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Conservation laws of a perturbed Kaup–Newell equation

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A new Lax pair is introduced for a perturbed Kaup–Newell equation and used to construct two series of conservation laws through a Riccati equation that a ratio of eigenfunctions satisfies. Both series of conservation laws are defined recursively, and the first two in each series are presented explicitly.

Keywords: Lax pair; conservation law; Riccati equation.

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1. Introduction

Many nonlinear models are studied and showed to possess infinitely many conservation laws, and bi-Hamiltonian structures play a crucial role in constructing conserved densities.[1] Associated with non-semisimple Lie algebras, the variational identities have been developed to formulate generating functions for conserved densities.[2,3] There are also other approaches to conservation laws including the method using adjoint symmetries[4–6] and the expansion technique of ratios of eigenfunctions of spectral problems.[7,8]

A perturbed Kaup–Newell equation was recently introduced from a generalized Kaup–Newell spectral problem with linear perturbation.[9] The generalized

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Kaup–Newell equation is defined by

$$\begin{cases} p_t = \frac{1}{2}(p_{xx} - 2pp_xq - p^2q_x - 4\alpha pp_x), \\ q_t = -\frac{1}{2}q_{xx} - \frac{1}{2}p_xq^2 - pqq_x - 2\alpha(pq)_x, \end{cases} \tag{1.1}$$

where α is an arbitrary constant. The case of $\alpha = 0$ reads to the standard Kaup–Newell equation,[10] which possesses a tri-Hamiltonian structure[11] and is the Euler–Poincaré flow on the space of first-order differential operators.[12]

In this paper, we would like to present a different Lax pair for the perturbed Kaup–Newell equation (1.1) and construct two series of conservation laws through a Riccati equation which a ratio of two eigenfunctions needs to satisfy.[7] Such a scheme of constructing conservation laws using Lax pairs has been also applied to other nonlinear integrable equations (see, e.g. Refs. 13 and 14). A few concluding remarks are given at the end of the paper.

2. A Riccati Equation and Conservation Laws

Motivated by a Lax pair for the Kaup–Newell equation (see, e.g. Ref. 15), we can derive a new Lax pair for the perturbed Kaup–Newell equation (1.1):

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad U = (U_{ij})_{2\times 2} = \begin{bmatrix} \lambda + \alpha p & \lambda p \\ q & -\lambda - \alpha p \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{2.1}$$

and

$$\phi_t = V\phi = V(u,\lambda)\phi, \quad V = (V_{ij})_{2\times 2}\,, \tag{2.2}$$

where

$$\begin{cases} V_{11} = -V_{22} = \lambda^2 - \frac{1}{2}\lambda pq - \frac{1}{2}\alpha p^2 q - \alpha^2 p^2 + \frac{1}{2}\alpha p_x\,, \\ V_{12} = -\frac{1}{2}\lambda(-2\lambda p - p_x + p^2 q + 2\alpha p^2)\,, \\ V_{21} = \lambda q - \frac{1}{2}q_x - \frac{1}{2}pq^2 - \alpha pq\,. \end{cases} \tag{2.3}$$

That is to say, the isospectral ($\lambda_t = 0$) zero curvature equation:

$$U_t - V_x + [U,V] = 0$$

exactly presents the perturbed Kaup–Newell equation (1.1).

To apply the scheme of generating conservation laws based on Lax pairs, we consider the ratio of the two eigenfunctions

$$\phi_{21} = \frac{\phi_2}{\phi_1}\,. \tag{2.4}$$

Obviously from the spectral problem in (2.1), we see that the ratio ϕ_{21} satisfies the following Riccati equation:

$$\phi_{21,x} = -2(\lambda + \alpha p)\phi_{21} + q - \lambda p \phi_{21}^2\,. \tag{2.5}$$

Expanding ϕ_{21} into a Laurent series

$$\phi_{21} = \sum_{i=1}^{\infty} \chi_i \lambda^{-i}\,, \tag{2.6}$$

we obtain from the above Riccati equation a recursion relation for defining χ_i:

$$\chi_1 = \frac{q}{2}, \quad \chi_{i+1} = -\frac{1}{2}\chi_{i,x} - \frac{1}{2}p\sum_{j=1}^{i}\chi_{i-j+1}\chi_j - \alpha p\chi_i, \quad i \geq 1\,. \tag{2.7}$$

