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Lump solutions of a new extended (2 + 1)-dimensional Boussinesq equation

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Through symbolic computation with Maple, two classes of lump solutions, rationally localized in all directions in space, are presented for a new extended $(2+1)$-dimensional Boussinesq equation. Analyticity of the solutions is naturally achieved, and particularly, taking special choices of the involved parameters will guarantee the positiveness of the constant term in the quadratic function f. Moreover, it deserves a note that one parameter in f plays an important role in order to maintain the positiveness of the quadratic function f. As illustrative examples, two particular lump solutions with specific values of the involved parameters are worked out and their three-dimensional plots, contour plots, x-curves and y-curves are made.

Keywords: Lump solution; Hirota bilinear method; (2 + 1)-dimensional Boussinesq equation.

1. Introduction

Searching for exact solutions of nonlinear partial differential equations is a hot research topic. One kind of exact solutions are soliton solutions, which are exponentially localized solutions in all directions in space. Recently, another kind of exact solutions has been widely discussed, called lump solutions. In contrast to soliton solutions, lump solutions are localized solutions, which are rationally decayed.

A class of lump solutions for the Kadomtsev–Petviashvili (KP) equation was presented by Manakov *et al.* in 1977.[1] Then, many high-dimensional nonlinear partial differential equations that have lump solutions were found, such as the three-dimensional three-wave resonant interaction equation,[2] Davey–Stewartson II equation,[3] BKP equation.[4] As lump waves were regarded as the appropriate prototypes to model rogue wave dynamics in both oceanography and nonlinear optics, many methods have been developed to search for lump solutions of nonlinear partial differential equations, such as the inverse scattering transformation,[5] the Darboux transformation,[6] the Bäckland transformation[2] and the Hirota bilinear method[7] and so on. Among them, the Hirota bilinear method based on a quadratic function plays an important role in presenting lump solutions of nonlinear partial differential equations.[8–12]

It is well known that the Boussinesq equation describes the propagation of shallow water waves with small amplitudes as they propagate at a uniform speed in a water canal of constant depth. It was proposed in 1871 by Boussinesq,[13] and was written as

$$u_{tt} - u_{xx} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0, \tag{1}$$

where β and α are arbitrary constants. It is also a soliton equation solvable by the inverse scattering method,[14] which arises in several other physical applications including one-dimensional nonlinear lattice-waves,[15] vibrations in a nonlinear string[14] and ion sound waves in a plasma.[16] In Ref. 17, solitons solution of the Boussinesq equation was obtained by using Darboux transformation of three-order eigenvalue differential equations. The rational solutions of the Boussinesq equation and applications to rogue waves have been studied in Ref. 18. Moreover, solitons, positrons and complexitons, including rational solutions, were systematically presented by the Wronskian technique.[19,20]

Recently, a new integrable (2+1)-dimensional Boussinesq equation was studied[21]

$$u_{tt} - 4u_{yt} + 4u_{yy} - 3\varepsilon u_{xy} - \frac{3}{2}\varepsilon(u^2)_{xx} + \frac{3}{4}\varepsilon^2 u_{xxxx} = 0, \quad \varepsilon^2 = \pm 1\,. \tag{2}$$

For the sake of generality, four arbitrary parameters were inserted in the original Eq. (2) in Ref. 22, which is called a new extended $(2+1)$-dimensional Boussinesq equation

$$u_{tt} + \alpha u_{yt} - \alpha u_{yy} + \alpha_1 \varepsilon u_{xy} + \alpha_2 \varepsilon(u^2)_{xx} + \alpha_3 \varepsilon^2 u_{xxxx} = 0, \quad \varepsilon^2 = \pm 1\,, \tag{3}$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3$ are arbitrary constants. The one-soliton solutions of both bright and dark types of Eq. (3) are derived by using the traveling wave method, and N-soliton, breather and rational solutions are also obtained by using the Hirota bilinear method and the long wave limit.[22]

In this paper, we focus on the new extended (2 + 1)-dimensional Boussinesq equation and present two classes of lump solutions of Eq. (3) by symbolic computation with Maple. Using the Hirota bilinear form (5) of Eq. (3), we will try to search for exact solutions with positive quadratic function solutions for Eq. (3). In the two classes of lump solutions, the positive quadratic function solutions contain five or six parameters except the parameters $\alpha, \alpha_1, \alpha_2, \alpha_3$, respectively, and it deserves a note that in order to get lump solutions of Eq. (3), the parameter α_3 needs to be greater than zero. Finally, a few concluding remarks will be given at the end of the paper.

