and it remains unclear whether they admit N -solitons or N -rogue waves.

Lump waves occupy a central position in the study of nonlinear dispersive systems, embodying both mathematical elegance and physical relevance. Their remarkable versatility underscores the ongoing challenges in modeling complex wave phenomena. Such structures have been identified in linear formulations (see, e.g., [18]) and a broad spectrum of nonlinear and nonintegrable models, extending across $(2 + 1)$ (see [27–34]), $(3 + 1)$ (see [22] [35–37]), and $(4 + 1)$ dimensions (see, e.g., [38]). Their analytical realization is commonly achieved via Hirota's bilinear method and its modern generalizations, providing a unified framework for identifying localized coherent waveforms [12].

The dynamics of lump waves are further enriched by their interactions with homoclinic, heteroclinic, and solitonic structures in $(2 + 1)$ -dimensional integrable systems [39–41]. These phenomena are closely related to reductions of multi-soliton configurations, whose analytical and integrability properties have been explored using Riemann–Hilbert analysis and bi-Hamiltonian theory (see, e.g., [42–47]).

Ongoing efforts to investigate the existence, algebraic structure, and nonlinear dynamics of lump and rogue waves in extended systems [48], whether scalar [49] or vectorial [50], remain a major area of interest in nonlinear wave research. Beyond their mathematical importance, these studies enhance our understanding of coherent energy localization [51, 52], offering insights with potential applications across diverse physical and engineering fields [53–55]

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Declarations

Competing interests The authors declare no competing interests.

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