Lump and rational solutions for weakly coupled generalized Kadomtsev–Petviashvili equation

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Received 11 May 2021
Revised 6 July 2021
Accepted 12 July 2021
Published 16 August 2021

Through the $Z_m$-KP hierarchy, we present a new $(3 + 1)$-dimensional equation called weakly coupled generalized Kadomtsev–Petviashvili (wc-gKP) equation. Based on Hirota bilinear differential equations, we get rational solutions to wc-gKP equation, and further we obtain lump solutions by searching for a symmetric positive semi-definite matrix. We do some numerical analysis on the trajectory of rational solutions and fit the trajectory equation of wave crest. Some graphics are illustrated to describe the properties of rational solutions and lump solutions. The method used in this paper to get lump solutions by constructing a symmetric positive semi-definite matrix can be applied to other integrable equations as well. The results expand the understanding of lump and rational solutions in soliton theory.

Keywords: WC-gKP equation; symmetric positive semi-definite matrix; lump solutions; rational solutions.

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1. Introduction

Nonlinear science plays an important role in fluid mechanics, plasma physics and other fields.\textsuperscript{1–4} The nonlinear evolution equation can explain the movement of fluid in shallow water waves well.\textsuperscript{3–10} Therefore, finding the exact solution of these integrable systems is a very significant work. In recent years, experts in physics and fluid mechanics have, for wave propagation in integrable systems, extended the generation mechanism and dynamic properties of rogue waves and lump waves.\textsuperscript{1–10}

In this paper, we consider the (3+1)-dimensional generalized Kadomtsev–Petviashvili (KP) equation\textsuperscript{11} as follows:

\[ u_{xt} - u_{xxxy} - 3(u_xu_y)_x - 2u_{xx} + u_{yy} + u_{zz} = 0, \]  

which is derived from the generalized bilinear equation. Here, \( u = u(x, y, z, t) \) denotes a scalar function of the space variables \( x, y, z \), and time variable \( t \).

Under the variable transformation, we have

\[ u(x, y, z, t) = 2(\ln f)_x, \]  

where \( f(x, y, z, t) \) is a real function. Inserting Eq. (2) into Eq. (1) yields

\[ (D_x D_t - D_x^3 D_y - 2D_x^2 + D_y^2 + D_z^2) f \cdot f = 0. \]  

The operator \( D \) is Hirota’s bilinear differential operator defined by

\[ D_x^m D_t^n f \cdot g = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, y, z, t) g(x, y, z, t)|_{x' = x, t' = t}. \]  

Through the \( Z_m \)-KP hierarchy which takes values in a maximal commutative subalgebra,\textsuperscript{12,13} a new (3+1)-dimensional equation called weakly coupled generalized Kadomtsev–Petviashvili (wc-gKP) is presented:

\[ \begin{cases} u_{xt} - u_{xxxy} - 3(u_xu_y)_x - 2u_{xx} + u_{yy} + u_{zz} = 0 \\ v_{tx} - v_{xxxy} - 3(v_xu_y)_x - 3(u_xv_y)_x - 2v_{xx} + v_{yy} + v_{zz} = 0. \end{cases} \]  

In addition, the consistent Riccati expansion method, generalized bilinear method and other ways can also be used to construct the new nonlinear systems which possess the rational and lump solutions.\textsuperscript{14–18}

The (3+1)-dimensional equation (5) reduces to the following equation in (2+1) dimensions under \( z = x \):

\[ \begin{cases} u_{xt} - u_{xxxy} - 3(u_xu_y)_x - u_{xx} + u_{yy} = 0, \\ v_{tx} - v_{xxxy} - 3(v_xu_y)_x - 3(u_xv_y)_x - v_{xx} + v_{yy} = 0. \end{cases} \]  

In most studies on the lump solutions, the bulk of Refs. 19–25 get lump solutions to a single equation by searching for positive quadratic functions.\textsuperscript{1–10} Lump and rational solutions also can be generated from soliton solutions by taking a long wave limit.\textsuperscript{29,30} In particular, Tian et al. use these two methods to investigate the breather wave and the lump wave of the KP equation.\textsuperscript{26–28} In this paper, we try to find lump and rational solutions of Eq. (6) by searching for the symmetric positive.
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semi-definite matrix. Furthermore, we discuss the trajectory of lump and rational solutions, and do some numerical analysis on rational solutions to Eq. (6).

The arrangement of this paper is organized as follows. In Sec. 2, we construct the bilinear equation to Eq. (6). In Sec. 3, based on the bilinear formalism to Eq. (6), we get rational solutions to Eq. (6), and further we obtain lump solutions by searching for a symmetric positive semi-definite matrix. We also do some numerical analysis on the trajectory of rational solutions and fit the trajectory equation of the wave crest. Finally, some conclusions are given in Sec. 4.