2. Lump Solution

In order to obtain lump solutions to the potential function u in (3), we first use the dependent variable transformation

$$u = \frac{6\alpha_3\varepsilon}{\alpha_2}(\ln f)_{xx}, \tag{4}$$

to transform Eq. (3) into the bilinear form

$$(D_t^2 + \alpha D_y D_t - \alpha D_y^2 + \alpha_1\varepsilon D_x D_y + \alpha_3\varepsilon^2 D_x^4) f\cdot f = 0, \tag{5}$$

where f is a real function and D is the Hirota bilinear differential operator.

Therefore, if f solves the bilinear Eq. (5), $u = \frac{6\alpha_3\varepsilon}{\alpha_2}(\ln f)_{xx}$ will solve the nonlinear Eq. (3). In order to derive quadratic function solutions f to Eq. (5), we assume

$$f = g^2 + h^2 + a_9\,, \tag{6}$$

with

$$g = (a_1x + a_2y + a_3t + a_4)^2, \quad h = (a_5x + a_6y + a_7t + a_8)^2\,, \tag{7}$$

where $a_i (1 \le i \le 8)$ are real parameters to be determined. A simple form of $h^2 + a_5$ can generate analytic solutions of Eq. (3), but those solutions are not rationally localized in all directions in the space. So, we start with (6) and submit Eqs. (6) and (7) into Eq. (5). Through symbolic computation, two sets of constraining equations on the parameters are generated.

Case one:

$$\begin{aligned}
&a_2 = 0, \quad a_5 = -\frac{1}{4}\frac{a_1^2a_6^2\alpha_1^2\varepsilon^2 - a_3^2a_6^2\alpha^2 - 4a_3^2a_6^2\alpha - 4a_3^4}{a_3^2a_6\alpha_1\varepsilon}\,,\\
&a_7 = -\frac{1}{2}\frac{a_6(a_1\alpha_1\varepsilon + a_3\alpha)}{a_3}\,,\\
&a_9 = -\frac{3}{256}\frac{1}{a_3^{10}a_6^4\alpha_1^4\varepsilon^2}[\alpha_3(a_1^4a_6^4\alpha_1^4\varepsilon^4 - 2a_1^2a_3^2a_6^4\alpha^2\alpha_1^2\varepsilon^2 - 8a_1^2a_3^2a_6^4\alpha\alpha_1^2\varepsilon^2\\
&\qquad + 8a_1^2a_3^4a_6^2\alpha_1^2\varepsilon^2 + a_3^4a_6^4\alpha^4 + 8a_3^4a_6^4\alpha^3 + 8a_3^6a_6^2\alpha^2 + 16a_3^4a_6^4\alpha^2\\
&\qquad + 32a_3^6a_6^2\alpha + 16a_3^8)^2]\,,
\end{aligned} \tag{8}$$

which needs to satisfy the following constraint conditions:

$$a_6 \neq 0, \quad a_3 \neq 0, \quad a_1a_6 - a_2a_5 \neq 0, \quad \alpha_1 \neq 0, \quad a_9 > 0\,, \tag{9}$$

to guarantee the well-definition of f, the positiveness of f and the localization of u in all directions in the (x, y)-plane. The parameters in the set of (8) generate the first class of positive quadratic function solutions f to Eq. (5) as follows:

$$\begin{aligned} f = (a_1x + a_2y + a_3t + a_4)^2 + \Bigg(a_6y - \frac{a_6(a_1\alpha_1\varepsilon + a_3\alpha)}{2a_3}t \\ - \frac{a_1^2a_6^2\alpha_1^2\varepsilon^2 - a_3^2a_6^2\alpha^2 - 4a_3^2a_6^2\alpha - 4a_3^4}{4a_3^2a_6\alpha_1\varepsilon}x + a_8\Bigg)^2 + a_9\,, \end{aligned} \tag{10}$$

