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Lump solutions to the BKP equation by symbolic computation

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Lump solutions are rationally localized in all directions in the space. A general class of lump solutions to the $(2+1)$-dimensional B-Kadomtsev–Petviashvili (BKP) equation is presented through symbolic computation with Maple. The Hirota bilinear form of the equation is the starting point in the computation process. Like the KP equation, the resulting lump solutions contain six arbitrary parameters. Two of the parameters are due to the translation invariances of the BKP equation with the independent variables, and the other four need to satisfy a nonzero determinant condition and the positivity condition, which guarantee analyticity and rational localization of the solutions.

Keywords: Hirota bilinear form; lump solution; BKP equation.

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1. Introduction

Since lump solutions to the Kadomtsev–Petviashviii (KP) equation[1]

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0\,, \tag{1}$$

were recognized as long-wave limits of soliton solutions,[2,3] lump solutions are presented for many other integrable equations such as the Davey–Stewartson-II equation,[3] the three-dimensional three-wave resonant interaction[4] and the Ishimori-I equation.[5] Almost all integrable equations possess Hirota bilinear forms.[6] It becomes an interesting question to take advantage of Hirota bilinear forms or even generalized bilinear forms[7] to construct lump solutions to nonlinear partial differential equations. We will, in this paper, consider the $(2 + 1)$-dimensional

B-Kadomtsev–Petviashviii (BKP) equation[8,9]

$$P_{\rm BKP}(u) := (u_t + 15uu_{xxx} + 15{u_x}^3 - 15u_xu_y + u_{5x})_x - 5u_{xxxy} - 5u_{yy} = 0\,. \quad (2)$$

The BKP equation (2) is a member of the BKP soliton hierarchy, and it is also a $(2+1)$-dimensional generalization of the Caudrey–Dodd–Gibbon–Sawada–Kotera (CDGSK) equation[10,11]

$$v_t + 15vv_{xxx} + 15v_xv_{xx} + 45v^2v_x + v_{5x} = 0\,, \quad (3)$$

when $v = u_x$ and v is a function of x and t. The associated spectral problem is of third-order:

$$-\phi_y + \phi_{xxx} + (3v - \lambda)\phi = 0\,, \quad (4)$$

which provides a basis for solving the Cauchy problem of the CDGSK equation (3)[12,13] by the inverse scattering transform.[14]

Soliton solutions are exponentially localized solutions in certain directions, and can be generated by using Hirota bilinear forms[6] (noting that some intelligent guesswork often needs to be made[15]). In contrast to soliton solutions, lump solutions are a kind of rational function solutions, localized in all directions in the space. By using Hirota bilinear forms, general rational function solutions were presented by the Wronskian and Casoratian determinant techniques,[16,17] particularly for the KdV equation, the Boussinesq equation and the Toda lattice equation (see, e.g., Refs. 18, 19 and 22).

In this paper, we would like to study the (2+1)-dimensional BKP equation, and construct a class of lump solutions by symbolic computation with Maple, supplementing soliton solutions.[20–22] Starting with the Hirota bilinear form of the BKP equation, we will carry out a search for positive quadratic function solutions to the corresponding bilinear BKP equation. The resulting quadratic function solutions contain a set of six arbitrary parameters. Taking special choices of the involved parameters leads to a particular class of lump solutions generated from long-wave limits of soliton solutions.[23] A few concluding remarks are given at the end of the paper.

2. Lump Solutions to the BKP Equation

Under the link between f and u:

$$u = 2(\ln f)_x\,, \quad (5)$$

the (2+1)-dimensional BKP equation (2) becomes the following (2+1)-dimensional Hirota bilinear equation:

$$\begin{aligned} B_{\rm BKP}(f) &:= (D_x^6 - 5D_x^3D_y + D_xD_t - 5D_y^2)f \cdot f \\ &= -10f_{yy}f + 10{f_y}^2 + 2f_{xt}f - 2f_tf_x - 10f_{xxxy}f + 30f_{xxy}f_x - 30f_{xy}f_{xx} \\ &\quad + 10f_yf_{xxx} + 2f_{6x}f - 12f_{5x}f_x + 30f_{4x}f_{xx} - 20{f_{xxx}}^2 = 0\,. \end{aligned} \quad (6)$$