2. Bilinear Formalism

Using the variable transformation, we have
\[
\begin{align*}
\begin{cases}
u = 2 \frac{(g)}{f} x, \\
u = 2 \frac{(f)}{g} x,
\end{cases}
\end{align*}
\]
the bilinear form of Eq. (6) is generated as
\[
\begin{align*}
\begin{cases}
(D_x D_t - D_x^3 D_y - D_x^2 + D_y^2) f \cdot f = 0, \\
(D_x D_t - D_x^3 D_y - D_x^2 + D_y^2) f \cdot g = 0.
\end{cases}
\end{align*}
\]
That is,
\[
\begin{align*}
\begin{cases}
2f_{tx} f - 2f_{xx} f - 2f_{xxy} f + 2f_{yy} f - 2f_x f_t + 2f_x^2 + 6f_{xxy} f_x - 6f_{xxx} f_{xy} \\
+ 2f_{xxx} f_y - 2f_x^2 = 0, \\
f g_{tx} - f g_{xx} - f g_{xxy} + f g_{yy} + f t_x g - f x x g - f x y g_y + f y y g - f t g_x \\
- f x g_t + 2f x g_x + 3f x g_{xy} - 3f x g_{xy} + f x x g_y \\
+ 3f_{xxy} g_x - 3f_{xy} g_{xx} + f g_{xxx} g_x - 2f y g_y = 0.
\end{cases}
\end{align*}
\]
Here, \( f = f(x, y, t), g = g(x, y, t) \) are real functions, and the operator \( D \) is Hirota’s bilinear differential operator defined by Eq. (4).

It is clear that if \( f, g \) solve Eq. (9), then \( u(x, y, t), v(x, y, t) \) are solutions to Eq. (6) through dependent variable transformation equations (7).

3. Rational Solutions and Lump Solutions

In order to find the rational solutions to Eq. (6), we make the following assumption:
\[
\begin{align*}
\begin{cases}
f = X^T A X + c_1, \\
g = X^T B X + c_2,
\end{cases}
\end{align*}
\]
and
\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ t \end{bmatrix}.
\]

Here, both $A$ and $B$ are symmetric matrices with real entries. $c_1, c_2$ are also real parameters to be determined.

Substituting Eq. (10) into Eqs. (9) and (12) can be derived by comparing the coefficients of the same power terms of $x, y$, and $t$. The following set of constraining equations for the parameters had been generated by performing a direct Maple symbolic computation with $f$:

\begin{align*}
c_1 &= c_1, \quad a_{11} = a_{11}, \quad a_{12} = a_{12}, \\
a_{13} &= \frac{3a_{11}^2a_{12} + a_{11}^2c_1 - a_{12}^2c_1}{a_{11}c_1}, \\
a_{22} &= \frac{a_{12}(3a_{11}^2 + a_{12}c_1)}{a_{11}c_1}, \\
a_{23} &= -\frac{a_{12}(3a_{11}^2a_{12} - a_{11}^2c_1 + a_{12}^2c_1)}{a_{11}^2c_1}, \\
a_{33} &= \frac{9a_{11}^4a_{12}^2 + 6a_{11}^4a_{12}c_1 + a_{11}^4c_1^2 + 6a_{11}^2a_{12}^2c_1^2 - 2a_{11}^2a_{12}^2c_1^2 + a_{12}^4c_1^2}{a_{11}^3c_1^2}.
\end{align*}

(12)

Combining Eqs. (7), (10) and (12), we can get the $f$

\begin{align*}
f &= a_{11}x^2 + 2a_{12}xy + \frac{(6a_{11}^2a_{12} + 2a_{11}^2c_1 - 2a_{12}^2c_1)xt}{a_{11}c_1} \\
&\quad + \frac{a_{12}(3a_{11}^2 + a_{12}c_1)y^2}{a_{11}c_1} - 2\frac{a_{12}(3a_{11}^2a_{12} - a_{11}^2c_1 + a_{12}^2c_1)yt}{a_{11}^2c_1} \\
&\quad + \frac{(9a_{11}^4a_{12}^2 + 6a_{11}^4a_{12}c_1 + a_{11}^4c_1^2 + 6a_{11}^2a_{12}^2c_1^2)}{a_{11}^3c_1^2} + c_1,
\end{align*}

(13)

that corresponds to the rational solutions $u$ to Eq. (6). To get lump solutions $u$ to Eq. (6), the matrix $A$ and parameter $c_1$ should satisfy the following constraints:

1. Matrix $A$ is a positive semi-definite matrix, in other words, the eigenvalues of matrix $A$ are all non-negative.
2. $c_1 > 0$.
3. The elements in matrix $A$ must also satisfy the constraints $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 > 0$.