where a_9 is the same as the parameter presented by equations in (8). So, the first class of lump solutions to Eq. (3) is as follows:

$$u = \frac{6\alpha_3}{\alpha_2}\frac{2(a_1^2 - a_5^2)(h^2 - g^2) - 8a_1a_5gh + 2(a_1^2 + a_5^2)a_9}{(g^2 + h^2 + a_9)^2}\,, \tag{11}$$

where

$$g = a_1x + a_2y + a_3t + a_4\,, \tag{12a}$$

$$\begin{aligned} h = -\frac{1}{4}\frac{a_1^2a_6^2\alpha_1^2\varepsilon^2 - a_3^2a_6^2\alpha^2 - 4a_3^2a_6^2\alpha - 4a_3^4}{a_3^2a_6\alpha_1\varepsilon}x + a_6y \\ - \frac{1}{2}\frac{a_6(a_1\alpha_1\varepsilon + a_3\alpha)}{a_3}t + a_8\,. \end{aligned} \tag{12b}$$

Note that in order to guarantee the positiveness of f, the value of the parameter a_9 must be greater than zero. So, the value of the corresponding parameter α_3 must be less than zero. If we take

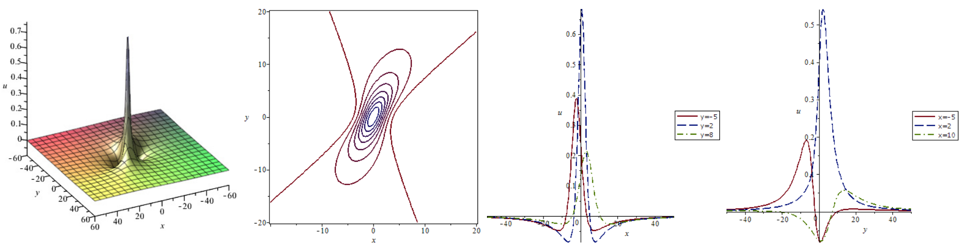
$$\begin{aligned} &a_1 = 1, \quad a_2 = 0, \quad a_3 = 0.5, \quad a_4 = 0, \quad a_5 = -0.5, \quad a_7 = 0.5, \quad a_8 = 0\,, \\ &a_9 = 18.75, \quad \alpha = 1, \quad \alpha_1 = -1, \quad \alpha_2 = -1, \quad \alpha_3 = -1, \quad \epsilon = 1, \quad a_6 = 1\,, \end{aligned} \tag{13}$$

this leads to

$$f = 1.25x^2 + y^2 + 0.5t^2 + 0.5tx - xy + ty + 18.75\,, \tag{14a}$$

$$\begin{aligned} u = \frac{3}{(m)^2}(-0.15625tx + 0.4375ty + 5.859375 + 0.125t^2 - 0.390625x^2 \\ + 0.1875y^2 + 0.3125xy)\,, \end{aligned} \tag{14b}$$

where $m = 0.25y^2 + 0.125t^2 + 0.3125x^2 + 4.6875 + 0.125tx - 0.25xy + 0.25ty$, and when $t = 0$, the corresponding three-dimensional plot, contour plot, x-curves and y-curves are depicted in Fig. 1. It is obvious that this kind of lump solution, as a type of rational solutions, are rationally localized in all directions in the space, which is different from soliton solutions exponentially localized in certain directions. From Fig. 1(a), it can be seen that the solution has one peak and two valleys. Because the



(a) (b) (c) (d)

Fig. 1. (Color online) The lump wave via the solution u in (15) with related parameters are chosen as $a_1 = 1$, $a_2 = 0$, $a_3 = 0.5$, $a_4 = 0$, $a_6 = 1$, $a_8 = 0$, $\alpha = 1$, $\alpha_1 = -1$, $\alpha_2 = -1$, $\alpha_3 = -1$, $\epsilon = 1$ and $t = 0$. (a) Three-dimensional plot; (b) contour plot; (c) x-curves; (d) y-curves.

height of the peak is larger than the depths of the valley bottoms, the solution can be called bright lump waves. In fact, there are also other kind of lump solutions, which have one peak and one valley and called bright-dark lump waves.[23] Figure 1(b) shows that the solution concentrate the energy of the background wave into a small region, so it is limited. Figures 1(c) and 1(d) show the different localized oscillations of the lump solution along the x- and y-axes, respectively.