The transformation (5) is also one of the characteristic transformations in establishing Bell polynomial theories of soliton equations,[24,25] and the exact relation between the BKP equation and the bilinear BKP equation is

$$P_{\rm BKP}(u) = \left[\frac{B_{\rm BKP}(f)}{f^2}\right]_x . \tag{7}$$

Therefore, if f solves the $(2+1)$-dimensional bilinear BKP equation (6), then $u = 2(\ln f)_x$ will solve the $(2+1)$-dimensional BKP equation (2).

In order to find quadratic function solutions to the bilinear BKP equation (6), similarly to the KP case,[26] we start with

$$f = g^2 + h^2 + a_9, \quad g = a_1x + a_2y + a_3t + a_4, \quad h = a_5x + a_6y + a_7t + a_8 , \tag{8}$$

where a_i, $1 \le i \le 9$, are real parameters to be determined. A simpler form $g^2 + a_5$ does not generate analytic solutions to the BKP equation, which are rationally localized in all directions in the space; and so, we take a sum of two squares of linear functions plus a constant function in (8). A direct Maple symbolic computation with such a class of functions f gives the following set of constraining equations for the parameters:

$$\begin{aligned} &\Big\{ a_1 = a_1, \quad a_2 = a_2, \quad a_3 = \frac{5(a_1{a_2}^2 - a_1{a_6}^2 + 2a_2a_5a_6)}{{a_1}^2 + {a_5}^2}, \quad a_4 = a_4 , \\ &a_5 = a_5, \quad a_6 = a_6, \quad a_7 = \frac{5(2a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2)}{{a_1}^2 + {a_5}^2}, \quad a_8 = a_8 , \\ &a_9 = -\frac{3(a_1a_2 + a_5a_6)(a_1^2 + {a_5}^2)^2}{(a_1a_6 - a_2a_5)^2} \Big\} . \end{aligned} \tag{9}$$

The function f is well defined on the whole space $\mathbb{R}^3$, if we require a nonzero determinant condition

$$\Delta := a_1a_6 - a_2a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \neq 0 , \tag{10}$$

which guarantees ${a_1}^2 + {a_5}^2 \neq 0$ and so every parameter in (9) makes sense. This set of parameters, in turn, generates positive quadratic function solutions to the bilinear BKP equation (6):

$$\begin{aligned} f = &\left(a_1x + a_2y + \frac{5(a_1{a_2}^2 - a_1{a_6}^2 + 2a_2a_5a_6)}{{a_1}^2 + {a_5}^2}t + a_4\right)^2 \\ &+ \left(a_5x + a_6y + \frac{5(2a_1a_2a_6 - {a_2}^2a_5 + a_5{a_6}^2)}{{a_1}^2 + {a_5}^2}t + a_8\right)^2 \\ &- \frac{3(a_1a_2 + a_5a_6)({a_1}^2 + {a_5}^2)^2}{(a_1a_6 - a_2a_5)^2} , \end{aligned} \tag{11}$$

when we impose a positivity condition for a_9:

$$a_1a_2 + a_5a_6 < 0 . \tag{12}$$

The resulting class of quadratic function solutions yield a class of lump solutions to the $(2+1)$-dimensional BKP equation (2) through the transformation (5):

$$u = \frac{4a_1 g + 4a_5 h}{f}\,, \tag{13}$$

where f is determined by (11), and g and h are defined by

$$g = a_1 x + a_2 y + \frac{5(a_1 {a_2}^2 - a_1 {a_6}^2 + 2\,a_2 a_5 a_6)}{{a_1}^2 + {a_5}^2} t + a_4\,, \tag{14}$$

$$h = a_5 x + a_6 y + \frac{5(2\,a_1 a_2 a_6 - {a_2}^2 a_5 + a_5 {a_6}^2)}{{a_1}^2 + {a_5}^2} t + a_8\,. \tag{15}$$

In this class of lump solutions, all six involved parameters, a_1, a_2, a_4, a_5, a_6 and a_8, are arbitrary provided that the solutions u are well defined on the whole space $\mathbb{R}^3$. This can be achieved, when the determinant condition (10) and the positivity condition (12) are satisfied. That determinant condition precisely means that two directions (a_1, a_2) and (a_5, a_6) in the xy-plane are not parallel.