Conditions (1) and (2) guarantee that $f$ is always greater than 0. Condition (3) guarantees that $f$ has only one minimum value for any time $t$.

It is found that there are three arbitrary parameters $a_{11}, a_{12}, c_1$ in Eq. (13). Under the constraints of the above three conditions, we may assign the values of the three free variables as follows:

\begin{align*}
a_{11} &= 1, \quad a_{12} = 1, \quad c_1 = 1,
\end{align*}

(14)

which implies that

\begin{align*}
f &= 21t^2 + 6xt - 6yt + x^2 + 2xy + 4y^2 + 1,
\end{align*}

(15)
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Fig. 1. (Color online) The lump solution $u$ (Eq. (16) to Eq. (6)) at $t = 0$. (a) Contour plot; (b) $x$-curve; (c) $y$-curve.

and

\[ u = \frac{12t + 4x + 4y}{21t^2 + 6xt - 6yt + x^2 + 2xy + 4y^2 + 1}. \]  

The contour plot, $x$-curve, and $y$-curve of the lump solution $u$ are depicted in Fig. 1 when $t = 0$. Lump solution $u$ has a maximum point $(-5t + 1, 2t)$ and a minimum point $(-5t - 1, 2t)$, and the corresponding maxima and minima are 2 and $-2$, respectively. The moving velocity of both extremum points with time is $\sqrt{29}$.

Although the constraints between parameters $\{a_{ij}, b_{ij}, c_1, c_2, i, j = 1, 2, 3\}$ can be obtained by substituting Eq. (13) and $g = X^T BX + c_2$ into Eq. (9), the constraint equations are too complex to further analyze the properties of the rational solutions $v$. To remedy this, we substitute Eq. (15) into Eq. (9) and get the following constraint equations:

\[
\begin{align*}
    b_{11} &= b_{11}, \quad b_{12} = b_{12}, \quad b_{13} = -3c_2 + b_{12} + 5b_{11}, \\
    b_{22} &= -3c_2 + 5b_{12} + 2b_{11}, \quad b_{23} = 3c_2 - 8b_{12} + 2b_{11}, \\
    b_{33} &= -30c_2 + 42b_{12} + 9b_{11}, \quad c_2 = c_2.
\end{align*}
\]  

Equation (17) contains three arbitrary parameters $b_{11}, b_{12}, c_2$, which are given as follows:

\[ b_{11} = 1, \quad b_{12} = 2, \quad c_2 = 1, \]  

which implies that

\[ g = 63t^2 + 8tx - 22ty + x^2 + 4xy + 9y^2 + 1, \]  

and

\[ v_r = \frac{-2x^2y - 2x^2t - 10xy^2 + 32xyt - 84xt^2 - 2y^3 - 2y^2t}{(21t^2 + 6tx - 6ty + x^2 + 2xy + 4y^2 + 1)^2}. \]  

Equations (16) and (20) are a set of rational solutions to Eq. (6).
We find that rational solution $v_r$ has some interesting properties. First of all, $v_r$ is an odd function, which means that $v(x, y, t) + v(-x, -y, -t) = 0$. So, if we can figure out the trajectory of the crest when $t > 0$, we can figure out where the crest is at any given moment. Second, the solutions obtained from equations $v_{r,x} = 0$ and $v_{r,y} = 0$ are unwieldy algebraic structures. From that we can be sure that the crest is not moving uniformly in a straight line with time. Therefore, in order to study the motion path of the crest clearly, some numerical analysis on rational solutions is appropriate.

Table 1 describes the position of the crest at different moments. Here, $h$ is the height of the crest, and $v_{r,x}$, $v_{r,y}$ are used to measure the error of the numerical calculation. The closer $v_{r,x}$ and $v_{r,y}$ are to 0, the more accurate the position of the calculated crest is.

![Table 1. The position of the crest of rational solution $v_r$.](image)

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According to the description in the previous paragraph, we conduct nonlinear fitting for the data in Table 1. In some time range of $t > 0$, the movement trajectory of the wave crest roughly conforms to the following equation:

$$
\begin{align*}
  x &= \frac{-0.0335241199228886}{t + 0.0396119992712996} - 5t, \\
  y &= \frac{0.102865205001053}{t + 0.181067231314590} + 2t,
\end{align*}
$$

(21)

and, the height of wave crest $h$ satisfies the following equation:

$$
  h = -0.0141135773333083 + 6.00024659784242t.
$$

(22)

It can be known from Eq. (21) that with the increase of time, the motion rule of wave crest is approximately uniform linear motion. In addition, every other unit of time, at the height of the crest, increases by about six. It is worth mentioning that although the height of the crest changes over time, the lump wave is always local in the $x-o-y$ plane. Figure 2 shows this feature intuitively and clearly. From Fig. 2, we can intuitively observe that $v_r$ is an odd function, which has been discussed previously. Combined with Eq. (21), we can get the trajectory of the minimum
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Fig. 2. (Color online) The rational solution $v_r$ (Eq. (20) to Eq. (6)).