Case two:

$$
\begin{aligned}
a_1 &= \frac{m_{11}}{\alpha_1\varepsilon(a_2^2+a_6^2)}, \quad a_5 = \frac{m_{12}}{\alpha_1\varepsilon(a_2^2+a_6^2)}, \\
a_9 &= -\frac{1}{\alpha_1^4\varepsilon^2(a_2^2+a_6^2)(a_2a_7-a_3a_6)^2}[3\alpha_3(a_2^4\alpha^2 - 2a_2^3a_3\alpha^2 + a_2^2a_3^2\alpha^2 \\
&\quad + 2a_2^2a_6^2\alpha^2 - 2a_2^2a_6a_7\alpha^2 + a_2^2a_7^2\alpha^2 - 2a_2a_3a_6^2\alpha^2 + a_3^2a_6^2\alpha^2 + a_6^4\alpha^2 \\
&\quad - 2a_6^3a_7\alpha^2 + a_6^2a_7^2\alpha^2 - 2a_2^2a_3^2\alpha + 2a_2^2a_7^2\alpha + 2a_2a_3^3\alpha - 8a_2a_3a_6a_7\alpha \\
&\quad + 2a_2a_3a_7^2\alpha + 2a_3^2a_6^2\alpha + 2a_3^2a_6a_7\alpha - 2a_6^2a_7^2\alpha + 2a_6a_7^3\alpha + a_3^4 \\
&\quad + 2a_3^2a_7^2 + a_7^4)^2],
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
m_{11} &= a_2^3\alpha - a_2^2a_3\alpha + a_2a_6^2\alpha - a_3a_6^2\alpha - a_2a_3^2 + a_2a_7^2 - 2a_3a_6a_7, \\
m_{12} &= a_2^2a_6\alpha - a_2^2a_7\alpha + a_6^3\alpha - a_6^2a_7\alpha - 2a_2a_3a_7 + a_3^2a_6 - a_6a_7^2,
\end{aligned}
\tag{16}
$$

which need to satisfy

$$
a_2a_7 - a_3a_6 \neq 0, \quad a_2a_6 \neq 0, \tag{17}
$$

to make the corresponding solutions f to be well defined, the conditions

$$
\alpha_3 < 0, \quad a_1a_6 - a_2a_5 \neq 0, \tag{18}
$$

to guarantee the positiveness of f and the localization of u in all directions in the space, respectively. The parameters in set (15) yield the second class of positive quadratic function solutions to Eq. (3) as

$$f = \left(\frac{m_{11}}{\alpha_1\varepsilon(a_2^2+a_6^2)}x + a_2y + a_3t + a_4\right)^2 + \left(\frac{m_{12}}{\alpha_1\varepsilon(a_2^2+a_6^2)}x + a_6y + a_7t + a_8\right)^2 + a_9\,, \tag{19}$$

where a_9 is the same as the parameter presented in Eq. (15). So, the corresponding lump solution of equation u is as follows:

$$u = \frac{6\alpha_3}{\alpha_2}\frac{2(a_1^2-a_5^2)(h^2-g^2) - 8a_1a_5gh + 2(a_1^2+a_5^2)a_9}{(g^2+h^2+a_9)^2}\,, \tag{20}$$

with the functions g and h being given as follows:

$$g = \frac{a_2^3\alpha - a_2^2a_3\alpha + a_2a_6^2\alpha - a_3a_6^2\alpha - a_2a_3^2 + a_2a_7^2}{\alpha_1\varepsilon(a_2^2+a_6^2)}x - \frac{2a_3a_6a_7}{\alpha_1\varepsilon(a_2^2+a_6^2)}x + a_2y + a_3t + a_4\,, \tag{21a}$$

$$h = \frac{a_2^2a_6\alpha - a_2^2a_7\alpha + a_6^3\alpha - a_6^2a_7\alpha - 2a_2a_3a_7 + a_3^2a_6}{\alpha_1\varepsilon(a_2^2+a_6^2)}x - \frac{a_6a_7^2}{\alpha_1\varepsilon(a_2^2+a_6^2)}x + a_6y + a_7t + a_8\,. \tag{21b}$$