Particularly, this gives rise to

$$\chi_2 = -\frac{1}{8}pq^2 - \frac{1}{2}\alpha pq - \frac{1}{4}q_x\,,$$

$$\chi_3 = \frac{1}{16}p^2q^3 + \frac{1}{4}pqq_x + \frac{1}{16}p_xq^2 + \frac{1}{2}\alpha^2p^2q + \frac{3}{8}\alpha p^2q^2 + \frac{1}{2}\alpha pq_x + \frac{1}{4}\alpha p_xq + \frac{1}{8}q_{xx}\,.$$

Expanding ϕ_{21} into another Laurent series

$$\phi_{21} = \sum_{i=0}^{\infty} \kappa_i \lambda^{-i}\,, \tag{2.8}$$

we obtain from the above Riccati equation a recursion relation for defining κ_i:

$$\begin{cases} \kappa_0 = -\dfrac{2}{p}, \quad \kappa_1 = \dfrac{1}{2}\kappa_{0,x} - \dfrac{1}{2}q + \alpha p\kappa_0 = -\left(\dfrac{1}{p}\right)_x - \dfrac{1}{2}q - 2\alpha\,, \\ \kappa_{i+1} = \dfrac{1}{2}\kappa_{i,x} + \dfrac{1}{2}p\displaystyle\sum_{j=1}^{i}\kappa_{i-j+1}\kappa_j + \alpha p\kappa_i, \quad i \geq 1\,. \end{cases} \tag{2.9}$$

Note that there are the opposite signs in defining χ_{i+1} and κ_{i+1} due to the existence of an additional term $2p\kappa_0\kappa_i\lambda^{-i}$ in the second recursive relation. The recursion relation (2.9) particularly produces

$$\kappa_2 = \frac{1}{8p^3}(p^4q^2 - 2p^3q_x - 4p^2p_xq + 4pp_{xx} - 4p_x^2 + 4\alpha p^4q - 8\alpha p^2p_x)\,,$$

$$\begin{aligned} \kappa_3 = \frac{1}{16p^4}(&-p^6q^3 + 4p^5qq_x + 7p^4p_xq^2 - 2p^4q_{xx} - 8p^3p_xq_x - 8p^3p_{xx}q \\ &+ 4p^2p_{xxx} - 8pp_xp_{xx} + 4p_x^3 - 6\alpha p^6q^2 - 8\alpha^2p^6q + 8\alpha p^5q_x \\ &+ 16\alpha^2p^4p_x + 28\alpha p^4p_xq - 16\alpha p^3p_{xx})\,. \end{aligned}$$

Now to construct conservation laws, we introduce

$$I = V_{11} + V_{12}\phi_{21}, \quad J = U_{11} + U_{12}\phi_{21} \tag{2.10}$$

and then, from the presented Lax pair (2.1) and (2.2), we obtain

$$I_x = \left(\frac{\phi_{1,t}}{\phi_1}\right)_x = \left(\frac{\phi_{1,x}}{\phi_1}\right)_t = J_t\,. \tag{2.11}$$

Substituting the two Laurent series expansions of ϕ_{21} into this basic formula and equating the same powers of λ will produce conservation laws for the perturbed Kaup–Newell equation (1.1). In what follows, we compute these two series of conservation laws.

First, using (2.6) and comparing different powers of λ from zeroth in (2.11) directly yields infinitely many conservation laws

$$\begin{cases}(p\chi_1+\alpha p)_t=\left(-\frac{1}{2}p^2q\chi_1+p\chi_2+\frac{1}{2}p_x\chi_1-\alpha p^2\chi_1-\frac{1}{2}\alpha p^2q-\alpha^2p^2+\frac{1}{2}\alpha p_x\right)_x\,,\\ (p\chi_{i+1})_t=\left(-\frac{1}{2}p^2q\chi_{i+1}+p\chi_{i+2}+\frac{1}{2}p_x\chi_{i+1}-\alpha p^2\chi_{i+1}\right)_x\,,\quad i\geq 1\,,\end{cases} \tag{2.12}$$

where χ_i, $i \geq 1$, are defined recursively by (2.7). The first two conservation laws in (2.12) can be worked out as follows:

$$\left(\frac{1}{2}pq+\alpha p\right)_t=\left(-\frac{3}{8}p^2q^2-\frac{1}{4}pq_x+\frac{1}{4}p_xq-\frac{3}{2}\alpha p^2q-\alpha^2p^2+\frac{1}{2}\alpha p_x\right)_x \tag{2.13}$$