When $t < 0$

$$\begin{align*}
x &= \frac{-0.0335241199228886}{t - 0.0396119992712996} - 5t, \\
y &= \frac{0.102865205001053}{t - 0.181067231314590} + 2t.
\end{align*}$$

(23)

When $t > 0$, the height of the wave peak increases with the increase of time, this qualitative conclusion conforms to Eq. (22). In addition, it can be clearly seen from Fig. 2 that rational solution $v_r$ does not change its shape significantly with time $t$. Rational solution $v_r$ looks particularly similar to the spatial structures of lump solutions.

Whether Eqs. (21) and (23) can accurately express the position of the wave peak at other moments, we have made the following test on Eqs. (21) and (23), and the results are shown in Table 2.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
<th>$h$</th>
<th>$v_{r,x}$</th>
<th>$v_{r,y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>700</td>
<td>-3500.000048</td>
<td>1400.000147</td>
<td>4172.000294</td>
<td>-0.022112116</td>
<td>0.008479364</td>
</tr>
<tr>
<td>800</td>
<td>-4000.000042</td>
<td>1600.000129</td>
<td>4798.000258</td>
<td>-0.029704090</td>
<td>0.003799510</td>
</tr>
<tr>
<td>900</td>
<td>-4500.000037</td>
<td>1800.000114</td>
<td>5380.000228</td>
<td>-0.017040070</td>
<td>-0.098400384</td>
</tr>
<tr>
<td>-700</td>
<td>3500.000048</td>
<td>-1400.000147</td>
<td>-4172.000294</td>
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<td>-5380.000228</td>
<td>-0.017040070</td>
<td>-0.098400384</td>
</tr>
</tbody>
</table>

Table 2. Test of fitting equations.
In Table 2, both \( v_{r,x} \) and \( v_{r,y} \) are very close to 0, so Eqs. (21) and (23) can accurately represent the position of wave peaks when \( t \in [-900, 900] \). From Table 2, we can also find that Eq. (22) is no longer able to accurately express the height of the wave peak, but we can still calculate the height of the wave peak through Eqs. (20), (21) and (23).

Finally, we give a class of lump solutions to Eq. (6). If \( u = 2(\ln f)_x \) is a lump solution to Eq. (6) and \( g = f_x \), then \( v = 2(\frac{g}{f})_x \) is also a lump solution to Eq. (6). In combination with Eqs. (13) and (14), we give a specific set of solutions to Eq. (6):

\[
\begin{align*}
    u &= \frac{12t + 4x + 4y}{21t^2 + 6xt - 6yt + x^2 + 2xy + 4y^2 + 1}, \\
    v &= \frac{6t^2 - 12tx - 36ty - 2x^2 - 4xy + 4y^2 + 2}{(21t^2 + 6tx - 6ty + x^2 + 2xy + 4y^2 + 1)^2}.
\end{align*}
\] (24)

Lump solution \( u \) has a maximum point \((-5t + 1, 2t)\) and a minimum point \((-5t - 1, 2t)\), and the corresponding maxima and minima are 2 and \(-2\), respectively. At any given moment, the peak of lump solution \( v \) is in position \((-5t, 2t)\). Lump wave \( u \) and lump wave \( v \) move at the same speed with time, both of which are \( \sqrt{29} \). This type of lump wave is illustrated in Fig. 3.

4. Conclusion

In this work, we obtain rational solutions and lump solutions to wc-gKP equations by using bilinear formalism and constructing symmetric positive semi-definite matrices. We also find an interesting set of rational solutions \( u, v_r \) to Eq. (6), as shown in Eqs. (15) and (20). Figure 2 shows that the rational solution \( v_r \) is an odd function, and the spatial structure of \( v_r \) is very similar to that of general lump solutions. With the increase of time, the motion trajectory equations (21) and (24) of \( v_r \) are approximately uniform linear motion, and the height equation (22) of the crest increases linearly. In the same way, we can get rational and lump solutions to Eq. (5). The method used in this paper to get lump solutions by constructing a symmetric positive semi-definite matrix can be applied to other integrable
equations as well. Whether general higher-order lump solutions can be obtained by constructing symmetric positive semi-definite matrices is an important direction of our future research. Meanwhile, we also hope that our results will provide some valuable information in the field of nonlinear science.

Acknowledgment

This work is supported by the National Natural Science Foundation of China under Grant No. 11775104.

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