If we take

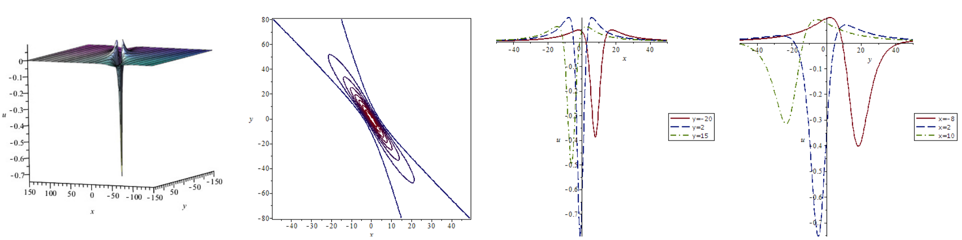
$$\begin{aligned} &a_1 = 1, \quad a_2 = 1, \quad a_3 = 1, \quad a_4 = 0, \quad a_5 = -3, \quad a_6 = -1, \quad a_7 = 1\,,\\ &a_8 = 0, \quad a_9 = 150, \quad \alpha = 1, \quad \alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = -1, \quad \epsilon = 1\,, \end{aligned} \tag{22}$$

we obtain

$$f = 2t^2 - 4tx + 10x^2 + 8xy + 2y^2 + 150\,, \tag{23a}$$

$$u = -\frac{12(3t^2 + 10tx + 8ty - 25x^2 - 20xy - 3y^2 + 375)}{(5x^2 + 4xy + y^2 + t^2 - 2tx + 75)^2}\,, \tag{23b}$$

and when $t = 0$, the localized characteristics and energy distribution of lump solution can be seen clearly in Fig. 2 including three-dimensional plot, contour plot,



(a) (b) (c) (d)

Fig. 2. (Color online) The lump wave via the solution v in (15) with related parameters are chosen as $a_2 = 1$, $a_3 = 1$, $a_4 = 0$, $a_6 = -1$, $a_7 = 1$, $a_8 = 0$, $\alpha = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = -1$, $\epsilon = 1$ and $t = 0$. (a) Three-dimensional plot; (b) contour plot; (c) x-curves; (d) y-curves.

x-curves and y-curves. It can be seen from Fig. 2(a) that the lump solution has two small peaks and one valley. The valley is hidden under the plane wave, and hence, it can be called a dark lump solution. Figure 2(b) shows the energy distribution of lump solution, and this solution concentrates the energy of the background wave into a small region. Figures 2(c) and 2(d) represent the localized characteristics of lump solution in the space, and the localized oscillation is presented.

3. Conclusions

In this paper, we obtained two classes of lump solutions to Eq. (3) based on the Hirota formulation. It deserves to take note that in order to keep the positiveness of f, the value of the parameter α_3 must be less than zero. It is also observed that at any given time t, all the above lump solutions $g^2 + h^2 \to \propto$, or equivalently, $x^2 + y^2 \to \propto$, which means that the exact solution $u(x, y, t)$ is analytic and localized in all directions.

It is known that the Hirota bilinear form provides an efficient tool to solve nonlinear differential equations of mathematical physics. By involving different prime numbers, Hirota bilinear operators have been generalized to generate diverse nonlinear differential equations possessing potential applications. So, it is interesting to investigate lump solutions for nonlinear differential equations which possess generalized bilinear forms, which will be one of our future work.

Meanwhile, compared with the extensive study of various exact solutions for nonlinear evolution equation, such as soliton solutions and so on, for both discrete and continuous cases,[24–28] the study of interaction solutions has aroused great interest of researchers.[29–34] When lump solutions of Eq. (3) is generated through symbolic computations, there is an interesting question that is what kind of interaction solutions can be formulated by combining other kinds of functions? There are already some results about interaction solutions between lump solutions and stripe solutions.[35–39] We will do some work in our next paper.

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