Note that the solutions defined by (13) are analytic in $\mathbb{R}^3$ if and only if the parameter $a_9 > 0$. The analyticity of the solutions in (13) is guaranteed if both the determinant condition (10) and the positivity condition (12) hold. It is easy to observe that at any given time t, all the above solutions $u \to 0$ if and only if the corresponding sum of squares $g^2 + h^2 \to \infty$, or equivalently, $x^2 + y^2 \to \infty$ due to (10). Therefore, the nonzero determinant condition (10) and the positivity condition (12) guarantee both analyticity and localization of the solutions in (13). There are various possibilities to take appropriate parameters to obtain lump solutions.

If we take a particular choice of the parameters

$$a_1 = 1, \quad a_2 = 3(\alpha^2 - \beta^2), \quad a_5 = 0, \quad a_6 = 6\alpha\beta\,, \tag{16}$$

where $\beta > \alpha$ to guarantee $a_9 > 0$, then we have

$$a_3 = 45(\alpha^4 - 6\alpha^2\beta^2 + \beta^4), \quad a_7 = 180\alpha\beta(\alpha^2 - \beta^2), \quad a_9 = \frac{\beta^2 - \alpha^2}{4\alpha^2\beta^2}\,. \tag{17}$$

So, the resulting lump solutions read

$$u = \frac{4g}{f}\,, \tag{18}$$

where

$$f = g^2 + [6\alpha\beta y + 180\alpha\beta(\alpha^2 - \beta^2)t + a_8]^2 + \frac{\beta^2 - \alpha^2}{4\alpha^2\beta^2}\,, \tag{19}$$

$$g = x + 3(\alpha^2 - \beta^2)y + 45(\alpha^4 - 6\alpha^2\beta^2 + \beta^4)t + a_4\,. \tag{20}$$

This is a particular class of lump solutions generated from taking long-wave limits of a 2-soliton solution in Ref. 23, as did for the KP equation (see, e.g., Refs. 2 and 3 for details).

We now take

$$a_1 = 3(\alpha^2 - \beta^2), \quad a_2 = 1, \quad a_5 = 6\alpha\beta, \quad a_6 = 0\,, \tag{21}$$

where $\beta > \alpha$ to guarantee $a_9 > 0$. Then we have

$$a_3 = \frac{5(\alpha^2 - \beta^2)}{3(\alpha^2 + \beta^2)^2}, \quad a_7 = -\frac{10\alpha\beta}{3(\alpha^2 + \beta^2)^2}, \quad a_9 = -\frac{81(\alpha^2 - \beta^2)(\alpha^2 + \beta^2)^4}{4\alpha^2\beta^2}\,. \tag{22}$$

The resulting lump solutions read

$$u = \frac{12(\alpha^2 - \beta^2)g + 24\alpha\beta h}{f}\,, \tag{23}$$

where

$$f = g^2 + h^2 - \frac{81(\alpha^2 - \beta^2)(\alpha^2 + \beta^2)^4}{4\,\alpha^2\beta^2}\,, \tag{24}$$

$$g = 3(\alpha^2 - \beta^2)x + y + \frac{5(\alpha^2 - \beta^2)}{3(\alpha^2 + \beta^2)^2}t + a_4\,, \tag{25}$$

$$h = 6\alpha\beta x - \frac{10\alpha\beta}{3(\alpha^2 + \beta^2)^2}t + a_8\,. \tag{26}$$

This is a new class of lump solutions to the $(2+1)$-dimensional BKP equation (2).

The solutions (18) have a dependence on the spatial variable x in the first function g, but not in the second function h, while the solution (23) has a dependence on the variable y in the first function g, but not in the second function h. Generally, based on (13), we see that the corresponding lump solution u by (13) goes to zero, when the determinant Δ in (10) goes to zero. This is really true in the following two cases of the parameters.