and

$$\left(-\frac{1}{8}p^2q^2-\frac{1}{2}\alpha p^2q-\frac{1}{4}pq_x\right)_t$$
$$=\left(\frac{1}{8}p^3q^3+\frac{3}{8}p^2qq_x+\frac{1}{8}pq_{xx}-\frac{1}{8}p_xq_x+\frac{3}{4}\alpha p^3q^2+\alpha^2p^3q+\frac{3}{4}\alpha p^2q_x\right)_x\,. \tag{2.14}$$

Second, by use of (2.8) in (2.11), the coefficients of λ presents a trivial conservation law, and comparing different powers of λ from zeroth exactly yields infinitely many conservation laws

$$\begin{cases}(p\kappa_1+\alpha p)_t=\left(-\frac{1}{2}p^2q\kappa_1+p\kappa_2+\frac{1}{2}p_x\kappa_1-\alpha p^2\kappa_1-\frac{1}{2}\alpha p^2q-\alpha^2p^2+\frac{1}{2}\alpha p_x\right)_x\,,\\ (p\kappa_{i+1})_t=\left(-\frac{1}{2}p^2q\kappa_{i+1}+p\kappa_{i+2}+\frac{1}{2}p_x\kappa_{i+1}-\alpha p^2\kappa_{i+1}\right)_x\,,\quad i\geq 1\,,\end{cases} \tag{2.15}$$

where κ_i, $i \geq 1$, are defined recursively by (2.9). The first two conservation laws in (2.15) can be computed as follows:

$$\left[-\frac{1}{2p}(p^2q-2p_x+2\alpha p^2)\right]_t$$
$$=\left[-\frac{1}{8p}(-3p^3q^2+2p^2q_x+10pp_xq-4p_{xx}-8\alpha^2p^3-12\alpha p^3q+20\alpha pp_x)\right]_x \tag{2.16}$$

and

$$\left[-\frac{1}{8p^2}(-p^4q^2+2p^3q_x+4p^2p_xq-4pp_{xx}+4p_x^2-4\alpha p^4q+8\alpha p^2p_x)\right]_t$$
$$=\left[-\frac{1}{8p^2}(p^5q^3-6p^3p_xq^2-3p^4qq_x+5p^2p_xq_x+6p^2p_{xx}q+p^3q_{xx}+2p_xp_{xx}\right.$$
$$\left.-2pp_{xxx}+6\alpha p^5q^2+8\alpha^2p^5q-24\alpha p^3p_xq-16\alpha^2p^3p_x-6\alpha p^4q_x+12\alpha p^2p_{xx})\right]_x. \tag{2.17}$$

Observing the two recursive formulas (2.7) and (2.9), we see that the first series of conservation laws in (2.12) is of differential polynomial type and the second series of conservation laws in (2.15) is of rational differential function type.

We point out that the other Riccati equation,

$$\phi_{12,x}=2(\lambda+\alpha p)\phi_{12}+\lambda p-q\phi_{12}^2, \tag{2.18}$$

which the other ratio $\phi_{12}=\frac{\phi_1}{\phi_2}$ of the two eigenfuntions satisfies, generates two equivalent classes of conservation laws, by use of similar Laurent expansions of ϕ_{12}.

3. Concluding Remarks

Based on a Lax pair, we constructed two series of conservation laws for the discussed perturbed Kaup–Newell equation. Both series of conservation laws were generated from taking two Laurent expansions of a ratio of eigenfunctions, which satisfy a Riccati equation, and the first two conservation laws in each series were explicitly presented.

Such Riccati equations are also used to present algebro-geometric solutions by expanding ratios of eigenfunctions, called the Baker–Akhiezer functions, into Laurent series. In 2×2 matrix spectral problems, hyperelliptical curves are adopted, and in 3×3 matrix spectral problems, trigonal curves are introduced to construct algebro-geometric solutions to soliton equations, by taking characteristic polynomials of Lax matrices.[16–18]

Moreover, by symbolic computations, rational and lump solutions are discussed recently for many interesting nonlinear equations,[19–22] and Hirota bilinear forms[23] or generalized bilinear forms[24–26] have been used to present rational and lump solutions by links of logarithmic derivatives.

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