First taking

$$a_1 = 1 + \varepsilon, \quad a_2 = 1, \quad a_4 = 0, \quad a_5 = 1, \quad a_6 = 1, \quad a_8 = 0\,, \tag{27}$$

which leads to $\Delta = \varepsilon$. Assume that $\varepsilon < 0$ to have $a_9 > 0$. From (13) in this case, we obtain the following lump solution

$$u = -\frac{4\varepsilon^2 p(\varepsilon)}{q(\varepsilon)}\,, \tag{28}$$

where

$$\begin{cases} p(\varepsilon) = x\varepsilon^3 + (4x + 2y)\varepsilon^2 + (10t + 6x + 4y)\varepsilon + 20t + 4x + 4y\,, \\ q(\varepsilon) = 3\varepsilon^7 - (x^2 - 24)\varepsilon^6 - 2(2x^2 + xy - 45)\varepsilon^5 - 2(4x^2 + 4xy + y^2 - 102)\varepsilon^4 \\ \quad -4(10tx + 5ty + 2x^2 + 3xy + y^2 - 75)\varepsilon^3 \\ \quad -4(25t^2 + 10tx + 10ty + x^2 + 2xy + y^2 - 72)\varepsilon^2 + 168\,\varepsilon + 48\,. \end{cases}$$

Obviously, we see that the limit of this solution is zero, when ε approaches zero.

Second taking

$$a_1 = 1, \quad a_2 = 1, \quad a_4 = 0, \quad a_5 = 1, \quad a_6 = 1 - \varepsilon, \quad a_8 = 0\,, \tag{29}$$

which leads to $\Delta = -\varepsilon$. Assume that $\varepsilon > 0$ to guarantee $a_9 > 0$. From (13) in this case, we obtain the following lump solution

$$u = \frac{8\varepsilon^2 p(\varepsilon)}{q(\varepsilon)}\,, \tag{30}$$

where

$$\begin{cases} p(\varepsilon) = -(10t+y)\varepsilon + 10t + 2x + 2y\,, \\ q(\varepsilon) = 25t^2\varepsilon^6 - 10(10t^2+ty)\varepsilon^5 + 2(100t^2+20ty+y^2)\varepsilon^4 \\ \quad -4(50t^2+10tx+15ty+xy+y^2)\varepsilon^3 \\ \quad +4(25t^2+10tx+10ty+x^2+2xy+y^2)\varepsilon^2 + 24\varepsilon - 48\,. \end{cases}$$

We also see that the limit of this solution is zero, when ε approaches zero.

3. Concluding Remarks

Based on the Hirota formulation and by symbolic computation with Maple, we constructed a class of lump solutions to the $(2+1)$-dimensional BKP equation. The analyticity and localization of the resulting lump solutions is guaranteed by a nonzero determinant condition and a positivity condition. A special class of lump solutions under reductions of the parameters involved covers the lump solutions previously presented by computing long-wave limits of soliton solutions.[23] A few particular classes of lump solutions with specific values of the parameters were computed.

Fortunately, for the bilinear BKP equation (6), we do not need to classify positive and negative equations, as we did for the KP case. Lumps solutions exist for both negative and positive BKP equations, but lumps solutions do not exist for the $(2+1)$-dimensional KPII equation (see, e.g., Ref. 26). Rational solutions to the (3+1)-dimensional KPII equation are linked to the good Boussinesq equation by a transformation of dependent variables.[27]

Very recently, Hirota bilinear forms have been generalized by introducing different assignments of signs[7] and applied to construction of rational solutions (see, e.g., Refs. 28–32). We point out that resonant solutions, in terms of exponential functions, to generalized bilinear and trilinear differential equations have been systematically discussed.[33,34] It would be very interesting to see when there exist positive polynomial solutions, even quadratic function solutions, to generalized bilinear and trilinear equations, which lead to lump solutions to the corresponding nonlinear equations, through the two characteristic transformations $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$ (see, e.g., Ref. 35). Particularly, rogue wave solutions could be generated as well in terms of positive polynomial solutions. An open question in the study of lump solutions by the Hirota bilinear technique is how to determine if a real multivariate polynomial is positive or nonnegative or possesses only one zero.[36]

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